# Absurd sentences and normal deductions: The case of a logic of demodalised analytic implication with *falsum*

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**Abstract**

The logic of demodalised Parry's analytic implication, **DAI**, was introduced by J. M. Dunn. In its original formulation, the language of **DAI** contained classical negation, classical conjunction, and demodalised analytic implication as primitive connectives. Subsequently, R. L. Epstein rediscovered **DAI** as a logic of content inclusion, using content-assignment models. Epstein called it the logic of dependence **D**. He also studied **DAI** expanded with the constants *falsum* and *verum* in the context of algebraic analysis. More recently, A. Ledda, F. Paoli, and M. Pra Baldi investigated **DAI** with constants and presented its algebraic semantics in terms of implicative involutive bisemilattices.

In this paper, we study **DAI** formulated in a language without negation but with the constant *falsum*. First, we compare several semantic treatments of *falsum*, which give rise to different definable negations and, consequently, to different logics. We then focus on a single one of these logics, namely negation-free **DAI** with *falsum*. Second, we introduce two natural deduction systems for the resulting logic and prove soundness and completeness for both. We show that, in the first system, the elimination of maximum formulas requires infinitely many rules. For the second system, we establish a Prawitz-style normalisation theorem.

**Keywords**: content-assignment semantics, demodalised analytic implication; *falsum* constant; natural deduction; normalisation theorem; relating semantics.

## 1 Introduction

The logic of demodalised analytic implication **DAI** was introduced by J. M. Dunn in 1972 [4] as a modification—or demodalisation—of W. T. Parry's logic of analytic (strict) implication [18, 19]; cf. [16, 22]. In its original formulation, the language of **DAI** contained classical negation, classical conjunction, and demodalised analytic implication. Subsequently, R. L. Epstein [5, 6] rediscovered **DAI** as content inclusion logic and called it the dependence logic **D**. Dunn presented **DAI** by means of matrices, whereas Epstein employed content-assignment semantics—which he called set-assignment semantics—designed to provide a formal representation of sentential content; cf. [7].

There is a substantial body of literature on both the model theory and the proof theory of **DAI**. In what follows, we focus on selected papers dealing, on the one hand, with algebraic approaches and, on the other, with non-axiomatic deductive systems, since these two strands are directly related to the issues investigated in this paper.

In 1987, Epstein [5] considered **DAI** expanded with the constants *falsum* and *verum* and gave the first algebraic treatment of the resulting system. Later, in 2019, A. Ledda, F. Paoli and M. Pra Baldi [15] analysed **DAI** with constants in an algebraic setting and presented the algebraic semantics for it in terms of implicative involutive bisemilattices.[1]

Besides providing algebraic insights into **DAI** with constants, these analyses bring to the fore the question of how absurd sentences, represented by *falsum*, and tautological sentences, represented by *verum*, should be understood in terms of content. Epstein and Ledda et al. adopt different conceptions, reflecting different answers to the question of whether the content of these two types of sentence should be represented by the entire universe of content or by the empty set. Epstein's content-assignment semantics provides a natural framework for analysing this issue formally. We discuss the problem in detail and indicate several possible solutions. The issue is significant not only from an algebraic perspective but also from a proof-theoretic one, particularly in connection with natural deduction systems.

As for proof theory, in 1987, W. A. Carnielli [1] introduced tableau systems for all of Epstein's logics, including **DAI**. In 2022, Jarmużek and Klonowski [10] proposed a different approach to tableau systems. In 1991, L. Fariñas del Cerro and V. Lugardon [3] developed sequent systems for certain modifications of Epstein's logics (including modification of **DAI**, though not **DAI**). In 1997, they [2] extended logics of this kind to the first-order case,

[1] M. Nowak [16] and N. Zamperlin [22] proposed different semantics for **DAI** based on algebraic structures in 2002 and 2025, respectively. Dunn, of course, had already provided the first algebraic semantics for the logic by means of matrices.

and provided corresponding sequent systems. Other interesting results were presented in 2007 by F. Paoli [17], who studied algebraic and proof-theoretic aspects of FDE-fragments of Epstein's logics, including **DAI**.

However, proof-theoretic work on **DAI** still lacks a natural deduction or sequent-calculus treatment enjoying, respectively, the standard properties of normalisation or cut elimination. An initial attempt to develop a natural-deduction-style system was made in [14], but the systems presented there do not admit a normalisation theorem. The present paper aims to fill this gap.

We study **DAI** formulated in a language without negation but with the constant *falsum*. Our goals are (i) to analyse different semantic treatments of *falsum*, which give rise to different definitions of negation, and (ii) to present natural deduction systems for **DAI** with *falsum* and investigate the elimination of maximum formulas in these systems.

# 2 Logic $\mathbf{DAI}_\perp$: Content-assignment semantics and a content of absurd sentences

In what follows, we consider a propositional language (zeroth-order) with countably many propositional variables $p_1, p_2, p_3, \ldots$, the constant $\perp$ (*falsum*), the connectives $\wedge$ (conjunction), $\rightarrow$ (demodalised (non-modal) analytic implication) and parentheses $), ($. The set of variables is denoted by $\mathsf{Var}$, and the set of formulas constructed in the usual way over this language is denoted by $\mathsf{For}$. For every formula $A \in \mathsf{For}$, we denote the set of variables occurring in $A$ by $\mathsf{var}(A)$. We occasionally use the following abbreviations: $A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$ and $A \looparrowright B = A \rightarrow (B \rightarrow B)$.

## 2.1 A content-assignment semantics

We define the logic of demodalised analytic implication with *falsum* using the *content-assignment semantics* proposed by Epstein [6], which is based on a function that enables us to represent sentential content formally.

A *content-assignment model* (a *ca-model*) is a triple $\langle v, c, R \rangle$ such that $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ is a logical value assignment, $c\colon \mathsf{For} \longrightarrow P(U)$ is a function (a *content-assignment function* or *content assignment*), where $U$ is a non-empty set (a *universe of content*) and $P(U)$ is the powerset of $U$, and $R \subseteq c(\mathsf{For}) \times c(\mathsf{For})$ is a *content relation.* We consider the following truth conditions:

+ $M \models A$ iff $v(A) = 1$, if $A \in \mathsf{Var}$,
+ $M \not\models \perp$,

+ $M \models A \wedge B$ iff $M \models A$ and $M \models B$,
+ $M \models A \rightarrow B$ iff $M \not\models A$ or $M \models B$, and $R(c(A), c(B))$.

We sometimes write $M \models X$ meaning that for all $B \in X$, $M \models B$.

As we can see, the truth condition for implication requires, for a conditional to be true, that its antecedent be false or its consequent be true—a classical extensional condition—and that the content of the antecedent and that of the consequent stand in a certain relation to one another.

Let *Mod* be a class of ca-models. We say that $A$ is a *semantic consequence of* $X$ *in Mod* ($X \models_{Mod} A$) iff for any sa-model $M \in Mod$, if $M \models X$, then $M \models A$. We say that $A$ is *valid in Mod* iff $\emptyset \models_{Mod} A$. By logic we will understand a consequence relation determined by means of a given class of models.

To characterise the logic under consideration, we must specify how sentential content is assigned to complex sentences and define the content relation in such a way as to obtain demodalised analytic implication.

Epstein assumed that logical connectives do not affect the content of a sentence, since they are syncategorematic. He introduced two types of content-assignment functions:

+ a *union content-assignment* is defined by the following condition:

  (uca) $c(A \star B) = c(A) \cup c(B)$,

  for every $\star \in \{\wedge, \rightarrow\}$,
+ an *intersection content-assignment* is defined by the following condition:

  (ica) $c(A \star B) = c(A) \cap c(B)$,

  for every $\star \in \{\wedge, \rightarrow\}$.

In the original formulation of **DAI** without constants, but with classical negation $\neg$, for each content-assignment function $s$ the following content neutrality condition was additionally assumed: $c(\neg A) = c(A)$.

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  (ica) $c(A \star B) = c(A) \cap c(B)$,

  for every $\star \in \wedge, \rightarrow$.

In the original formulation of **DAI**, whose language contained classical negation $\neg$ but no constants, every content-assignment function $c$ was additionally required to satisfy the content-neutrality condition: $c(\neg A) = c(A)$.

The conditions (uca) and (ica) correspond to the two definitions of the content relation adopted by Epstein in ca-models. For a ca-model $\langle v, c, R\rangle$ in which $c$ satisfies (uca), the relation $R$ is defined by:

(ucaR) $R(c(A), c(B))$ iff $c(B) \subseteq c(A)$.

For a ca-model $\langle v, c, R\rangle$ in which $c$ satisfies (ica), the relation $R$ is defined by:

(icaR) $R(c(A), c(B))$ iff $c(A) \subseteq c(B)$.

The conditions (uca) and (ica) do not specify the content assigned to the constant *falsum*. This raises the question of how the content-assignment function should operate on *falsum*, or, equivalently, what content should be assigned to the absurd sentence represented by this constant. At least three approaches to this problem can be found in the literature directly concerned with demodalised analytic implication. Each has certain advantages and disadvantages from the perspective of the logical analysis of language.

## 2.2 A content of absurd sentences

Ledda, Paoli, and Pra Baldi [15] proposed the following condition on the content-assignment function for *falsum*:

(LPP$\bot$) $c(\bot) = \emptyset$.

They also assumed (uca) for binary connectives.

On this approach, we assume that there are certain absurd sentences with empty content—that is, contentless absurd sentences. This assumption allows classical negation to be defined straightforwardly.

Let us consider ca-models whose content assignments satisfy both (uca) and (LPP$\bot$). For implication, we adopt the truth condition (ucaR). Such models will be called *uca-LPP-models*, and the class of all such models will be denoted by **uca-LPP**.

We define negation in the standard way:

$(\mathrm{Def}_1\neg)$ $\neg A = A \to \bot$ .

We then obtain:

$$
\begin{aligned}
c(\neg A) =& c(A \to \bot) \\
=& c(A) \cup c(\bot) \\
=& c(A).
\end{aligned}
$$

Thus, the content-neutrality condition is satisfied. Moreover, the negation defined by $(\mathrm{Def}_1\neg)$ is classical:

$$
\begin{aligned}
M \models \neg A \text{ iff } & M \models A \to \bot \\
\text{iff } & M \not\models A \text{ or } M \models \bot, \text{ and } c(\bot) \subseteq c(A) \\
\text{iff } & M \not\models A.
\end{aligned}
$$

Prior to Ledda et al., Epstein [5] proposed the following condition on the content-assignment function for *falsum*:

$(\mathrm{E}\bot)$ $c(\bot) = U.$

He also assumed condition (uca) for binary connectives.

On this approach, we postulate the existence of certain absurd sentences with maximal content. Their content exhibits a kind of content overflow: it contains every possible content.

However, under this approach, the definition of negation becomes problematic. Consider models whose content assignments satisfy both (uca) and $(\mathrm{E}\bot)$, and in which implication is interpreted according to the truth condition (ucaR). We call such models *uca-E-models*. In these models, classical negation cannot be defined in the standard way by $(\mathrm{Def}_1\neg)$, since:

$$
\begin{aligned}
M \models \neg A \text{ iff } & M \models A \to \bot \\
\text{iff } & M \not\models A \text{ or } M \models \bot, \text{ and } c(\bot) \subseteq c(A) \\
\text{iff } & M \not\models A \text{ and } c(\bot) \subseteq c(A) \\
\text{iff } & M \not\models A \text{ and } c(A) = U.
\end{aligned}
$$

Under this approach, the connective defined by $(\mathrm{Def}_1\neg)$ yields a form of non-classical negation that may be worthy of further investigation. For example, there is a uca-E-model $M$ such that $M \not\models A$ and $M \not\models A \to \bot$, where $A \in \mathsf{Var}$. We leave a systematic investigation of this form of negation for a separate paper.

This does not mean, however, that classical negation cannot be defined under this approach. We may instead introduce it by the following definition:

(Def$_2\neg$) $\neg A = (A \wedge (\bot\!\to\!\bot)) \to\!\bot$ .

The negation defined by (Def$_2\neg$) is classical:

$$
\begin{aligned}
M \models \neg A \text{ iff } & M \models (A \wedge (\bot\!\to\!\bot)) \to\!\bot \\
\text{iff } & M \not\models A \wedge (\bot\!\to\!\bot) \text{ or } M \models\!\bot, \text{ and } c(\bot) \subseteq c(A \wedge (\bot\!\to\!\bot)) \\
\text{iff } & M \not\models A \wedge (\bot\!\to\!\bot) \text{ and } c(\bot) \subseteq c(A) \cup c(\bot) = c(\bot) \\
\text{iff } & M \not\models A \text{ or } M \not\models\!\bot\!\to\!\bot \\
\text{iff } & M \not\models A \text{ or } [M \models\!\bot \text{ and } M \not\models\!\bot] \text{ or } c(\bot) \not\subseteq c(\bot) \\
\text{iff } & M \not\models A.
\end{aligned}
$$

Although (Def$_2\neg$) allows us to define classical negation, it does not preserve content neutrality. Indeed, for every formula $A$, $c(\neg A) = U$. Thus, there is a uca-E-model $\langle v, c, R\rangle$ and a formula $A$ such that $c(\neg A) \neq c(A)$.

However, the foregoing analysis does not rule out the possibility of using (E$\bot$) together with (Def$_1\neg$) to obtain a classical negation satisfying the content-neutrality condition.

Let us consider ca-models whose content-assignment functions satisfy (ica) and (E$\bot$), and in which implication is interpreted according to the truth condition (icaR). We call such models *ica-E-models* and denote the class of all such models by **ica-E**. For these models, we obtain:

$$
\begin{aligned}
c(\neg A) =& c(A \to\!\bot) \\
=& c(A) \cap c(\bot) \\
=& c(A).
\end{aligned}
$$

Moreover, the following holds:

$$
\begin{aligned}
M \models A \to\!\bot \text{ iff } & M \not\models A \text{ or } M \models\!\bot, \text{ and } c(A) \subseteq c(\bot) \\
\text{iff } & M \not\models A \text{ or } M \models\!\bot \\
\text{iff } & M \not\models A.
\end{aligned}
$$

Of course, if we replace (uca) and (ucaR) with (ica) and (icaR), respectively, then the condition (LPP$\bot$) does not allow classical negation to be defined by (Def$_1\neg$), just as (E$\bot$) fails to do so under (uca) and (ucaR).

Consider ca-models whose content-assignment functions satisfy (ica) and (LPP$\bot$), and in which implication is interpreted according to the truth condition (icaR). We call such models *ica-LPP-models*. We then obtain:

$$\begin{aligned} M \models A \to\perp \text{ iff } & M \not\models A \text{ or } M \models\perp, \text{ and } c(A) \subseteq c(\perp) \\ \text{iff } & M \not\models A \text{ and } c(A) \subseteq c(\perp) \\ \text{iff } & M \not\models A \text{ and } c(A) = \emptyset. \end{aligned}$$

Thus, once again, we obtain a form of non-classical negation. The question remains whether classical negation can nevertheless be expressed in the ica-LPP-models under consideration. It can: it suffices to use the definition ($\mathrm{Def}_2\neg$) once again. We then obtain:

$$\begin{aligned} M \models \neg A \text{ iff } & M \models (A \wedge (\perp\to\perp)) \to\perp \\ \text{iff } & M \not\models A \wedge (\perp\to\perp) \text{ and } c(A \wedge (\perp\to\perp)) \subseteq c(\perp) \\ \text{iff } & M \not\models A \text{ or } M \not\models\perp\to\perp, \text{ and } c(A) \cap \emptyset \subseteq \emptyset \\ \text{iff } & M \not\models A. \end{aligned}$$

As in the earlier case involving (E$\perp$), the definition ($\mathrm{Def}_2\neg$) does not yield, in the present setting, a negation satisfying the content-neutrality condition. Indeed, there is an ica-LPP-model $\langle v, c, R\rangle$ and a formula $A$ such that $c(\neg A) \neq c(A)$.

In the foregoing discussion, we did not address the problem of determining the content of the constant *verum*, focusing exclusively on *falsum*. In the approach adopted by Ledda et al., the content-assignment function assigns the same value to *verum* and *falsum*. By contrast, in Epstein's approach, the content assigned to *verum* is the complement of the content assigned to *falsum*. Accordingly, when both constants and the two conditions governing the assignment of content to compound formulas are taken into account, eight combinations can be distinguished, as shown in Table 1.

In the remainder of this paper, we focus on the constant *falsum* and the classes **uca-LPP** and **ica-E**. We leave the consideration on *verum* and other classes of models, as well as the logics they define, as a topic for further research.

Before proceeding to the subsequent analysis, however, it is worth considering one further approach to representing the content of absurd sentences, which differs in kind from those presented above. This approach requires replacing the single constant $\perp$ with infinitely many *falsum* constants, each indexed by a finite set of propositional variables. It may be developed along the lines suggested by the sequent calculi introduced by del Cerro and Lugardon [2] for certain variants of Epstein's logics of content relations, including a variant of **DAI**.

| | *falsum* | *verum* |
|---|---|---|
| uca | $\emptyset$ | $\emptyset$ |
| ica | $\emptyset$ | $\emptyset$ |
| uca | $\emptyset$ | $U$ |
| ica | $\emptyset$ | $U$ |
| uca | $U$ | $\emptyset$ |
| ica | $U$ | $\emptyset$ |
| uca | $U$ | $U$ |
| ica | $U$ | $U$ |

Table 1: Combinations of content-assignments

Suppose that we modify the language introduced above. As before, it contains countably many propositional variables and the binary connectives of conjunction and implication. In addition, for every $n \geq 1$ and every set of $\{A_1, \ldots, A_n\}$ of propositional variables, the language contains a constant $\bot_{A_1 \ldots A_n}$. We impose the following conditions on the content-assignment function for these *falsum* constants:

(DL$\bot$) $c(\bot_{A_1 \ldots A_n}) = c(A_1) \cup \ldots \cup c(A_n)$.

On this approach, the content of an absurd sentence is determined by the contents of the atomic sentences occurring in it. Consequently, from the point of view of content, we are not dealing with a single absurdity, but with infinitely many extensionally indistinguishable absurdities. In view of the foregoing considerations, however, classical negation can be defined by modifying (Def$_1\neg$) as follows:

(Def$_3\neg$) $\neg A = A \rightarrow \bot_{A_1 \ldots A_n}$,

where $\mathsf{var}(A) = A_1, \ldots, A_n$.

The definition (Def$_3\neg$) yields a classical negation satisfying the content-neutrality condition under both treatments of complex content considered above. More precisely, it does so both in ca-models whose content-assignment functions satisfy (uca) and (DL$\bot$) and in which implication is interpreted according to (ucaR), and in ca-models whose content-assignment functions satisfy (ica) and (DL$\bot$) and in which implication is interpreted according to (icaR). The principal drawback of this solution is that it requires an infinite family of *falsum* constants.

## 2.3 Logic $\mathbf{DAI}_\perp$: The negation-free DAI with *falsum*

Let us now characterise the logic under consideration. The *negation-free logic of demodalised analytic implication with falsum*, $\mathbf{DAI}_\perp$, can be characterised using either the class **uca-LPP** or the class **ica-E**, with negation defined by (Def$_1\neg$). Indeed, the two classes determine the same consequence relation:

$$\models_{\textbf{uca-LPP}} = \models_{\textbf{ica-E}}.$$

It is easy to see that either class can be used to define $\mathbf{DAI}_\perp$, provided that a relating semantics is employed in the metalogical study of the logic; see [12, 11, 13].

A *relating model* is a pair $\langle v, R\rangle$ such that $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ is a logical value assignment and $R \subseteq \mathsf{For} \times \mathsf{For}$ is a *relating relation.* We adopt the same truth conditions as above:

+ $M \models A$ iff $v(A) = 1$, if $A \in \mathsf{Var}$,
+ $M \not\models \perp$,
+ $M \models A \wedge B$ iff $M \models A$ and $M \models B$,
+ $M \models A \rightarrow B$ iff $M \not\models A$ or $M \models B$, and $R(A, B)$.

Let **rmD** be the class of all relating models with $R$ satisfying the following conditions, cf. [13]:

(tran) $\forall_{A,B,C\in\mathsf{For}}((R(A, B)$ and $R(B, C))$ only if $R(A, C))$,
(d1) $\forall_{A\in\mathsf{For}}R(A, \neg A)$,
(d2) $\forall_{A\in\mathsf{For}}R(\neg A, A)$,
(d3) $\forall_{A,B,C\in\mathsf{For}}(R(A, B \wedge C)$ iff $(R(A, B)$ and $R(A, C)))$,
(d4) $\forall_{A,B,C\in\mathsf{For}}(R(A, B \rightarrow C)$ iff $(R(A, B)$ and $R(A, C)))$.

Let $\models_{\textbf{rmD}}$ be defined in the standard way by means of the class **rmD**. Then, modifying [6, pp. 120–123, 131–133] (cf. [13, pp. 9–18, 20–25]) we obtain:

$$\models_{\textbf{uca-LPP}} = \models_{\textbf{rmD}} = \models_{\textbf{ica-E}}.$$

Indeed, suppose that $R \subseteq \mathsf{For} \times \mathsf{For}$ satisfies (tran) and (d1)–(d2). For every $A \in \mathsf{For}$, let

$$c_R(A) = \{B \in \mathsf{For} : R(B, A)\},$$

and define

$c'_R(A) = \mathsf{For} \setminus c_R(A)$.

It is easy to show that $c_R$ is a content-assignment function satisfying (ica) and (E⊥), whereas $c'_R$ is a content-assignment function satisfying (uca) and (LPP⊥).

Conversely, the relation $R_c \subseteq \mathsf{For} \times \mathsf{For}$ satisfies (tran) and (d1)–(d2) under either of the following definitions:

$R_c(A, B)$ iff $c(B) \subseteq c(A)$, where $c$ satisfies (uca) and (LPP⊥),
$R_c(A, B)$ iff $c(A) \subseteq c(B)$, where $c$ satisfies (ica) and (E⊥).

In view of the correspondences established above, it is easy to see that $\mathbf{DAI}_\bot$ is deductively equivalent to the conservative expansion of **DAI** obtained by adding *falsum* to its language.[2] It is not, however, deductively equivalent to the original **DAI**. Indeed, no formula of the original language of **DAI** can define *falsum* in such a way that either (LPP⊥) or (E⊥) is satisfied.

On the basis of the foregoing considerations, $\mathbf{DAI}_\bot$ can also be axiomatized straightforwardly. It suffices to appeal to the definability of the relating relation by means of an appropriate schema; see [6, p. 125] and cf. [11]. More precisely, for every model $M = \langle v, R\rangle \in \mathbf{rmD}$ and all formulas $A, B \in \mathsf{For}$, $R(A, B)$ iff $M \models A \looparrowright B$. We can therefore axiomatize $\mathbf{DAI}_\bot$ by means of the following schemata; see [11, 13] and cf. [6, 16]:

(CL1) $\neg(A \wedge \neg(A \wedge A))$,
(CL2) $\neg((A \wedge B) \wedge \neg A)$,
(CL3) $\neg(\neg(A \wedge \neg B) \wedge \neg\neg(\neg(B \wedge C) \wedge \neg\neg(C \wedge A)))$,
(MD) $\dfrac{\neg(A \wedge \neg B) \quad A}{B}$
(DAI) $(A \to B) \leftrightarrow ((\neg(A \wedge \neg B)) \wedge (A \looparrowright B))$,
(DAI1) $((A \looparrowright B) \wedge (B \looparrowright C)) \to (A \looparrowright C)$,
(DAI2) $A \looparrowright \neg A$,
(DAI3) $\neg A \looparrowright A$,
(DAI4) $(A \looparrowright (B \to C)) \leftrightarrow ((A \looparrowright B) \wedge (A \looparrowright C))$,
(DAI5) $(A \looparrowright (B \wedge C)) \leftrightarrow ((A \looparrowright B) \wedge (A \looparrowright C))$.

[2] This expansion can be characterised by the uca-LPP-models, or equivalently by the ica-E-models, satisfying the content-neutrality condition for negation $c(\neg A) = c(A)$.

The proofs of soundness and completeness for the axiomatic system presented above are straightforward adaptations of the corresponding proofs given in [6, 16, 11]; cf. [9].

# 3 Natural deduction systems for $\mathbf{DAI}_\perp$

We now present natural deduction systems for $\mathbf{DAI}_\perp$, obtained by modifying the systems introduced in [14]. To formulate these systems precisely and investigate the problem of eliminating maximum formulas, we adapt the framework developed by D. Prawitz [20]; cf. [8, 21].

Both systems include the following standard rules:

$$\text{(RAA)}\ \frac{\begin{matrix}(\neg A)\\ \perp\end{matrix}}{A} \qquad\qquad \text{(EFQ)}\ \frac{\perp}{A}$$

$$(\wedge\mathrm{I})\ \frac{A \quad B}{A\wedge B} \qquad (\wedge\mathrm{E}_1)\ \frac{A\wedge B}{A} \qquad (\wedge\mathrm{E}_2)\ \frac{A\wedge B}{B}$$

In the case of the rule (RAA), we asssume that $A \neq \perp$ and $A \neq \neg B$ for any $B \in \mathsf{For}$ (cf. [20]).

## 3.1 First approach

We now turn to the first natural deduction system for $\mathbf{DAI}_\perp$. This system is formulated entirely in the language of the logic and therefore contains no labelled formulas. Moreover, the rules for implication do not employ formulas encoding, within the object language, the fact that the content-inclusion relation holds.

The system is, however, problematic with respect to the elimination of maximum formulas. As we shall see, the form of the implication rules gives rise to maximum formulas that cannot be eliminated. This difficulty can be overcome by introducing additional rules of an appropriate form. However, infinitely many such rules are required.

For the first system, in addition to the standard rules presented above, we adopt a single introduction rule for implication and two specific rules that simultaneously introduce certain implications and eliminate others; we refer to them as introduction–elimination rules for implication:

$$(\to\mathrm{Iv})\ \frac{\begin{matrix}(A)\\ B\end{matrix}}{A \to B}$$

where $\mathsf{var}(B) \subseteq \mathsf{var}(A)$

$$(\to\mathrm{IEv1})\ \frac{E \to D \quad D \to C \quad \begin{matrix}(A)\\ B\end{matrix}}{A \to B} \qquad (\to\mathrm{IEv2})\ \frac{A \to D \quad A \to C \quad \begin{matrix}(A)\\ B\end{matrix}}{A \to B}$$

where $\mathsf{var}(B) \subseteq \mathsf{var}(C)$ and $\mathsf{var}(E) \subseteq \mathsf{var}(A)$ where $\mathsf{var}(B) \subseteq \mathsf{var}(C) \cup \mathsf{var}(D)$

In addition, we adopt a single elimination rule for implication:

$$(\to\mathrm{E})\ \frac{A \to B \quad A}{B}$$

In [14], the following introduction–elimination rules for implication were adopted in place of $(\to\mathtt{IEv1})$ and $(\to\mathtt{IEv2})$:

$$(\to\mathrm{IEv1+})\ \frac{E \to D' \quad D \to C \quad \begin{matrix}(A)\\ B\end{matrix}}{A \to B} \qquad (\to\mathrm{IEv2+})\ \frac{D' \to D \quad D' \to C \quad \begin{matrix}(A)\\ B\end{matrix}}{A \to B}$$

where $\mathsf{var}(B) \subseteq \mathsf{var}(C)$, $\mathsf{var}(D) \subseteq \mathsf{var}(D')$ and $\mathsf{var}(E) \subseteq \mathsf{var}(A)$

where $\mathsf{var}(B) \subseteq \mathsf{var}(C) \cup \mathsf{var}(D)$ and $\mathsf{var}(D') \subseteq \mathsf{var}(A)$

These rules are, of course, generalisations of the introduction–elimination rules for implication adopted above. However, we can show that they are derivable in our system:

**Fact 1.** $(\to\mathrm{IEv1+})$ and $(\to\mathrm{IEv2+})$ are derivable.

*Proof.* Assume $\mathsf{var}(B) \subseteq \mathsf{var}(C)$, $\mathsf{var}(D) \subseteq \mathsf{var}(D')$ and $\mathsf{var}(E) \subseteq \mathsf{var}(A)$. Let $\mathcal{D}$ be the following deduction:

$$(\to\mathrm{IEv1})\ \frac{\dfrac{\Sigma_1}{E \to D'} \qquad (\to\mathrm{Iv})\ \dfrac{(\to\mathrm{Iv})\ \dfrac{D}{D \to D}}{D' \looparrowright D} \qquad (\to\mathrm{Iv})\ \dfrac{D}{D \to D}}{E \looparrowright D}$$

We obtain then:

$$(\to\mathrm{IEv1})\,\dfrac{\mathcal{D}\qquad (\to\mathrm{IEv1})\,\dfrac{(\to\mathrm{Iv})\,\dfrac{D}{D\to D}\qquad \dfrac{\Sigma_2}{D\to C}\qquad (\to\mathrm{Iv})\,\dfrac{C}{C\to C}}{(D\to D)\looparrowright C}\qquad \begin{array}{c}[A]\\ \Sigma_3\\ \hline B\end{array}}{A\to B}$$

Assume $\mathsf{var}(B)\subseteq\mathsf{var}(C)\cup\mathsf{var}(D)$ and $\mathsf{var}(D')\subseteq\mathsf{var}(A)$. Let $\mathcal{D}$ be the following deduction:

$$(\to\mathrm{IEv1})\,\dfrac{(\to\mathrm{Iv})\,\dfrac{(\to\mathrm{Iv})\,\dfrac{D'}{D'\to D'}}{A\looparrowright D'}\qquad \dfrac{\Sigma_1}{D'\to D}\qquad (\to\mathrm{Iv})\,\dfrac{D}{D\to D}}{A\looparrowright D}$$

We obtain then:

$$(\to\mathrm{IEv2})\,\dfrac{\mathcal{D}\qquad (\to\mathrm{IEv1})\,\dfrac{(\to\mathrm{Iv})\,\dfrac{(\to\mathrm{Iv})\,\dfrac{D'}{D'\to D'}}{A\looparrowright D'}\qquad \dfrac{\Sigma_2}{D'\to C}\qquad (\to\mathrm{Iv})\,\dfrac{C}{C\to C}}{A\looparrowright C}\qquad \begin{array}{c}[A]\\ \Sigma_3\\ \hline B\end{array}}{A\to B}$$

□

The notion of a rule can be made precise within the metatheory of natural deduction systems by adapting Prawitz's account [20]. Let us focus on selected rules. An instance of the inference rule (→!E) is specified as follows:

(→E): $(A\to B, A/B)$.

The instances of the other inference rules are specified in a similar way.

The instances of the deduction rules (→Iv), (→IE1v) and (→IE2v) are specified as follows:

(→Iv): $\langle\langle X,B\rangle,\langle Y,A\to B\rangle\rangle$, where $Y=X\setminus\{A\}$ and $\mathsf{var}(B)\subseteq\mathsf{var}(A)$.

(→IEv1): $\langle\langle X_1,E\to D\rangle,\langle X_2,D\to C\rangle,\langle Y,B\rangle,\langle Z,A\to B\rangle\rangle$, where $Z=(X_1\cup X_2)\cup(Y\setminus\{A\})$, $\mathsf{var}(B)\subseteq\mathsf{var}(C)$ and $\mathsf{var}(E)\subseteq\mathsf{var}(A)$.

(→IEv1): $\langle\langle X_1,A\to D\rangle,\langle X_2,A\to C\rangle,\langle Y,B\rangle,\langle Z,A\to B\rangle\rangle$, where $Z=(X_1\cup X_2)\cup(Y\setminus\{A\})$ and $\mathsf{var}(B)\subseteq\mathsf{var}(C)\cup\mathsf{var}(D)$.

The instances of the other deduction rules are specified in a similar way.

The definition of deduction is a simple modification of Prawitz's definition of deduction (see [20]). We say that $A$ is *deducible from* $X$ *in* $\mathbf{DAI}_\perp$ ($X \mid -_{\mathbf{DAI}_\perp} A$) iff there is a deduction $\Pi$ of $A$ depending on $X$ or a subset $Y\subset X$. As usual, we say that $A$ is *deducible in* $\mathbf{DAI}_\perp$ iff $\emptyset\vdash_{\mathbf{DAI}_\perp} A$.

### 3.1.1 Soundness and completeness

The soundness and completeness theorems for the system studied in this section follow directly from the corresponding results for the system presented in [14]. In that paper, however, soundness was established by means of a translation into monadic predicate logic. Here, we provide a soundness lemma that allows us to prove the result in a more standard, direct manner.

We begin by introducing an auxiliary notion. We say that a model $M = \langle v, c, \subseteq\rangle \in$ **ica-E** is *sound for the pair* $\langle X, A\rangle$ ($M \Vdash \langle X, A\rangle$) iff $M \not\models X \cup \{\neg A\}$. We can now state the soundness lemma:

**Lemma 2.** 1. For every rule $(R)$, if the instance of $(R)$ is of the form $(A_1, \ldots, A_n/A)$, then for every model $M \in$ **ica-E**, $M \Vdash \langle\{A_i\}_{i\leq n}, A\rangle$.

2. For every rule $(R)$, if the instance of $(R)$ is of the form $\langle\langle X_1, A_1\rangle, \ldots, \langle X_n, A_n\rangle, \langle X, A\rangle\rangle$, then for every model $M \in$ **ica-E**, if for every $i \leq n$, $M \Vdash \langle X_i, A_i\rangle$, then $M \Vdash \langle X, A\rangle$.

*Proof.* Let us consider only selected rules; for others we reason similarly.

For ($\to$Iv). Consider an arbitrary model $M = \langle v, c, \subseteq\rangle \in$ **ica-E**. Assume that $M \not\models X \cup \{\neg B\}$. Suppose that $M \models Y \cup \{A\}$. Since $Y = X \setminus \{A\}$, then $M \models B$.

Suppose $x \notin c(B)$. Hence, by (ica) and (E$\bot$), we find $C \in \mathsf{var}(B)$ such that $x \notin c(C)$. Since $\mathsf{var}(B) \subseteq \mathsf{var}(A)$, $C \in \mathsf{var}(A)$. Again by (ica) and (E$\bot$), $x \notin c(A)$. Thus, $c(A) \subseteq c(B)$.

Therefore, by truth condition for $\to$, $M \models A \to B$.

For ($\to$IEv1). Consider an arbitrary model $M = \langle v, c, \subseteq\rangle \in$ **ica-E**. Assume that $M \not\models X_1 \cup \{\neg(E \to D)\}$, $M \not\models X_2 \cup \{\neg(D \to C)\}$ and $M \not\models Y \cup \{\neg B\}$. Suppose that $M \models Z \cup \{A\}$. Since $Z = (X_1 \cup X_2)(Y \setminus \{A\})$, then $M \models E \to D$, $M \models D \to C$ and $M \models B$. Thus, by truth condition for $\to$, $c(E) \subseteq c(D)$ and $c(D) \subseteq c(C)$, and so $c(E) \subseteq c(C)$.

Suppose $x \notin c(B)$. Hence, by (ica) and (E$\bot$), we find $F \in \mathsf{var}(B)$ such that $x \notin c(F)$. Since $\mathsf{var}(B) \subseteq \mathsf{var}(C)$, $F \in \mathsf{var}(C)$. By (ica) and (E$\bot$), $x \notin c(C)$, and so $x \notin c(E)$. Hence, by (ica) and (E$\bot$), we find $F \in \mathsf{var}(E)$ such that $x \notin c(F)$. Since $\mathsf{var}(E) \subseteq \mathsf{var}(A)$, $F \in \mathsf{var}(A)$. By (ica) and (E$\bot$), $x \notin c(A)$. Thus, $c(A) \subseteq c(B)$.

Therefore, by truth condition for $\to$, $M \models A \to B$.

For ($\to$IEv2). Consider an arbitrary model $M = \langle v, c, \subseteq\rangle \in$ **ica-E**. Assume that $M \not\models X_1 \cup \{\neg(A \to D)\}$, $M \not\models X_2 \cup \{\neg(A \to C)\}$ and $M \not\models Y \cup \{\neg B\}$. Suppose that $M \models Z \cup \{A\}$. Since $Z = (X_1 \cup X_2)(Y \setminus \{A\})$, then $M \models A \to D$, $M \models A \to C$ and $M \models B$. Thus, by truth condition for $\to$, $c(A) \subseteq c(D)$ and $c(A) \subseteq c(C)$, and so $c(A) \subseteq c(D) \cap c(C)$.

Suppose $x \notin c(B)$. Hence, by (ica) and (E$\bot$), we find $F \in \mathsf{var}(B)$ such that $x \notin c(F)$. Since $\mathsf{var}(B) \subseteq \mathsf{var}(C) \cup \mathsf{var}(D)$, $F \in \mathsf{var}(C)$ or $F \in \mathsf{var}(D)$. By (ica) and (E$\bot$), $x \notin c(C)$ or $x \notin c(D)$. Thus, $x \notin c(C) \cap c(D)$, and so $x \notin c(A)$. Thus, $c(A) \subseteq c(B)$.

Therefore, by truth condition for $\to$, $M \models A \to B$. $\square$

By carrying out the same deductions as in [14], we can establish the following lemma:

**Lemma 3.** 1. All formulas of the form of (CL1)–(CL3) are deducible in $\mathbf{DAI}_\bot$.

2. All formulas of the form of (DAI), (DAI1)–(DAI5) are deducible in $\mathbf{DAI}_\bot$.

By Lemmas 2 and 3 we obtain the following theorem in the standard way:

**Theorem 4.** For all $X \cup \{A\} \subseteq \mathsf{For}$, $X \vdash_{\mathbf{DAI}_\bot} A$ iff $X \models_{\textbf{ica-E}} A$.

### 3.1.2 Elimination of maximum formulas

Let us begin by presenting an example of a deduction containing a maximum formula that cannot be eliminated. Let $\mathcal{D}$ be the following deduction:

$$(\to\mathtt{IEv2})\ \dfrac{A \looparrowright B \qquad (\to\mathtt{Iv})\ \dfrac{(\to\mathtt{Iv})\ \dfrac{A}{A \to A}}{A \looparrowright A} \qquad (\to\mathtt{Iv})\ \dfrac{A \wedge B}{(A \wedge B) \to (A \wedge B)}}{A \looparrowright (A \wedge B)}$$

Consider the following deduction of the formula $A \to \neg(\neg A \wedge \neg B)$ depending on $A \looparrowright B$.

$$(\to\mathtt{IEv1})\ \dfrac{\mathcal{D} \qquad (\to\mathtt{Iv})\ \dfrac{(\to\mathtt{E})\ \dfrac{A \qquad (\wedge\mathtt{E1})\ \dfrac{\neg A \wedge \neg B}{\neg A}}{\bot}}{\neg(\neg A \wedge \neg B)}}{A \to \neg(\neg A \wedge \neg B)}$$

The deduction above contains two maximum formulas that cannot be eliminated: $A \looparrowright A$ and $A \looparrowright (A \wedge B)$. It is clear that, in deductions carried out within the system under consideration, maximum formulas may arise through applications of the rules $(\to\mathtt{IEv1})$ and $(\to\mathtt{IEv2})$.

The occurrence of maximum formulas resulting from applications of the introduction–elimination rules for implication can be prevented, but doing so requires the introduction of further rules. To specify the form of these rules, let us introduce some auxiliary notions.

We say $\mathcal{D}$ is a *basic deduction of implication* (*bdi*) iff there are $A, B \in \mathsf{For}$ such that at least one of the following cases holds:

$$\mathcal{D} = \begin{array}{c}(\neg(A \to B))\\ \bot\end{array}, \quad \mathcal{D} = \begin{array}{c}(A)\\ B\end{array} \quad \text{or } \mathcal{D} = A \to B.$$

We define inductively the family of rules for implications.

+ Rules $(\to\mathrm{Iv})$, $(\to\mathrm{IEv1})$, $(\to\mathrm{IEv2})$ are *rules for implication of level* 0.
+ Let the following rule be a rule for implication of level $n$:

$$(\mathrm{R})\quad \frac{\mathcal{D}_1 \quad \dots \quad A_i \to B_i \quad \dots \quad \mathcal{D}_m \quad \begin{array}{c}(A)\\ B\end{array}}{A \to B}$$

for some $m \geq 2$ and where for any $j \leq m \setminus \{i\}$, $\mathcal{D}_j$ is a bdi. Then, the following rules are *rules for implication of level* $n+1$:

$$(\mathrm{RAA}^R)\quad \frac{\mathcal{D}_1 \quad \dots \quad \begin{array}{c}(\neg(A_i \to B_i))\\ \bot\end{array} \quad \dots \quad \mathcal{D}_m \quad \begin{array}{c}(A)\\ B\end{array}}{A \to B}$$

$$(\to\mathrm{Iv}^R)\quad \frac{\mathcal{D}_1 \quad \dots \quad \begin{array}{c}(A_i)\\ B_i\end{array} \quad \dots \quad \mathcal{D}_m \quad \begin{array}{c}(A)\\ B\end{array}}{A \to B}$$

where $\mathsf{var}(B_i) \subseteq \mathsf{var}(A_i)$

$$(\to\mathrm{IEv1}^R)\quad \frac{\mathcal{D}_1 \quad \dots \quad E \to D \quad D \to C \quad \begin{array}{c}(A_i)\\ B_i\end{array} \quad \dots \quad \mathcal{D}_m \quad \begin{array}{c}(A)\\ B\end{array}}{A \to B}$$

where $\mathsf{var}(B_i) \subseteq \mathsf{var}(C)$ and $\mathsf{var}(E) \subseteq \mathsf{var}(A_i)$,

$$(\to\mathrm{IEv1}^R)\quad \frac{\mathcal{D}_1 \quad \dots \quad A_i \to D \quad A_i \to C \quad \begin{array}{c}(A_i)\\ B_i\end{array} \quad \dots \quad \mathcal{D}_m \quad \begin{array}{c}(A)\\ B\end{array}}{A \to B}$$

where $\mathsf{var}(B_i) \subseteq \mathsf{var}(C) \cup \mathsf{var}(D)$.

The notation for the rules introduced above is somewhat inconvenient from a notational point of view. However, it allows us to identify the rule from which each new rule is, in a sense, obtained. This will make the following example of the elimination of a maximum formula easier to follow.

Let us illustrate how our rules for implications operate. Consider the following form of deduction:

$$(\to\mathrm{IEv2})\,\dfrac{(\to\mathrm{IEv1})\,\dfrac{(\to\mathrm{Iv})\,\dfrac{\dfrac{[E] \atop \Sigma_1}{C}}{E\to C} \qquad C\to D \qquad \dfrac{[A] \atop \Sigma_2}{B_2}}{A\to B_2} \qquad (\to\mathrm{Iv})\,\dfrac{\dfrac{[A] \atop \Sigma_3}{B_1}}{A\to B_1} \qquad \dfrac{[A] \atop \Sigma_4}{B}}{A\to B}$$

where $\mathsf{var}(C) \subseteq \mathsf{var}(E)$; $\mathsf{var}(B_2) \subseteq \mathsf{var}(D)$ and $\mathsf{var}(E) \subseteq \mathsf{var}(A)$; $\mathsf{var}(B_1) \subseteq \mathsf{var}(A)$; $\mathsf{var}(B) \subseteq \mathsf{var}(B_1) \cup \mathsf{var}(B_2)$.

In the example above, the maximum formulas are: $E \to C$, $A \to B_1$ and $A \to B_2$. Using the new rules, we can transform the deduction in the following steps:

1. Step 1. The application of $(\to\mathrm{Iv1}^{\to\mathrm{IEv2}})$:

$$(\to\mathrm{Iv1}^{\to\mathrm{IEv2}})\,\dfrac{(\to\mathrm{IEv1})\,\dfrac{(\to\mathrm{Iv})\,\dfrac{\dfrac{[E] \atop \Sigma_1}{C}}{E\to C} \qquad C\to D \qquad \dfrac{[A] \atop \Sigma_2}{B_2}}{A\to B_2} \qquad \dfrac{[A] \atop \Sigma_3}{B_1} \qquad \dfrac{[A] \atop \Sigma_4}{B}}{A\to B}$$

2. Step 2. The application of $(\to\mathrm{IEv1}^{\to\mathrm{Iv1}^{\to\mathrm{IEv2}}})$:

$$(\to\mathrm{IEv1}^{\to\mathrm{Iv1}^{\to\mathrm{IEv2}}})\,\dfrac{(\to\mathrm{Iv})\,\dfrac{\dfrac{[E] \atop \Sigma_1}{C}}{E\to C} \qquad C\to D \qquad \dfrac{[A] \atop \Sigma_2}{B_2} \qquad \dfrac{[A] \atop \Sigma_3}{B_1} \qquad \dfrac{[A] \atop \Sigma_4}{B}}{A\to B}$$

3. Step 3. The application of $(\to\mathrm{Iv1}^{\to\mathrm{IEv1}^{\to\mathrm{Iv1}^{\to\mathrm{IEv2}}}})$:

$$(\to\mathrm{Iv1}^{\to\mathrm{IEv1}^{\to\mathrm{Iv1}^{\to\mathrm{IEv2}}}})\,\dfrac{\dfrac{[E] \atop \Sigma_1}{C} \qquad C\to D \qquad \dfrac{[A] \atop \Sigma_2}{B_2} \qquad \dfrac{[A] \atop \Sigma_3}{B_1} \qquad \dfrac{[A] \atop \Sigma_4}{B}}{A\to B}$$

We can see that the maximum formulas identified above are eliminated at step 3.

This solution—or, more precisely, this strategy for solving the problem—is not satisfactory from a proof-theoretic point of view, since it requires the introduction of infinitely many rules. We therefore present an alternative approach below.

## 3.2 Second approach

Let us now turn to the second natural deduction system. This time, our aim is to obtain a system in which maximum formulas can be eliminated without introducing infinitely many rules. Achieving this goal, however, requires a modification of the language of the deductive system. The language of the second system is not simply the language of the logic, but an extension of it containing labelled formulas (signed formulas).

By a *labelled formula* we mean a pair $\langle A, \bullet \rangle$ such that $A \in \mathsf{For}$. Labelled formulas allow us to express, within deductions, that an element belonging to the content of one expression also belongs to the content of another specified expression. We denote the set of labelled formulas by $\mathsf{For}^\bullet$. We usually write $A^\bullet$ instead of $\langle A, \bullet \rangle$, and also we write $A \star B^\bullet$ instead of $(A \star B)^\bullet$. For every $X \subseteq \mathsf{For}$, we put $X^\bullet = \{A^\bullet : A \in X\}$.

We define the set of expressions as the union of the set of formulas and the set of labelled formulas: $\mathsf{Ex} = \mathsf{For} \cup \mathsf{For}^\bullet$. We use $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \ldots$ as variables ranging over individual d-expressions, and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ as variables ranging over sets of d-expressions. In addition, for every $\mathcal{X} \subseteq \mathsf{Ex}$, we define $\mathsf{for}(\mathcal{X}) = \{A \in \mathsf{For} : A \in \mathcal{X}\}$ and $\mathsf{for}^\bullet(\mathcal{X}) = \{A \in \mathsf{For} : A^\bullet \in \mathcal{X}\}$.

The first group of rules in the second system consists of the standard rules introduced above. The second group consists of auxiliary rules, which we call $\bullet$*-rules.* They have the following form:

$$(\bot\bullet)\ \frac{A^\bullet}{\bot^\bullet}$$

$$(\wedge^\bullet\mathrm{E}_1)\ \frac{A \wedge B^\bullet}{A^\bullet} \qquad (\wedge\bullet\mathrm{E}_2)\ \frac{A \wedge B^\bullet}{B^\bullet} \qquad (\wedge\bullet\mathrm{I})\ \frac{A^\bullet \quad B^\bullet}{A \wedge B^\bullet}$$

$$(\to\bullet\mathrm{E}_1)\ \frac{A \to B^\bullet}{A^\bullet} \qquad (\to\bullet\mathrm{E}_2)\ \frac{A \to B^\bullet}{B^\bullet} \qquad (\to^\bullet\mathrm{I})\ \frac{A^\bullet \quad B^\bullet}{A \to B^\bullet}$$

In what follows, we show that $\mathsf{var}(A) \subseteq \mathsf{var}(B)$ iff $A^\bullet$ is deducible from $B^\bullet$ using only $\bullet$-rules.

Let us now define the rules for implication. We begin with a single introduction rule for implication:

$$(\to\mathrm{I})\ \dfrac{\begin{matrix}(A) & (A^{\bullet}) \\ B & B^{\bullet}\end{matrix}}{A \to B}$$

In an application of the rule $(\to\mathrm{I})$ the deduction of $B^{\bullet}$ depends on a set of assumptions in which $A^{\bullet}$ is the only labelled formula.

We adopt a single elimination rule for implication, namely *modus ponens*:

$$(\to\mathrm{E})\ \dfrac{A \to B \quad A}{B}$$

Another elimination rule for implication of the form:

$$(\to\mathrm{E}')\ \dfrac{A \to B \quad A^{\bullet}}{B^{\bullet}}$$

is not adopted as a primitive rule. Instead of $(\to\mathrm{E}')$, we adopt the following modifications of the rule (RAA), which we call RAA$^{\bullet}$-rules:

$$(\mathrm{RAA}^{\bullet}1)\ \dfrac{\begin{matrix}(\neg(A \to B)) \\ \bot\end{matrix} \qquad A_1^{\bullet} \quad \dots \quad A_n^{\bullet}}{C^{\bullet}}$$

where $\mathsf{var}(A) = \{A_1, \dots, A_n\}$
and $C \in \mathsf{var}(B)$

$$(\mathrm{RAA}^{\bullet}2)\ \dfrac{\begin{matrix}(\neg(\bot \to B)) \\ \bot\end{matrix} \qquad \bot^{\bullet}}{C^{\bullet}}$$

where $C \in \mathsf{var}(B)$

In what follows, we will sometimes use the notation $(\mathrm{RAA}^{\bullet}_{1,2})$ to refer to either of the two RAA$^{\bullet}$-rules. The context will make clear which rule is intended.

Let us explain why we adopt the RAA$^{\bullet}$-rules rather than $(\to\mathrm{E}')$ as primitive rules of our system. It is true that the more natural rule $(\to\mathrm{E}')$ could be used instead of the RAA$^{\bullet}$-rules to obtain a complete system. The problem, however, is that its application may give rise to a maximum formula that cannot be eliminated. Consider the following deduction of $r^{\bullet}$ depending on $p, \neg(p \wedge \neg(q \to r)), q^{\bullet}$:

$$\dfrac{\dfrac{\dfrac{\neg(p \wedge \neg(q \to r)) \qquad \dfrac{p \qquad \neg(q \to r)}{p \wedge \neg(q \to r)}\,(\wedge\mathrm{I})}{\bot}\,(\to\mathrm{E})}{q \to r}\,(\mathrm{RAA}) \qquad q^\bullet}{r^\bullet}\,(\to\mathrm{E}')$$

Let us also note that instead of (RAA$^\bullet$1), (RAA$^\bullet$2), we could introduce a single rule constituting a general form of the RAA$^\bullet$-rules:

$$(\mathrm{RAA}^\bullet)\quad \dfrac{\begin{array}{c}(\neg(A \to B))\\ \bot\end{array} \qquad A^\bullet}{B^\bullet}$$

The approach employing two RAA$^\bullet$-rules seems more natural from the perspective of normalisation. In the generalised rule (RAA$^\bullet$), the premise occupying the position of $A^\bullet$ may be introduced by the $\bullet$-rules and subsequently, in a sense, eliminated by an application of (RAA$^\bullet$). It may therefore be regarded as a kind of maximum formula.

In what follows, we shall show that the rule (RAA$^\bullet$) is derivable. To this end, we use several auxiliary results, which will be established after adapting further metatheoretical notions from the theory of natural deduction systems to the present setting.

The instances of inference rules ($\bot^\bullet$), ($\star^\bullet$E1), ($\star^\bullet$E2), ($\star^\bullet$I) and ($\to$E) are specified as follows:

($\bot^\bullet$): $(A^\bullet / \bot^\bullet)$,

($\star^\bullet$E1): $(A \star B^\bullet / A^\bullet)$,

($\star^\bullet$E2): $(A \star B^\bullet / B^\bullet)$,

($\star^\bullet$I): $(A^\bullet, B^\bullet / A \star B^\bullet)$,

($\to$E): $(A \to B, A / B)$.

The instances of the other inference rules are specified in a similar way.

The instances of the deduction rules ($\to$I), (RAA$^\bullet$1), and (RAA$^\bullet$2) are specified as follows:

($\to$I): $\langle\langle \mathcal{X}, B\rangle, \langle \mathcal{Y}, B^\bullet\rangle, \langle \mathcal{Z}, (A \to B)\rangle\rangle$, where $\mathcal{Z} = (\mathcal{X} \setminus \{A\}) \cup (\mathcal{Y} \setminus \{A^\bullet\})$ and $\mathsf{for}^\bullet(\mathcal{Y}) = \{A\}$,

(RAA$^\bullet$1): $\langle\langle \mathcal{X}, \bot\rangle, \langle \mathcal{Y}_1, A_1^\bullet\rangle, \ldots, \langle \mathcal{Y}_n, A_n^\bullet\rangle, \langle \mathcal{Z}, C^\bullet\rangle\rangle$, where $\mathcal{Z} = (\mathcal{X} \setminus \{\neg(A \to B)\}) \cup \bigcup_{i \leq n} \mathcal{Y}_i$, $\mathsf{var}(A) = \{A_1, \ldots, A_n\}$ and $C \in \mathsf{var}(B)$, (RAA$^\bullet$2): $\langle\langle\mathcal{X},\bot\rangle,\langle\mathcal{Y},\bot^\bullet\rangle,\langle\mathcal{Z},C^\bullet\rangle\rangle$, where $\mathcal{Z}=(\mathcal{X}\setminus\{\neg(\bot\to B)\})\cup\mathcal{Y}$ and $C\in\mathsf{var}(B)$.

The instances of the other deduction rules are specified in a similar way.

The definition of a deduction is obtained by a straightforward modification of Prawitz's definition [20]: variables ranging over formulas and sets of formulas are replaced by variables ranging over d-expressions and sets of d-expressions, respectively.

Let $X\cup\{A\}\subseteq\mathsf{For}$. We adapt the definition of the deducibility (derivability) relation introduced in the preceding section, retaining the same notation. As usual, we say that $A$ is *deducible from $X$ in* $\mathbf{DAI}_\bot$ ($X\vdash_{\mathbf{DAI}_\bot}A$) iff there is a deduction $\Pi$ of $A$ depending on $X$ or a subset $Y\subseteq X$. As usual, we say that $A$ is *deducible in* $\mathbf{DAI}_\bot$ iff $\emptyset\vdash_{\mathbf{DAI}_\bot}A$.

By a $\bullet$-deduction, we mean a deduction constructed using only applications of $\bullet$-rules. If there is a $\bullet$-deduction $\Pi$ of $A^\bullet$ depending on $X^\bullet$, we write $X\vdash^\bullet_{\mathbf{DAI}_\bot}A$.

The following results clarify the role of the $\bullet$-rules.

**Fact 5.** For every $A,B\in\mathsf{For}$, $A\vdash^\bullet_{\mathbf{DAI}_\bot}B$ in $\mathbf{DAI}_\bot$ iff $\mathsf{var}(B)\subseteq\mathsf{var}(A)$.

*Proof.* ($\Rightarrow$) Let $\Pi$ be a $\bullet$-deduction of $B^\bullet$ depending on $A^\bullet$. We use an induction on the length of sub-deductions of $\Pi$ depending on $A^\bullet$, i.e. the length of sub-trees of $\Pi$ which are $\bullet$-deductions depending on $A^\bullet$.

*Base case.* Obvious for sub-deductions of the length 1.

*Induction hypothesis.* Let $k\geq 1$. For any sub-deduction $\Pi'$ of $\Pi$, if $\Pi'$ is a $\bullet$-deduction of $C^\bullet$ depending on $A^\bullet$ in $\mathbf{DAI}_\bot$ of length not greater than $k$, then $\mathsf{var}(C)\subseteq\mathsf{var}(A)$.

*Inductive step.* Suppose $\Pi'$ is a sub-deduction of $\Pi$ of length $k+1$ and a $\bullet$-deduction of $C^\bullet$ depending on $A^\bullet$ in $\mathbf{DAI}_\bot$. Thus, $\Pi'$ has the form $(\Pi_1,\dots,\Pi_m/C^\bullet)$, where $m\leq 2$ and for any $i\leq m$, $\Pi_i$ is a deduction of $A_i^\bullet$ depending on $A^\bullet$ in $\mathbf{DAI}_\bot$. Hence, there is an instance of a $\bullet$-rule $(A_1^\bullet,\dots,A_m^\bullet/C^\bullet)$ such that, for any $i\leq m$, $\Pi_i$ is a deduction of $A_i^\bullet$ depending on $A^\bullet$ in $\mathbf{DAI}_\bot$.

Let $(A_1^\bullet,\dots,A_m^\bullet/C^\bullet)=(D^\bullet/\bot^\bullet)$. Then, by inductive hypothesis we have: $\mathsf{var}(\bot)\subseteq\mathsf{var}(D)\subseteq\mathsf{var}(A)$.

Let $(A_1^\bullet,\dots,A_m^\bullet/C^\bullet)=(D\star E^\bullet/D^\bullet)$. Then, by inductive hypothesis we have: $\mathsf{var}(D)\subseteq\mathsf{var}(D)\cup\mathsf{var}(E)=\mathsf{var}(D\star E)\subseteq\mathsf{var}(A)$. Similarly, for $(A_1^\bullet,\dots,A_m^\bullet/C^\bullet)=(D\star E^\bullet/E^\bullet)$.

Let $(A_1^\bullet,\dots,A_m^\bullet/C^\bullet)=(D^\bullet,E^\bullet/D\star E^\bullet)$. Then, by inductive hypothesis $\mathsf{var}(D),\mathsf{var}(E)\subseteq\mathsf{var}(A)$ and so $\mathsf{var}(D\star E)\subseteq\mathsf{var}(D)\cup\mathsf{var}(E)\subseteq\mathsf{var}(A)$.

($\Leftarrow$) Let $A\in\mathsf{For}$. We prove the implication by an induction on the degree of formulas.

*Base case.* Let $B \in \mathsf{For}$ be a formula of degree 0.

Let $B = \bot$. Suppose $\mathsf{var}(B) \subseteq \mathsf{var}(A)$. And so $\mathsf{var}(B) = \emptyset \subseteq \mathsf{var}(A)$. We have:

$$(\bot^\bullet)\,\frac{A^\bullet}{\bot^\bullet}$$

Let $B \in \mathsf{Var}$. We use an induction on the degrees of subformulas of $A$.

+ *Base case.* Let $C \in \mathsf{sub}(A)$ be of degree 0. Suppose $\mathsf{var}(B) \subseteq \mathsf{var}(C)$. Then, of course, $B = C$.

+ *Induction hypothesis.* Let $k \geq 0$. For any $C \in \mathsf{sub}(A)$, if the degree of $C$ is not greater than $k$ and $\mathsf{var}(B) \subseteq \mathsf{var}(C)$, then there is a $\bullet$-deduction of $B^\bullet$ depending on $C^\bullet$ in $\mathbf{DAI}_\bot$.

+ *Inductive step.* Let $C \in \mathsf{sub}(A)$ be of degree $k+1$. Let $C = D \star E$, for some $D, E \in \mathsf{For}$ and $\star \in \{\wedge, \rightarrow\}$. Suppose $\mathsf{var}(B) \subseteq \mathsf{var}(C) = \mathsf{var}(D) \cup \mathsf{var}(E)$. Then $B \in \mathsf{var}(D) \neq \emptyset$ or $B \in \mathsf{var}(E) \neq \emptyset$. Suppose that the first case holds. (The second case can be considered in a similar way.) By inductive hypothesis, we have a $\bullet$-deduction $\mathit{\Pi}$ of $B^\bullet$ depending on $D^\bullet$. We have:

$$(\star^\bullet\mathrm{E1})\,\cfrac{C^\bullet}{\cfrac{(D^\bullet)}{\cfrac{\mathit{\Pi}}{B^\bullet}}}$$

*Induction hypothesis.* Let $k \geq 0$. For any $B \in \mathsf{For}$, if the degree of $B$ is not greater than $k$ and $\mathsf{var}(B) \subseteq \mathsf{var}(A)$, then there is a $\bullet$-deduction $\mathit{\Pi}$ of $B^\bullet$ depending on $A^\bullet$ in $\mathbf{DAI}_\bot$.

*Inductive step.* Let $B \in \mathsf{For}$ be of degree $k+1$ and $\mathsf{var}(B) \subseteq \mathsf{var}(A)$. Let $B = D \star E$, for some $D, E \in \mathsf{For}$ and $\star \in \{\wedge, \rightarrow\}$. By inductive hypothesis, we have $\bullet$-deductions $\mathit{\Pi}_1$ of $D^\bullet$ depending on $A^\bullet$ and $\mathit{\Pi}_2$ of $E^\bullet$ depending on $A^\bullet$. Hence, we have:

$$(\star^\bullet\mathrm{I})\,\cfrac{\cfrac{\cfrac{[A^\bullet]}{\mathit{\Pi}_1}}{D^\bullet}\qquad\cfrac{\cfrac{[A^\bullet]}{\mathit{\Pi}_2}}{E^\bullet}}{B^\bullet}$$

□

By Fact 5, we obtain the following corollaries:

**Corollary 6.** For all $A_1, \ldots, A_n, A \in \mathsf{For}$, $\{A_1, \ldots, A_n\} \vdash^\bullet_{\mathbf{DAI}_\bot} A$ iff $\mathsf{var}(A) \subseteq \bigcup_{i \leq n} \mathsf{var}(A_i)$.

*Proof.* We obviously have $\{A_1, \ldots, A_m\} \vdash^{\bullet}_{\mathbf{DAI}_\perp} A_1 \star (\ldots(A_{m-1} \star A_m)\ldots)$ and for any $i \leq m$, $A_1 \star (\ldots(A_{m-1} \star A_m)\ldots) \vdash^{\bullet}_{\mathbf{DAI}_\perp} A_i$. By Fact 5, $A_1 \star (\ldots(A_{n-1} \star A_m)\ldots) \vdash^{\bullet}_{\mathbf{DAI}_\perp} A$ iff $\mathsf{var}(A) \subseteq \mathsf{var}(A_1 \star (\ldots(A_{n-1} \star A_m)\ldots)) = \bigcup_{i \leq m} \mathsf{var}(A_i)$. □

**Corollary 7.** For all $A_1, \ldots, A_n \in \mathsf{Var}$ and every $A \in \mathsf{For}$:

1. $A \vdash^{\bullet}_{\mathbf{DAI}_\perp} A_i$, for any $i \leq n$ iff $\{A_1, \ldots, A_n\} \subseteq \mathsf{var}(A)$,
2. $\{A_1, \ldots, A_n\} \vdash^{\bullet}_{\mathbf{DAI}_\perp} A$ iff $\mathsf{var}(A) = \{A_1, \ldots, A_n\}$.

*Proof.* For 1. Directly by Fact 5.

For 2. By Corollary 6, $\{A_1, \ldots, A_n\} \vdash^{\bullet}_{\mathbf{DAI}_\perp} A$ iff $\mathsf{var}(A) \subseteq \{A_1, \ldots, A_n\}$. Thus, if there is a deduction of $A^\bullet$ depending on $A_1^\bullet, \ldots, A_n^\bullet$ and $A_1, \ldots, A_n \in \mathsf{Var}$, then $A_1, \ldots, A_n \in \mathsf{var}(A)$. □

Let us now show that rules (RAA$^\bullet$) and ($\to$E$'$) are derivable in our system.

**Fact 8.** (RAA$^\bullet$) is derivable in the deductive system for $\mathbf{DAI}_\perp$.

*Proof.* In the case where $B = \perp$. We apply the rule ($\perp^\bullet$).

Assume $B \neq \perp$. If $\mathsf{var}(A) = \{A_1, \ldots, A_n\}$, then by Corollary 7.1, we obtain a $\bullet$-deduction $\Pi_i$ of $A_i^\bullet$ depending on $A^\bullet$. Let $\Sigma$ be the sequence of the following form:

$$\dfrac{\begin{array}{c}[\neg(A \to B)]\\ \Pi_0\end{array}}{\perp} \qquad \dfrac{\begin{array}{c}[A^\bullet]\\ \Pi_1\end{array}}{A_1^\bullet} \qquad \ldots \qquad \dfrac{\begin{array}{c}[A^\bullet]\\ \Pi_n\end{array}}{A_n^\bullet}$$

If $\mathsf{var}(A) = \emptyset$, because $A = \perp$, then $\Sigma$ has the following form:

$$\dfrac{\begin{array}{c}[\neg(\perp \to B)]\\ \Pi_0\end{array}}{\perp} \qquad \perp^\bullet$$

By Corollary 7.2, we consider a $\bullet$-deduction $\Pi$ of $B^\bullet$ depending on $\mathsf{var}(B)^\bullet$. Then, the deduction $\Pi$ has one of the following forms:

$$C_1^\bullet \qquad \dfrac{C_1^\bullet \quad C_2^\bullet}{D_1^\bullet} \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{C_1^\bullet \quad C_2^\bullet}{D_1^\bullet} \quad C_3^\bullet}{\vdots}\end{array}}{D_{k-1}^\bullet} \quad C_{k+1}^\bullet}{D_k^\bullet}$$

where $k \geq 2$, for any $i \leq k+1$, $C_i \in \mathsf{var}(B)$, $C_1, C_2 \vdash^{\bullet}_{\mathbf{DAI}_\bot} D_1$, for any $1 < i \leq k$, $D_{i-1}, C_{i+1} \vdash^{\bullet}_{\mathbf{DAI}_\bot} D_i$.

We then have, respectively:

$$(\mathrm{RAA}^{\bullet}_{1,2})\,\dfrac{\Sigma}{(C_1^{\bullet})} \qquad\qquad \dfrac{(\mathrm{RAA}^{\bullet}_{1,2})\,\dfrac{\Sigma}{(C_1^{\bullet})} \qquad (\mathrm{RAA}^{\bullet}_{1,2})\,\dfrac{\Sigma}{(C_2^{\bullet})}}{D_1^{\bullet}}$$

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{(\mathrm{RAA}^{\bullet}_{1,2})\,\dfrac{\Sigma}{(C_1^{\bullet})} \qquad (\mathrm{RAA}^{\bullet}_{1,2})\,\dfrac{\Sigma}{(C_2^{\bullet})}}{D_1^{\bullet}} \qquad (\mathrm{RAA}^{\bullet}_{1,2})\,\dfrac{\Sigma}{C_3^{\bullet}}}{} \\ \vdots \\ D_{k-1}^{\bullet}\end{array} \qquad (\mathrm{RAA}^{\bullet}_{1,2})\,\dfrac{\Sigma}{(C_{k+1}^{\bullet})}}{D_k^{\bullet}}$$

□

We obtain the following corollary.

**Corollary 9.** $(\to\mathrm{E}')$ is derivable.

*Proof.* We use Fact 8.

$$(\mathrm{RAA}^{\bullet})\,\dfrac{(\to\mathrm{E})\,\dfrac{\neg(A \to B) \qquad \dfrac{\Pi_1}{(A \to B)}}{\bot} \qquad \dfrac{\Pi_2}{A^{\bullet}}}{B^{\bullet}}$$

□

In the remainder of the paper, we will sometimes use derived rules to abbreviate proofs.

Let us also show that suitably modified versions of the rules $(\to\mathrm{I}_2)$ and $(\to\mathrm{I}_3)$ from [14] are derivable in our system. Consider the following counterparts of these rules:

$$(\to\mathrm{I}_2')\quad \dfrac{\begin{array}{c}(A)\\ B\end{array} \quad E \to C \quad C \to D \quad \begin{array}{c}(A^{\bullet})\\ E^{\bullet}\end{array} \quad \begin{array}{c}(D^{\bullet})\\ B^{\bullet}\end{array}}{A \to B}$$

where the deduction of $E^\bullet$ is from a non-empty set of assumptions and $A^\bullet$ is the only undischarged assumption with $\bullet$ in it; the deduction of $B^\bullet$ is from a non-empty set of assumptions and $D^\bullet$ is the only undischarged assumption with $\bullet$ in it.

$$(\to\mathrm{I}'_3)\ \dfrac{\begin{matrix}(A)\\ B\end{matrix}\quad A\to C\quad A\to D\quad \begin{matrix}(C\star D^\bullet)\\ B^\bullet\end{matrix}}{A\to B}$$

where the deduction of $B^\bullet$ is from a non-empty set of assumptions and $C\star D^\bullet$ is the only undischarged assumption with $\bullet$ in it.

We obtain the following fact:

**Fact 10.** $(\to\mathrm{I}'_2)$ and $(\to\mathrm{I}'_3)$ are derivable.

*Proof.* For $(\to\mathrm{I}'_2)$:

$$(\to\mathrm{I})\ \dfrac{\dfrac{\begin{matrix}[A]\\ \Pi_1\end{matrix}}{B}\qquad \dfrac{\begin{matrix}(\to\mathrm{E}')\ \dfrac{C\to D\qquad (\to\mathrm{E}')\ \dfrac{E\to C\qquad \dfrac{\begin{matrix}[A^\bullet]\\ \Pi_2\end{matrix}}{E^\bullet}}{C^\bullet}}{[D^\bullet]}\\ \Pi_3\end{matrix}}{B^\bullet}}{A\to B}$$

For $(\to\mathrm{I}'_3)$:

$$(\to\mathrm{I})\ \dfrac{\dfrac{\begin{matrix}[A]\\ \Pi_1\end{matrix}}{B}\qquad \dfrac{\begin{matrix}(\star^\bullet\mathrm{I})\ \dfrac{(\to\mathrm{E}')\ \dfrac{A\to C\qquad A^\bullet}{C^\bullet}\qquad (\to\mathrm{E}')\ \dfrac{A\to D\qquad A^\bullet}{D^\bullet}}{[C\star D^\bullet]}\\ \Pi_2\end{matrix}}{B^\bullet}}{A\to B}$$

$\square$

### 3.2.1 Soundness and completeness

To prove soundness, we begin by introducing some auxiliary notions and establishing a preliminary result. We say that a model $M=\langle v,c,\subseteq\rangle\in\textbf{ica-E}$ is *sound for a pair* $\langle\mathcal{X},\mathcal{A}\rangle$ ($M\Vvdash\langle\mathcal{X},\mathcal{A}\rangle$) iff the following cases hold:

+ $M\not\models\mathsf{for}(\mathcal{X})\cup\{\neg A\}$, if $\mathcal{A}=A$, for some $A\in\mathsf{For}$,

+ $M\models\mathsf{for}(\mathcal{X})$ only if $\bigcap\{c(B):B\in\mathsf{for}^\bullet(\mathcal{X})\}\subseteq c(A)$, if $\mathcal{A}=A^\bullet$, for some $A\in\mathsf{For}$.

Let us now proceed to the formulation and proof of the soundness lemma.

**Lemma 11.** 1. For any rule $(R)$, if the instance of $(R)$ is of the form $(\mathcal{A}_1, \ldots, \mathcal{A}_n / \mathcal{A})$, then for every model $M \in$ **ica-E**, $M \Vdash \langle \{\mathcal{A}_i\}_{i \leq n}, \mathcal{A} \rangle$.

2. For any rule $(R)$, if the instance of $(R)$ is of the form $\langle \langle \mathcal{X}_1, \mathcal{A}_1 \rangle, \ldots, \langle \mathcal{X}_n, \mathcal{A}_n \rangle, \langle \mathcal{X}, \mathcal{A} \rangle \rangle$, then for every model $M \in$ **ica-E**, if for every $i \leq n$, $M \Vdash \langle \mathcal{X}_i, \mathcal{A}_i \rangle$, then $M \Vdash \langle \mathcal{X}, \mathcal{A} \rangle$.

*Proof.* Let us consider only selected rules; for others we reason similarly.

For $(\perp^\bullet)$, $(\star^\bullet\mathrm{E1})$, $(\star^\bullet\mathrm{E2})$ and $(\star^\bullet\mathrm{I})$. Consider the instance of a rule of the form $(A_1^\bullet, \ldots, A_n^\bullet / A^\bullet)$. Obviously, we have $\{A_1, \ldots, A_n\} \vdash^\bullet_{\mathbf{DAI}_\perp} A$. By Corollary 7.2 $\mathsf{var}(A) \subseteq \bigcup_{i \leq n} \mathsf{var}(A_i)$. Thus, for every model $M = \langle v, c, \subseteq \rangle \in$ **ica-E**, $\bigcap_{i \leq n} c(A_i) \subseteq c(A)$.

For $(\to\mathrm{I})$. Consider an arbitrary model $M = \langle v, c, \subseteq \rangle \in$ **ica-E**. Assume that $M \not\models \mathsf{for}(\mathcal{X}) \cup \{\neg B\}$, and $M \models \mathsf{for}(\mathcal{Y})$ only if $\bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Y})\} \subseteq c(B)$. Suppose $M \models \mathsf{for}(\mathcal{Z}) \cup \{A\}$. Since $\mathcal{Z} = (\mathcal{X} \setminus \{A\}) \cup (\mathcal{Y} \setminus \{A^\bullet\})$, then $M \models B$ and $\bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Y})\} \subseteq c(B)$. Since $\mathsf{for}^\bullet(\mathcal{Y}) = \{A\}$, we have $c(A) \subseteq c(B)$. Therefore, $M \models A \to B$.

For $(\mathrm{RAA}^\bullet 1)$. Consider an arbitrary model $M = \langle v, c, \subseteq \rangle \in$ **ica-E**. Assume that for $M \not\models \mathsf{for}(\mathcal{X}) \cup \{\neg \perp\}$ and for every $i \leq n$, $M \models \mathsf{for}(\mathcal{Y}_i)$ only if $\bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Y}_i)\} \subseteq c(A_i)$. Suppose $M \models \mathsf{for}(\mathcal{Z})$. Since $\mathcal{Z} = (\mathcal{X} \setminus \{\neg(A \to B)\}) \cup \bigcup_{i \leq n} \mathcal{Y}_i$, then $M \not\models \neg(A \to B)$, so $M \models A \to B$, and for every $i \leq n$, $\bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Y}_i)\} \subseteq c(A_i)$. Since $M \models A \to B$, we have $c(A_1) \cap \ldots \cap c(A_n) = c(A) \subseteq c(B) = c(B_1) \cap \ldots \cap c(B_m)$, where $\mathsf{var}(B) = \{B_1, \ldots, B_m\}$. And since for every $i \leq n$, $\bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Z})\} \subseteq \bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Y}_i)\} \subseteq c(A_i)$, we have for every $j \leq m$, $\bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Z})\} \subseteq c(B_j)$.

For $(\mathrm{RAA}^\bullet 2)$. Consider an arbitrary model $M = \langle v, c, \subseteq \rangle \in$ **ica-E**. Assume that $M \not\models \mathsf{for}(\mathcal{X}) \cup \{\neg \perp\}$, and $M \models \mathsf{for}(\mathcal{Y})$ only if $\bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Y})\} \subseteq c(\perp)$. Suppose $M \models \mathsf{for}(\mathcal{Z})$. Since $\mathcal{Z} = (\mathcal{X} \setminus \{\neg(\perp \to B)\}) \cup \mathcal{Y}$, then $M \not\models \neg(\perp \to B)$, so $M \models \perp \to B$. Thus, $c(B) = U$. Since $c(B) = c(B_1) \cap \ldots \cap c(B_m)$, where $\mathsf{var}(B) = \{B_1, \ldots, B_m\}$, we also have for every $j \leq m$, $\bigcap\{c(C) : C \in \mathsf{for}^\bullet(\mathcal{Z})\} \subseteq c(B_j)$. □

To prove completeness, it is enough to show that, in the system we are analysing, all axioms of logic $\mathbf{DAI}_\perp$ are deducible.

**Lemma 12.** 1. All formulas of the form of (CL1)–(CL3) are deducible in $\mathbf{DAI}_\perp$.

2. All formulas of the form of (DAI), (DAI1)–(DAI5) are deducible in $\mathbf{DAI}_\perp$.

*Proof.* In what follows, we present only deductions of characteristic axioms.

For (DAI).

($\rightarrow$) Let $\Pi_1$ be the following deduction:

$$
(\rightarrow\mathrm{E})\ \dfrac{A \rightarrow B \qquad (\wedge\mathrm{E}_1)\ \dfrac{(\mathrm{RAA})\ \dfrac{(\rightarrow\mathrm{E})\ \dfrac{\neg\neg(A \wedge \neg B) \qquad \neg(A \wedge \neg B)}{\bot}}{A \wedge \neg B}}{A}}{B}
$$

Let $\Pi_2$ be the following deduction:

$$
(\wedge\mathrm{E}_1)\ \dfrac{(\mathrm{RAA})\ \dfrac{(\rightarrow\mathrm{E})\ \dfrac{\neg\neg(A \wedge \neg B) \qquad \neg(A \wedge \neg B)}{\bot}}{A \wedge \neg B}}{\neg B}
$$

Let $\Pi_3$ be the following deduction:

$$
(\rightarrow\mathrm{I})\ \dfrac{(\rightarrow\mathrm{I})\ \dfrac{B \qquad B^{\bullet}}{B \rightarrow B} \qquad (\rightarrow^{\bullet}\mathrm{I})\ \dfrac{(\rightarrow\mathrm{E}')\ \dfrac{A \rightarrow B \qquad A^{\bullet}}{B^{\bullet}} \qquad (\rightarrow\mathrm{E}')\ \dfrac{A \rightarrow B \qquad A^{\bullet}}{B^{\bullet}}}{B \rightarrow B^{\bullet}}}{A \hookrightarrow B}
$$

Let $\Pi_4$ be the following deduction:

$$
(\rightarrow^{\bullet}\mathrm{I})\ \dfrac{(\wedge^{\bullet}\mathrm{I})\ \dfrac{(\rightarrow^{\bullet}\mathrm{E1})\ \dfrac{A \rightarrow B^{\bullet}}{A^{\bullet}} \qquad (\rightarrow^{\bullet}\mathrm{I})\ \dfrac{(\rightarrow^{\bullet}\mathrm{E2})\ \dfrac{A \rightarrow B^{\bullet}}{B^{\bullet}} \qquad (\bot^{\bullet})\ \dfrac{A \rightarrow B^{\bullet}}{\bot^{\bullet}}}{\neg B^{\bullet}}}{A \wedge \neg B^{\bullet}} \qquad (\bot^{\bullet})\ \dfrac{A \rightarrow B^{\bullet}}{\bot^{\bullet}}}{\neg(A \wedge \neg B)^{\bullet}}
$$

and let $\Pi_5$ be the following deduction:

$$
(\rightarrow^{\bullet}\mathrm{I})\ \dfrac{(\rightarrow^{\bullet}\mathrm{E1})\ \dfrac{A \rightarrow B^{\bullet}}{A^{\bullet}} \qquad (\rightarrow^{\bullet}\mathrm{I})\ \dfrac{(\rightarrow^{\bullet}\mathrm{E2})\ \dfrac{A \rightarrow B^{\bullet}}{B^{\bullet}} \qquad (\rightarrow^{\bullet}\mathrm{E2})\ \dfrac{A \rightarrow B^{\bullet}}{B^{\bullet}}}{B \rightarrow B^{\bullet}}}{A \hookrightarrow B^{\bullet}}
$$

Finally, we obtain the following deduction:

$$
(\rightarrow\mathrm{I})\ \dfrac{(\wedge\mathrm{I})\ \dfrac{(\mathrm{RAA})\ \dfrac{(\rightarrow\mathrm{E})\ \dfrac{\Pi_1 \qquad \Pi_2}{\bot}}{\neg(A \wedge \neg B)} \qquad \Pi_3}{\neg(A \wedge \neg B) \wedge (A \hookrightarrow B)} \qquad (\wedge^{\bullet}\mathrm{I})\ \dfrac{\Pi_4 \qquad \Pi_5}{\neg(A \wedge \neg B) \wedge (A \hookrightarrow B)^{\bullet}}}{(A \rightarrow B) \rightarrow (\neg(A \wedge \neg B) \wedge (A \hookrightarrow B))}
$$

($\leftarrow$) Let $\Pi_1$ be the following deduction:

$$(\mathrm{RAA})\,\dfrac{(\to\mathrm{E})\,\dfrac{(\wedge\mathrm{E1})\,\dfrac{\neg(A\wedge\neg B)\wedge(A\looparrowright B)}{\neg(A\wedge\neg B)}\qquad(\wedge\mathrm{I})\,\dfrac{A\qquad\neg(B)}{A\wedge\neg B}}{\bot}}{B}$$

and let $\mathit{\Pi}_2$ be the following deduction:

$$(\to\mathrm{I})\,\dfrac{\mathit{\Pi}_1\qquad(\to^{\bullet}\mathrm{E1})\,\dfrac{(\to\mathrm{E}')\,\dfrac{(\wedge\mathrm{E2})\,\dfrac{\neg(A\wedge\neg B)\wedge(A\looparrowright B)}{A\looparrowright B}\qquad A^{\bullet}}{B\to B^{\bullet}}}{B^{\bullet}}}{A\to B}$$

Finally, we obtain the following deduction:

$$(\to\mathrm{I})\,\dfrac{\mathit{\Pi}_2\qquad(\to^{\bullet}\mathrm{I})\,\dfrac{(\to^{\bullet}\mathrm{E1})\,\dfrac{(\wedge^{\bullet}\mathrm{E2})\,\dfrac{\neg(A\wedge\neg B)\wedge(A\looparrowright B)^{\bullet}}{A\looparrowright B^{\bullet}}}{A^{\bullet}}\qquad(\to^{\bullet}\mathrm{E2})\,\dfrac{(\to^{\bullet}\mathrm{E2})\,\dfrac{(\wedge^{\bullet}\mathrm{E2})\,\dfrac{\neg(A\wedge\neg B)\wedge(A\looparrowright B)^{\bullet}}{A\looparrowright B^{\bullet}}}{B\to B^{\bullet}}}{B^{\bullet}}}{A\to B^{\bullet}}}{(\neg(A\wedge\neg B)\wedge(A\looparrowright B))\to(A\to B)}$$

For (DAI1). Let $\mathit{\Pi}_1$ be the following deduction:

$$(\to\mathrm{E}')\,\dfrac{(\wedge\mathrm{E2})\,\dfrac{(A\looparrowright B)\wedge(B\looparrowright C)}{B\looparrowright C}\qquad(\to^{\bullet}\mathrm{E1})\,\dfrac{(\to\mathrm{E}')\,\dfrac{(\wedge\mathrm{E1})\,\dfrac{(A\looparrowright B)\wedge(B\looparrowright C)}{A\looparrowright B}\qquad A^{\bullet}}{B\to B^{\bullet}}}{B^{\bullet}}}{C^{\bullet}}$$

and let $\mathit{\Pi}_2$ be the following deduction:

$$(\to\mathrm{I})\,\dfrac{(\to\mathrm{I})\,\dfrac{C\qquad C^{\bullet}}{C\to C}\qquad(\to^{\bullet}\mathrm{I})\,\dfrac{\mathit{\Pi}_1\qquad\mathit{\Pi}_1}{C\to C^{\bullet}}}{A\looparrowright C}$$

We obtain the following deduction:

$$(\to\mathrm{I})\,\dfrac{\mathit{\Pi}_2\qquad(\to^{\bullet}\mathrm{I})\,\dfrac{(\to^{\bullet}\mathrm{E1})\,\dfrac{(\wedge^{\bullet}\mathrm{E1})\,\dfrac{(A\looparrowright B)\wedge(B\looparrowright C)^{\bullet}}{A\looparrowright B^{\bullet}}}{A^{\bullet}}\qquad(\to^{\bullet}\mathrm{E2})\,\dfrac{(\wedge^{\bullet}\mathrm{E2})\,\dfrac{(A\looparrowright B)\wedge(B\looparrowright C)^{\bullet}}{B\looparrowright C^{\bullet}}}{C\to C^{\bullet}}}{A\looparrowright C^{\bullet}}}{((A\looparrowright B)\wedge(B\looparrowright C))\to(A\looparrowright C)}$$

The deductions for (DAI2) and (DAI3) are similar. For (DAI2), we obtain the following deduction:

$$
(\to\mathrm{I})\,\dfrac{(\to\mathrm{I})\,\dfrac{\neg A \qquad \neg A^{\bullet}}{\neg A \to \neg A} \qquad (\to^{\bullet}\mathrm{I})\,\dfrac{\neg A^{\bullet} \qquad (\to^{\bullet}\mathrm{I})\,\dfrac{A^{\bullet} \qquad (\bot^{\bullet})\,\dfrac{A^{\bullet}}{\bot^{\bullet}}}{\neg A^{\bullet}}}{\neg A \to \neg A^{\bullet}}\ \text{with}\ \neg A^{\bullet} = (\to^{\bullet}\mathrm{I})\,\dfrac{A^{\bullet} \qquad (\bot^{\bullet})\,\dfrac{A^{\bullet}}{\bot^{\bullet}}}{\neg A^{\bullet}}}{A \to (\neg A \to \neg A)}
$$

The deductions for (DAI4) and (DAI5) are similar.

($\to$) Let $\Pi_1$ be the following deduction:

$$
(\to^{\bullet}\mathrm{E1})\,\dfrac{(\to\mathrm{E}')\,\dfrac{A \looparrowright (B \star C) \qquad A^{\bullet}}{(B \star C) \to (B \star C)^{\bullet}}}{(\star^{\bullet}\mathrm{E1})\,\dfrac{B \star C^{\bullet}}{B^{\bullet}}}
$$

and let $\Pi_2$ be the following one:

$$
(\to^{\bullet}\mathrm{E1})\,\dfrac{(\to\mathrm{E}')\,\dfrac{A \looparrowright (B \star C) \qquad A^{\bullet}}{(B \star C) \to (B \star C)^{\bullet}}}{(\star^{\bullet}\mathrm{E2})\,\dfrac{B \star C^{\bullet}}{C^{\bullet}}}
$$

Moreover, let $\Pi_3$ be the following deduction:

$$
(\wedge\mathrm{I})\,\dfrac{(\to\mathrm{I})\,\dfrac{(\to\mathrm{I})\,\dfrac{B \qquad B^{\bullet}}{B \to B} \qquad (\to^{\bullet}\mathrm{I})\,\dfrac{\Pi_1 \qquad \Pi_1}{B \to B^{\bullet}}}{A \looparrowright B} \qquad (\to\mathrm{I})\,\dfrac{(\to\mathrm{I})\,\dfrac{C \qquad C^{\bullet}}{C \to C} \qquad (\to^{\bullet}\mathrm{I})\,\dfrac{\Pi_2 \qquad \Pi_2}{C \to C^{\bullet}}}{A \looparrowright C}}{(A \looparrowright B) \wedge (A \looparrowright C)}
$$

Let $\Pi_4$ be the following deduction:

$$
(\to^{\bullet}\mathrm{I})\,\dfrac{(\to^{\bullet}\mathrm{E2})\,\dfrac{(\to^{\bullet}\mathrm{E2})\,\dfrac{A \looparrowright (B \star C)^{\bullet}}{(B \star C) \to (B \star C)^{\bullet}}}{(\star^{\bullet}\mathrm{E1})\,\dfrac{B \star C^{\bullet}}{B^{\bullet}}} \qquad (\to^{\bullet}\mathrm{E2})\,\dfrac{(\to^{\bullet}\mathrm{E2})\,\dfrac{A \looparrowright (B \star C)^{\bullet}}{(B \star C) \to (B \star C)^{\bullet}}}{(\star^{\bullet}\mathrm{E1})\,\dfrac{B \star C^{\bullet}}{B^{\bullet}}}}{B \to B^{\bullet}}
$$

and $\Pi_5$ be the following deduction:

$$
(\to^{\bullet}\mathrm{I})\,\dfrac{(\to^{\bullet}\mathrm{E2})\,\dfrac{(\to^{\bullet}\mathrm{E2})\,\dfrac{A \looparrowright (B \star C)^{\bullet}}{(B \star C) \to (B \star C)^{\bullet}}}{(\star^{\bullet}\mathrm{E2})\,\dfrac{B \star C^{\bullet}}{C^{\bullet}}} \qquad (\to^{\bullet}\mathrm{E2})\,\dfrac{(\to^{\bullet}\mathrm{E2})\,\dfrac{A \looparrowright (B \star C)^{\bullet}}{(B \star C) \to (B \star C)^{\bullet}}}{(\star^{\bullet}\mathrm{E2})\,\dfrac{B \star C^{\bullet}}{C^{\bullet}}}}{C \to C^{\bullet}}
$$

Moreover, let $\Pi_6$ be the following deduction:

$$
(\wedge^{\bullet}\mathrm{I})\,\dfrac{(\to^{\bullet}\mathrm{I})\,\dfrac{(\to^{\bullet}\mathrm{E1})\,\dfrac{A \looparrowright (B \star C)^{\bullet}}{A^{\bullet}} \qquad \Pi_4}{A \looparrowright B^{\bullet}} \qquad (\to^{\bullet}\mathrm{I})\,\dfrac{(\to^{\bullet}\mathrm{E1})\,\dfrac{A \looparrowright (B \star C)^{\bullet}}{A^{\bullet}} \qquad \Pi_5}{A \looparrowright C^{\bullet}}}{(A \looparrowright B) \wedge (A \looparrowright C)^{\bullet}}
$$

Finally, we obtain the following deduction:

$$(\to\mathrm{I})\ \dfrac{\Pi_3 \qquad \Pi_6}{(A \looparrowright (B \star C)) \to ((A \looparrowright B) \wedge (A \looparrowright C))}$$

($\leftarrow$) Let $\Pi_1$ be the following deduction:

$$(\to^\bullet\mathrm{E1})\ \dfrac{(\to\mathrm{E}')\ \dfrac{(\wedge\mathrm{E1})\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C)}{A \looparrowright B} \qquad A^\bullet}{B \to B^\bullet}}{B^\bullet}$$

and let $\Pi_2$ be the following deduction:

$$(\to^\bullet\mathrm{E1})\ \dfrac{(\to\mathrm{E}')\ \dfrac{(\wedge\mathrm{E2})\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C)}{A \looparrowright C} \qquad A^\bullet}{C \to C^\bullet}}{C^\bullet}$$

then $\Pi_3$ is the following deduction

$$(\to\mathrm{I})\ \dfrac{(\to\mathrm{I})\ \dfrac{B \star C \qquad B \star C^\bullet}{(B \star C) \to (B \star C)} \qquad (\to^\bullet\mathrm{I})\ \dfrac{(\star^\bullet\mathrm{I})\ \dfrac{\Pi_1 \qquad \Pi_2}{B \star C^\bullet} \qquad (\star^\bullet\mathrm{I})\ \dfrac{\Pi_1 \qquad \Pi_2}{B \star C^\bullet}}{(B \star C) \to (B \star C)^\bullet}}{A \looparrowright (B \star C)}$$

Let $\Pi_4$ be the following deduction:

$$\dfrac{(\to^\bullet\mathrm{E1})\ \dfrac{(\to^\bullet\mathrm{E2})\ \dfrac{(\wedge^\bullet\mathrm{E1})\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C)^\bullet}{A \looparrowright B^\bullet}}{B \to B^\bullet}}{B^\bullet} \qquad (\to^\bullet\mathrm{E1})\ \dfrac{(\to^\bullet\mathrm{E2})\ \dfrac{(\wedge^\bullet\mathrm{E2})\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C)^\bullet}{A \looparrowright C^\bullet}}{C \to C^\bullet}}{C^\bullet}}{B \star C^\bullet}$$

then $\Pi_5$ is the following deduction:

$$(\to^\bullet\mathrm{I})\ \dfrac{(\to^\bullet\mathrm{E1})\ \dfrac{(\wedge^\bullet\mathrm{E1})\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C)^\bullet}{A \looparrowright B^\bullet}}{A^\bullet} \qquad (\to^\bullet\mathrm{I})\ \dfrac{\Pi_4 \qquad \Pi_4}{(B \star C) \to (B \star C)^\bullet}}{A \looparrowright (B \star C)^\bullet}$$

Finally, we obtain the following deduction:

$$(\to\mathrm{I})\ \dfrac{\Pi_3 \qquad \Pi_5}{((A \looparrowright B) \wedge (A \looparrowright C)) \to (A \looparrowright (B \star C))}$$

$\square$

As before, Lemmas 11 and 12 yield the following theorem:

**Theorem 13.** For all $X \cup \{A\} \subseteq \mathsf{For}$, $X \vdash_{\mathbf{DAI}_\perp} A$ iff $X \models_{\textbf{ica-E}} A$.

### 3.2.2 Normalisation

To prove the normalisation theorem, we first introduce the following reductions of deductions at a given d-expression:

+ $\wedge$-reduction:

$$\begin{array}{c} (\wedge\mathrm{E}_i)\,\dfrac{(\wedge\mathrm{I})\,\dfrac{\dfrac{\Sigma_1}{A_1} \qquad \dfrac{\Sigma_2}{A_2}}{A_1 \wedge A_2}}{(A_i)} \\ \Pi \end{array} \quad \curvearrowright \quad \begin{array}{c} \dfrac{\Sigma_i}{(A_i)} \\ \Pi \end{array}$$

+ $\star^\bullet$-reduction:

$$\begin{array}{c} (\star^\bullet\mathrm{E}i)\,\dfrac{(\star^\bullet\mathrm{I})\,\dfrac{\dfrac{\Sigma_1}{A_1^\bullet} \qquad \dfrac{\Sigma_2}{A_2^\bullet}}{A_1 \star A_2^\bullet}}{(A_i^\bullet)} \\ \Pi \end{array} \quad \curvearrowright \quad \begin{array}{c} \dfrac{\Sigma_i}{(A_i^\bullet)} \\ \Pi \end{array}$$

+ $\rightarrow$-reduction:

$$\begin{array}{c} (\rightarrow\mathrm{E})\,\dfrac{(\rightarrow\mathrm{I})\,\dfrac{\begin{array}{c}[A]\\ \dfrac{\Sigma_1}{B}\end{array} \qquad \begin{array}{c}[A^\bullet]\\ \dfrac{\Sigma_2}{B^\bullet}\end{array}}{A \rightarrow B} \qquad \dfrac{\Sigma_3}{A}}{(B)} \\ \Pi \end{array} \quad \curvearrowright \quad \begin{array}{c} \dfrac{\Sigma_3}{[A]} \\ \dfrac{\Sigma_1}{(B)} \\ \Pi \end{array}$$

Let us note that, in the case of $\wedge$-reduction, the deductions of the conjuncts cannot contain a labelled formula as an undischarged assumption. If one of the conjuncts were an implication, the labelled assumption would have to be discharged.[3]

Let us also observe that, in the case of $\star^\bullet$-reduction, the deductions of the respective components must contain the same labelled formula as an undischarged assumption whenever an implication is introduced at a later stage of the deduction.

Let us now proceed to show that the applications of the rule (RAA) within deduction can be restricted to cases where its consequence is a propositional variable.

[3] I am very grateful to Nils Kürbis for drawing my attention to an error in the conjunction reduction in an earlier version of this paper.

**Fact 14.** If $X \vdash_{\mathbf{DAI}_\bot} A$, then there is a deduction in $\mathbf{DAI}_\bot$ of $A$ from $X$ in which the consequence of every application of (RAA) is variable.

*Proof.* Let us make standard assumptions. Let $\Pi$ have the following form:

$$\begin{array}{c} \text{(RAA)}\,\dfrac{\begin{array}{c}[\neg B]\\ \Sigma \\ \hline \bot\end{array}}{(B)} \\ \Pi' \end{array}$$

Let us consider only the case where $B = C \to D$, for some $C, D \in \mathsf{For}$. Then, we modify the standard method to eliminate the application of (RAA) for formulas with implication as the main connective:

$$\begin{array}{c} \text{(}\to\text{I)}\,\dfrac{\text{(RAA)}\,\dfrac{\begin{array}{c} \left[\neg(C \to D)\right] \\ \Sigma \\ \hline \bot \end{array}}{D} \qquad \text{(RAA}^\bullet\text{)}\,\dfrac{\begin{array}{c} \left[\neg(C \to D)\right] \\ \Sigma \\ \hline \bot \end{array} \quad C^\bullet}{D^\bullet}}{(C \to D)} \\ \Pi' \end{array}$$

where the assumption $\neg(C \to D)$ is obtained as

$$\text{(}\to\text{I)}\,\dfrac{\text{(}\to\text{E)}\,\dfrac{\text{(}\to\text{E)}\,\dfrac{C \quad C \to D}{D} \quad \neg D}{\bot} \qquad (\bot^\bullet)\,\dfrac{C \to D^\bullet}{\bot^\bullet}}{[\neg(C \to D)]}$$

$\square$

Thus, we can prove the following theorem in the standard way:

**Theorem 15.** If $X \vdash_{\mathbf{DAI}_\bot} A$, then there is a normal deduction in $\mathbf{DAI}_\bot$ of $A$ from $X$.

# 4 Summary

Our paper consists of two parts. The first is semantic and philosophical: we discuss how to represent the content of absurd sentences represented by *falsum* and how various notions of negation can be defined using demodalised analytic implication together with *falsum*. The second part is proof-theoretic. Having fixed one interpretation of *falsum*, we present two deductive systems, establish soundness and completeness for both, and investigate the problem of eliminating maximum formulas.

In the first system, deductions are formulated entirely in the language of the logic. In the second system, they contain both formulas and labelled formulas and employ only a single label. We show that maximum formulas arising from applications of certain introduction–elimination rules for implication adopted in the first system can be eliminated, although only at the cost of introducing infinitely many rules. We also prove that deductions in the second system can be transformed so as to eliminate maximum formulas. In this case, infinitely many rules are not required, although certain special modifications of the *reductio ad absurdum* rule are needed.

The next step in our proof-theoretic investigation of logics of content relations will be to develop sequent calculi for **DAI** and other such logics.

# Conflict of interest and data availability statement

The author declares that there are no conflicts of interest related to this manuscript.

No datasets were generated or analysed in the course of this study, as it is conducted within a theoretical and logical-philosophical framework.

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