CLASSIFICATION OF MODULES NOT LYING
ON SHORT CHAINS

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Abstract

We give a complete description of finitely generated modules over artin algebras which are not the middle of a short chain of modules, using injective and tilting modules over hereditary artin algebras.

1. Introduction

Throughout the paper, by an algebra we mean an artin algebra over a fixed commutative artin ring R, that is, an R-algebra (associative, with identity) which is finitely generated as an R-module. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, by $K_0(A)$ the Grothendieck group of A, and by $[M]$ the image of a module M from $\operatorname{mod} A$ in $K_0(A)$. Then $[M] = [N]$ for two modules M and N in $\operatorname{mod} A$ if and only if M and N have the same (simple) composition factors including the multiplicities. A module M in $\operatorname{mod} A$ is called sincere if every simple right A-module occurs as a composition factor of M. Further, we denote by D the standard duality $\operatorname{Hom}_R(-, E)$ on $\operatorname{mod} A$, where E is a minimal injective cogenerator in $\operatorname{mod} R$. Moreover, for a module X in $\operatorname{mod} A$ and its minimal projective presentation $P_1 \xrightarrow{f} P_0 \to X \to 0$ in $\operatorname{mod} A$, the transpose $\operatorname{Tr} X$ of X is the cokernel of the homomorphism $\operatorname{Hom}_A(f, A)$ in $\operatorname{mod} A^{\mathrm{op}}$, where A^{op} is the opposite algebra of A. Then we obtain the homological operator $\tau_A = D\operatorname{Tr}$ on modules in $\operatorname{mod} A$, called the Auslander-Reiten translation, playing a fundamental role in the modern representation theory of artin algebras.

The aim of this article is to provide a complete description of all modules M in $\operatorname{mod} A$ satisfying the condition: for any module X in $\operatorname{ind} A$, we have $\operatorname{Hom}_A(X, M) = 0$ or $\operatorname{Hom}_A(M, \tau_A X) = 0$. We note that, by [2], [23], a sequence $X \rightarrowtail M \rightarrowtail \tau_A X$ of nonzero homomorphisms in $\operatorname{mod} A$ with X being indecomposable is called a short chain, and M the middle of this short chain. Therefore, we are concerned with the classification of all modules in $\operatorname{mod} A$ which are not the middle of a short chain. We also mention that, if M is a module in $\operatorname{mod} A$ which is not the middle of a short chain, then $\operatorname{Hom}_A(M, \tau_A M) = 0$, and hence the number of pairwise nonisomorphic indecomposable direct summands of M is less than or equal to the rank of $K_0(A)$, by [32, Lemma 2]. Further, by [23, Theorem 1.6] and [10, Lemma 1], an indecomposable module X in $\operatorname{mod} A$ is not the middle of a short chain if and only if X does not lie on a short cycle $Y \rightarrowtail X \rightarrowtail Y$ of nonzero nonisomorphisms in $\operatorname{ind} A$. Hence, every indecomposable direct summand Z of a module M in $\operatorname{mod} A$ which is not the middle of a short chain is uniquely determined (up to isomorphism) by the composition factors (see [23, Corollary 2.2]). Finally, we point out that the class of modules which are not the middle of a short chain contains the class of directing modules investigated in [5], [12], [25], [32], [33], [34].

Following [7], [11], by a tilted algebra we mean an algebra of the form $\text{End}_H(T)$, where H is a hereditary algebra and T is a tilting module in $\text{mod}\,H$, that is, $\text{Ext}^1_H(T, T) = 0$ and the number of pairwise nonisomorphic indecomposable direct summands of T is equal to the rank of $K_0(H)$. The tilted algebras play a prominent role in the representation theory of algebras and have attracted much attention (see [1], [9], [22], [25], [26], [27], [28] and their cited papers).

The following theorem is the main result of the paper.

THEOREM 1.1. *Let A be an algebra and M a module in $\text{mod}\,A$ which is not the middle of a short chain. Then there exists a hereditary algebra H, a tilting module T in $\text{mod}\,H$, and an injective module I in $\text{mod}\,H$ such that the following statements hold:*

(i) *the tilted algebra $B = \text{End}_H(T)$ is a quotient algebra of A;*

(ii) *M is isomorphic to the right B-module $\text{Hom}_H(T, I)$.*

We note that for a hereditary algebra H, T a tilting module in $\text{mod}\,H$, I an injective module in $\text{mod}\,H$, and $B = \text{End}_H(T)$, the right B-module $\text{Hom}_H(T, I)$ is not the middle of a short chain in $\text{mod}\,B$ (see Lemma 3.1). An important role in the proof of the main theorem plays the following characterization of tilted algebras established recently in the authors paper [13]: an algebra B is a tilted algebra if and only if $\text{mod}\,B$ admits a sincere module M which is not the middle of a short chain.

The following fact is a consequence of Theorem 1.1.

COROLLARY 1.2. *Let A be an algebra and M a module in $\text{mod}\,A$ which is not the middle of a short chain. Then $\text{End}_A(M)$ is a hereditary algebra.*

In Sections 2 and 3, after recalling some background on module categories and tilted algebras, we prove preliminary facts playing an essential role in the proof of Theorem 1.1. Section 4 is devoted to the proofs of Theorem 1.1 and Corollary 1.2. In the final Section 5 we present examples illustrating the main theorem.

For background on the representation theory applied here we refer to [1], [3], [25], [27], [28].

2. Preliminaries on module categories

Let A be an algebra. We denote by Γ_A the Auslander-Reiten quiver of A. Recall that Γ_A is a valued translation quiver whose vertices are the isomorphism classes $\{X\}$ of modules X in $\text{ind}\,A$, the valued arrows of Γ_A correspond to irreducible homomorphisms between indecomposable modules (and describe minimal left almost split homomorphisms with indecomposable domains and minimal right almost split homomorphisms with indecomposable codomains) and the translation is given by the Auslander-Reiten translations $\tau_A = D\text{Tr}$ and $\tau_A^- = \text{Tr}D$. We shall not distinguish between a module X in $\text{ind}\,A$ and the corresponding vertex $\{X\}$ of Γ_A. By a component of Γ_A we mean a connected component of the quiver Γ_A. Following [30], a component $\mathcal{C}$ of Γ_A is said to be generalized standard if $\text{rad}^\infty_A(X, Y) = 0$ for all modules X and Y in $\mathcal{C}$, where rad^∞_A is the infinite Jacobson radical of $\text{mod}\,A$. Moreover, two components $\mathcal{C}$ and $\mathcal{D}$ of Γ_A are said to be orthogonal if $\text{Hom}_A(X, Y) =$

0 and $\mathrm{Hom}_A(Y, X) = 0$ for all modules X in $\mathcal{C}$ and Y in $\mathcal{D}$. A family $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ of components of Γ_A is said to be (strongly) separating if the components in Γ_A split into three disjoint families $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$ such that the following conditions are satisfied:

(S1) $\mathcal{C}^A$ is a sincere family of pairwise orthogonal generalized standard components;

(S2) $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\mathrm{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;

(S3) any homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in $\mathrm{mod}\, A$ factors through $\mathrm{add}(\mathcal{C}_i)$ for any $i \in I$.

We then say that $\mathcal{C}^A$ separates $\mathcal{P}^A$ from $\mathcal{Q}^A$ and write

$$\Gamma_A = \mathcal{P}^A \vee \mathcal{C}^A \vee \mathcal{Q}^A.$$

A component $\mathcal{C}$ of Γ_A is said to be preprojective if $\mathcal{C}$ is acyclic (without oriented cycles) and each module in $\mathcal{C}$ belongs to the τ_A-orbit of a projective module. Dually, $\mathcal{C}$ is said to be preinjective if $\mathcal{C}$ is acyclic and each module in $\mathcal{C}$ belongs to the τ_A-orbit of an injective module. Further, $\mathcal{C}$ is called regular if $\mathcal{C}$ contains neither a projective module nor an injective module. Finally, $\mathcal{C}$ is called semiregular if $\mathcal{C}$ does not contain both a projective module and an injective module. By a general result of S. Liu [18] and Y. Zhang [37], a regular component $\mathcal{C}$ contains an oriented cycle if and only if $\mathcal{C}$ is a stable tube, that is, an orbit quiver $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some integer $r \geq 1$. Important classes of semiregular components with oriented cycles are formed by the ray tubes, obtained from stable tubes by a finite number (possibly empty) of ray insertions, and the coray tubes obtained from stable tubes by a finite number (possibly empty) of coray insertions (see [25], [28]).

The following characterizations of ray and coray tubes of Auslander-Reiten quivers of algebras have been established by S. Liu in [20].

THEOREM 2.1. *Let A be an algebra and $\mathcal{C}$ be a semiregular component of Γ_A. The following equivalences hold:*

(i) *$\mathcal{C}$ contains an oriented cycle but no injective module if and only if $\mathcal{C}$ is a ray tube;*

(ii) *$\mathcal{C}$ contains an oriented cycle but no projective module if and only if $\mathcal{C}$ is a coray tube.*

The following lemma from [13, Lemma 1.2] will play an important role in the proof of our main theorem.

LEMMA 2.2. *Let A be an algebra and M a sincere module in $\mathrm{mod}\, A$ which is not the middle of a short chain. Then the following statements hold:*

(i) *$\mathrm{Hom}_A(M, X) = 0$ for any A-module X in $\mathcal{T}$, where $\mathcal{T}$ is an arbitrary ray tube of Γ_A containing a projective module;*

(ii) *$\mathrm{Hom}_A(X, M) = 0$ for any A-module X in $\mathcal{T}$, where $\mathcal{T}$ is an arbitrary coray tube of Γ_A containing an injective module.*

LEMMA 2.3. *Let A be an algebra, $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ a separating family of stable tubes of Γ_A, and $\Gamma_A = \mathcal{P}^A \vee \mathcal{C}^A \vee \mathcal{Q}^A$ the associated decomposition of Γ_A with $\mathcal{C}^A = \mathcal{C}$. Then for arbitrary modules $M \in \mathcal{P}^A$, $N \in \mathcal{Q}^A$, and $i \in I$, the following statements hold:*

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(i) $\text{Hom}_A(M, X) \neq 0$ *for all but finitely many modules* $X \in \mathcal{C}_i$;

(ii) $\text{Hom}_A(X, N) \neq 0$ *for all but finitely many modules* $X \in \mathcal{C}_i$.

Proof. Let M be a module in $\mathcal{P}^A$, N a module in $\mathcal{Q}^A$, $i \in I$, and r_i be the rank of the stable tube $\mathcal{C}_i$. Consider an injective hull $M \to E_A(M)$ of M in $\text{mod}\, A$ and a projective cover $P_A(N) \to N$ of N in $\text{mod}\, A$. Applying the separating property of $\mathcal{C}$, we conclude that there exist indecomposable modules U and V in $\mathcal{C}_i$ such that $\text{Hom}_A(M, U) \neq 0$ and $\text{Hom}_A(V, N) \neq 0$. Then $\text{Hom}_A(M, X) \neq 0$ and $\text{Hom}_A(X, N) \neq 0$ for all indecomposable modules X in $\mathcal{C}_i$ of quasi-length greater than or equal to r_i, by [31, Lemma 3.9]. Since such modules X exhaust all but finitely many modules in $\mathcal{C}_i$, the claims (i) and (ii) hold. $\square$

We also have the following known fact.

LEMMA 2.4. *Let A be an algebra and $\mathcal{T}$ a stable tube of Γ_A. Then every indecomposable module X in $\mathcal{T}$ is the middle of a short chain in $\text{mod}\, A$.*

A path $X_0 \rightarrowtail X_1 \rightarrowtail ... \rightarrowtail X_{t-1} \rightarrowtail X_t$ in the Auslander-Reiten quiver Γ_A of an algebra A is called sectional if $\tau_A X_i \ncong X_{i-2}$ for all $i \in \{2, ..., t\}$. Then we have the following result proved by R. Bautista and S. O. Smalø [6].

LEMMA 2.5. *Let A be an algebra and*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} ... \rightarrow X_{t-1} \xrightarrow{f_t} X_t$$

be a path of irreducible homomorphisms $f_1, f_2, ..., f_t$ corresponding to a sectional path of Γ_A. Then $f_t...f_2 f_1 \neq 0$.

Let A be an algebra, $\mathcal{C}$ a component of Γ_A and V, W be A-modules in $\mathcal{C}$ such that V is a predecessor of W (respectively, a successor of W). If V lies on a sectional path from V to W (respectively, from W to V), then we say that V is a sectional predecessor of W (respectively, a sectional successor of W). Otherwise, we say that V is a nonsectional predecessor of W (respectively, a nonsectional successor of W). Moreover, denote by $\mathcal{S}_W$ the set of all indecomposable modules X in $\mathcal{C}$ such that there is a sectional path in $\mathcal{C}$ (possibly of length zero) from X to W, and by $\mathcal{S}_W^*$ the set of all indecomposable modules Y in $\mathcal{C}$ such that there is a sectional path in $\mathcal{C}$ (possibly of length zero) from W to Y.

PROPOSITION 2.6. *Let A be an algebra and $\mathcal{C}$ be an acyclic component of Γ_A with finitely many τ_A-orbits. Then the following statements hold:*

(i) *if V and W are modules in $\mathcal{C}$ such that V is a predecessor of W, V does not belong to $\mathcal{S}_W$, and W has no injective nonsectional predecessors in $\mathcal{C}$, then we have $\text{Hom}_A(V, \tau_A U) \neq 0$ for some module U in $\mathcal{S}_W$;*

(ii) *if V and W are modules in $\mathcal{C}$ such that V is a successor of W, V does not belong to $\mathcal{S}_W^*$, and W has no projective nonsectional successors in $\mathcal{C}$, then we have $\text{Hom}_A(\tau_A^- U, V) \neq 0$ for some module U in $\mathcal{S}_W^*$.*

Proof. We shall prove only (i), because the proof of (ii) is dual. Let V and W be modules in $\mathcal{C}$ such that V is a predecessor of W, V does not belong to $\mathcal{S}_W$, and W has no injective nonsectional predecessors in $\mathcal{C}$. Moreover, let $n(V)$ be the length of the shortest path in $\mathcal{C}$ from V to W. We prove first by induction on $n(V)$ that

then every path in $\mathcal{C}$ of sufficiently large length starting at V is passing through a module in $\tau_A \mathcal{S}_W$.

We may assume that V does not belong to $\tau_A \mathcal{S}_W$ and hence $n(V) \geq 3$. Because W has no injective nonsectional predecessors in $\mathcal{C}$ and $\mathcal{S}_W$ does not contain the module V, we conclude that there exists $\tau_A^- V$ and it is a predecessor of W in $\mathcal{C}$. Moreover, $n(\tau_A^- V) = n(V) - 2$. Indeed, if it is not the case, then we get a contradiction with the minimality of $n(V)$. Let $\{U_1, U_2, \ldots, U_t\}$ be the set of all direct predecessors of $\tau_A^- V$ in $\mathcal{C}$. Then, for any $i \in \{1, \ldots, t\}$, U_i is a predecessor of W in $\mathcal{C}$ and $n(U_i) = n(V) - 1$. Hence, by the induction hypothesis, every path of sufficiently large length starting at U_i is passing through a module in $\tau_A \mathcal{S}_W$. Since $\{U_1, U_2, \ldots, U_t\}$ is also the set of all direct successors of V, we have that every path in $\mathcal{C}$ of nonzero length starting at V is passing through U_i for some $i \in \{1, \ldots, t\}$. Therefore, the required property holds.

Let now $u : V \to E_A(V)$ be an injective hull of V in $\mathrm{mod}\, A$. Then there exists an indecomposable injective A-module I such that $\mathrm{Hom}_A(V, I) \neq 0$. Since W has no injective nonsectional predecessors in $\mathcal{C}$, applying [1, Chapter IV, Lemma 5.1], we conclude that there exists a path of irreducible homomorphisms

$$V = V_0 \xrightarrow{g_1} V_1 \xrightarrow{g_2} V_2 \longrightarrow \cdots \longrightarrow V_{r-1} \xrightarrow{g_r} V_r$$

with $V_r = \tau_A U$ for some $U \in \mathcal{S}_W$ and a homomorphism $h_r : V_r \to I$ such that $h_r g_r \ldots g_1 \neq 0$. Hence, we conclude that $\mathrm{Hom}_A(V, \tau_A U) = \mathrm{Hom}_A(V, V_r) \neq 0$. $\qquad \square$

3. Preliminaries on tilted algebras

Let H be an indecomposable hereditary algebra and Q_H the valued quiver of H. Recall that the vertices of Q_H are the numbers $1, 2, \ldots, n$ corresponding to a complete set $S_1, S_2, \ldots, S_n$ of pairwise nonisomorphic simple modules in $\mathrm{mod}\, H$ and there is an arrow from i to j in Q_H if $\mathrm{Ext}_H^1(S_i, S_j) \neq 0$, and then to this arrow is assigned the valuation $(\dim_{\mathrm{End}_H(S_j)} \mathrm{Ext}_H^1(S_i, S_j), \dim_{\mathrm{End}_H(S_i)} \mathrm{Ext}_H^1(S_i, S_j))$. Recall that the Auslander-Reiten quiver Γ_H of H has a disjoint union decomposition of the form

$$\Gamma_H = \mathcal{P}(H) \vee \mathcal{R}(H) \vee \mathcal{Q}(H),$$

where $\mathcal{P}(H)$ is the preprojective component containing all indecomposable projective H-modules, $\mathcal{Q}(H)$ is the preinjective component containing all indecomposable injective H-modules, and $\mathcal{R}(H)$ is the family of all regular components of Γ_H. More precisely, we have:

- if Q_H is a Dynkin quiver, then $\mathcal{R}(H)$ is empty and $\mathcal{P}(H) = \mathcal{Q}(H)$;

- if Q_H is a Euclidean quiver, then $\mathcal{P}(H) \cong (-\mathbb{N})Q_H^{\mathrm{op}}$, $\mathcal{Q}(H) \cong \mathbb{N}Q_H^{\mathrm{op}}$ and $\mathcal{R}(H)$ is a separating infinite family of stable tubes;

- if Q_H is a wild quiver, then $\mathcal{P}(H) \cong (-\mathbb{N})Q_H^{\mathrm{op}}$, $\mathcal{Q}(H) \cong \mathbb{N}Q_H^{\mathrm{op}}$ and $\mathcal{R}(H)$ is an infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$.

Let T be a tilting module in $\mathrm{mod}\, H$ and $B = \mathrm{End}_H(T)$ the associated tilted algebra. Then the tilting H-module T determines the torsion pair $(\mathcal{F}(T), \mathcal{T}(T))$ in $\mathrm{mod}\, H$, with the torsion-free part $\mathcal{F}(T) = \{X \in \mathrm{mod}\, H | \mathrm{Hom}_H(T, X) = 0\}$ and the torsion part $\mathcal{T}(T) = \{X \in \mathrm{mod}\, H | \mathrm{Ext}_H^1(T, X) = 0\}$, and the splitting torsion pair $(\mathcal{Y}(T), \mathcal{X}(T))$ in $\mathrm{mod}\, B$, with the torsion-free part $\mathcal{Y}(T) = \{Y \in$

mod $B|\operatorname{Tor}_1^B(Y,T) = 0\}$ and the torsion part $\mathcal{X}(T) = \{Y \in \operatorname{mod} B|Y \otimes_B T = 0\}$. Then, by the Brenner-Butler theorem, the functor $\operatorname{Hom}_H(T,-) : \operatorname{mod} H \to \operatorname{mod} B$ induces an equivalence of $\mathcal{T}(T)$ with $\mathcal{Y}(T)$, and the functor $\operatorname{Ext}_H^1(T,-) : \operatorname{mod} H \to \operatorname{mod} B$ induces an equivalence of $\mathcal{F}(T)$ with $\mathcal{X}(T)$ (see [8], [11]). Further, the images $\operatorname{Hom}_H(T,I)$ of the indecomposable injective modules I in $\operatorname{mod} H$ via the functor $\operatorname{Hom}_H(T,-)$ belong to one component $\mathcal{C}_T$ of Γ_B, called the connecting component of Γ_B determined by T, and form a faithful section Δ_T of $\mathcal{C}_T$, with Δ_T the opposite valued quiver Q_H^{op} of Q_H. Recall that a full connected valued subquiver Σ of a component $\mathcal{C}$ of Γ_B is called a section if Σ has no oriented cycles, is convex in $\mathcal{C}$, and intersects each τ_B-orbit of $\mathcal{C}$ exactly once. Moreover, the section Σ is faithful provided the direct sum of all modules lying on Σ is a faithful B-module. The section Δ_T of the connecting component $\mathcal{C}_T$ of Γ_B has the distinguished property: it connects the torsion-free part $\mathcal{Y}(T)$ with the torsion part $\mathcal{X}(T)$, because every predecessor in ind B of a module $\operatorname{Hom}_H(T,I)$ from Δ_T lies in $\mathcal{Y}(T)$ and every successor of $\tau_B^- \operatorname{Hom}_H(T,I)$ in ind B lies in $\mathcal{X}(T)$.

LEMMA 3.1. *Let H be an indecomposable algebra, T a tilting module in $\operatorname{mod} H$, and $B = \operatorname{End}_H(T)$ the associated tilted algebra. Then for any injective module I in $\operatorname{mod} H$, $M_I = \operatorname{Hom}_H(T,I)$ is a module in $\operatorname{mod} B$ which is not the middle of a short chain.*

Proof. Consider the connecting component $\mathcal{C}_T$ of Γ_B determined by T and its canonical section Δ_T given by the images of a complete set of pairwise nonisomorphic injective H-modules via the functor $\operatorname{Hom}_H(T,-) : \operatorname{mod} H \to \operatorname{mod} B$. Then M_I is isomorphic to a direct sum of indecomposable modules lying on Δ_T. Suppose M_I is the middle of a short chain $X \rightarrowtail M_I \twoheadrightarrow \tau_B X$ in $\operatorname{mod} B$. Then X is a predecessor in ind B of an indecomposable module Y lying on Δ_T, and consequently $Y \in \mathcal{Y}(T)$ forces $X \in \mathcal{Y}(T)$. Hence $\tau_B X$ also belongs to $\mathcal{Y}(T)$ since $\mathcal{Y}(T)$ is closed under predecessors in ind B. In particular, $\tau_B X$ does not lie on Δ_T. Then $\operatorname{Hom}_B(M_I, \tau_B X) \neq 0$ implies that there is an indecomposable module Z on Δ_T such that $\tau_B X$ is a successor of $\tau_B^{-1} Z$ in ind B. But then $\tau_B^{-1} Z \in \mathcal{X}(T)$ forces $\tau_B X \in \mathcal{X}(T)$, because $\mathcal{X}(T)$ is closed under successors in ind B. Hence the indecomposable B-module $\tau_B X$ is simultaneously in $\mathcal{Y}(T)$ and $\mathcal{X}(T)$, a contradiction. Therefore, M_I is indeed a module in $\operatorname{mod} B$ which is not the middle of a short chain. $\square$

Recently, the authors established in [13] the following characterization of tilted algebras.

THEOREM 3.2. *An algebra B is a tilted algebra if and only if $\operatorname{mod} B$ admits a sincere module M which is not the middle of a short chain.*

We exhibit now a handy criterion for an indecomposable algebra to be a tilted algebra established independently in [21] and [29].

THEOREM 3.3. *Let B be an indecomposable algebra. Then B is a tilted algebra if and only if the Auslander-Reiten quiver Γ_B of B admits a component $\mathcal{C}$ with a faithful section Δ such that $\operatorname{Hom}_B(X, \tau_B Y) = 0$ for all modules X and Y in Δ. Moreover, if this is the case and T_Δ^* is the direct sum of all indecomposable modules lying on Δ, then $H_\Delta = \operatorname{End}_B(T_\Delta^*)$ is an indecomposable hereditary algebra, $T_\Delta = D(T_\Delta^*)$ is a tilting module in $\operatorname{mod} H_\Delta$, and the tilted algebra $B_\Delta = \operatorname{End}_{H_\Delta}(T_\Delta)$ is the basic algebra of B.*

Let H be an indecomposable hereditary algebra not of Dynkin type, that is, the valued quiver Q_H of H is a Euclidean or wild quiver. Then by a concealed algebra of type Q_H we mean an algebra $B = \operatorname{End}_H(T)$ for a tilting module T in $\operatorname{add}(\mathcal{P}(H))$ (equivalently, in $\operatorname{add}(\mathcal{Q}(H))$). If Q_H is a Euclidean quiver, B is said to be a tame concealed algebra. Similarly, if Q_H is a wild quiver, B is said to be a wild concealed algebra. Recall that the Auslander-Reiten quiver Γ_B of a concealed algebra B is of the form:

$$\Gamma_B = \mathcal{P}(B) \vee \mathcal{R}(B) \vee \mathcal{Q}(B),$$

where $\mathcal{P}(B)$ is a preprojective component containing all indecomposable projective B-modules, $\mathcal{Q}(B)$ is a preinjective component containing all indecomposable injective B-modules and $\mathcal{R}(B)$ is either an infinite family of stable tubes separating $\mathcal{P}(B)$ from $\mathcal{Q}(B)$ or an infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$.

PROPOSITION 3.4. *Let B be a wild concealed algebra, $\mathcal{C}$ a regular component of Γ_B, M a module in $\mathcal{P}(B)$ and N a module in $\mathcal{Q}(B)$. Then the following statements hold:*

(i) $\operatorname{Hom}_B(M, X) \neq 0$ *for all but finitely many modules X in $\mathcal{C}$;*

(ii) $\operatorname{Hom}_B(X, N) \neq 0$ *for all but finitely many modules X in $\mathcal{C}$.*

In particular, all but finitely many modules in $\mathcal{C}$ are sincere.

Proof. (i) Let H be a wild hereditary algebra and T a tilting module in $\operatorname{add}(\mathcal{P}(H))$ such that $B = \operatorname{End}_H(T)$. Recall that the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$ induces an equivalence of the torsion part $\mathcal{T}(T)$ of $\operatorname{mod} H$ and the torsion-free part $\mathcal{Y}(T)$ of $\operatorname{mod} B$. Moreover, we have the following facts:

(a) the images under the functor $\operatorname{Hom}_H(T, -)$ of the regular components from $\mathcal{R}(H)$ form the family $\mathcal{R}(B)$ of all regular components of Γ_B;

(b) the images under the functor $\operatorname{Hom}_H(T, -)$ of all indecomposable modules in $\mathcal{P}(H) \cap \mathcal{T}(T)$ form the unique preprojective component $\mathcal{P}(B)$ of Γ_B.

Since $\mathcal{C}$ is in $\mathcal{R}(B)$, there exists a component $\mathcal{D}$ in $\mathcal{R}(H)$ such that $\mathcal{C} = \operatorname{Hom}_H(T, \mathcal{D})$. We note that $\mathcal{C}$ and $\mathcal{D}$ are of the form $\mathbb{Z}\mathbb{A}_\infty$. It follows from [4] (see also [28, Corollary XVIII.2.4]) that all but finitely many modules in $\mathcal{D}$ are sincere H-modules. We may choose an indecomposable module U in $\mathcal{P}(H) \cap \mathcal{T}(T)$ such that $M = \operatorname{Hom}_H(T, U)$. Further, there exists an indecomposable projective module P in $\mathcal{P}(H)$ such that $U = \tau_H^{-m} P$ for some integer $m \geq 0$. Take now an indecomposable module Z in $\mathcal{D}$. Then we obtain isomorphisms of R-modules

$$\operatorname{Hom}_H(U, Z) \cong \operatorname{Hom}_H(\tau_H^{-m} P, Z) \cong \operatorname{Hom}_H(P, \tau_H^m Z),$$

because H is hereditary (see [1, Corollary IV.2.15]). Since $\operatorname{Hom}_H(P, R) \neq 0$ for all but finitely many modules R in $\mathcal{D}$, we conclude that $\operatorname{Hom}_H(U, Z) \neq 0$ for all but finitely many modules Z in $\mathcal{D}$. Applying now the equivalence of categories $\operatorname{Hom}_H(T, -) : \mathcal{T}(T) \to \mathcal{Y}(T)$ and the equalities $\mathcal{P}(B) = \operatorname{Hom}_H(T, \mathcal{P}(H) \cap \mathcal{T}(T))$, $\mathcal{C} = \operatorname{Hom}_H(T, \mathcal{D})$, and $M = \operatorname{Hom}_H(T, U)$, we obtain that $\operatorname{Hom}_B(M, X) \neq 0$ for all but finitely many modules X in $\mathcal{C}$.

(ii) We note that the preinjective component $\mathcal{Q}(B)$ is the connecting component $\mathcal{C}_T$ of Γ_B determined by T, and is obtained by gluing the image $\operatorname{Hom}_H(T, \mathcal{Q}(H))$ of the preinjective component $\mathcal{Q}(H)$ of Γ_H with a finite part consisting of all indecomposable modules of the torsion part $\mathcal{X}(T) = \operatorname{Ext}_H^1(T, \mathcal{F}(T))$ of $\operatorname{mod} B$ (see [1, Theorem VIII.4.5]). But the wild concealed algebra B is also of the form $B = \operatorname{End}_{H^*}(T^*)$,

where H^* is a wild hereditary algebra and T^* is a tilting module in $\mathrm{add}(\mathcal{Q}(H^*))$. Then the functor $\mathrm{Ext}^1_{H^*}(T^*, -) : \mathrm{mod}\, H^* \to \mathrm{mod}\, B$ induces an equivalence of the torsion-free part $\mathcal{F}(T^*)$ of $\mathrm{mod}\, H^*$ and the torsion part $\mathcal{X}(T^*)$ of $\mathrm{mod}\, B$. Moreover, we have the following facts:

(a*) the images under the functor $\mathrm{Ext}^1_{H^*}(T^*, -)$ of the regular components from $\mathcal{R}(H^*)$ form the family $\mathcal{R}(B)$ of all regular components of Γ_B;

(b*) the images under the functor $\mathrm{Ext}^1_{H^*}(T^*, -)$ of all indecomposable modules in $\mathcal{Q}(H) \cap \mathcal{F}(T)$ form the unique preinjective component $\mathcal{Q}(B)$ of Γ_B.

In particular, we have that $\mathcal{C} = \mathrm{Ext}^1_{H^*}(T^*, \mathcal{D}^*)$ for a component $\mathcal{D}^*$ in $\mathcal{R}(H^*)$. We then conclude that $\mathrm{Hom}_B(X, N) \neq 0$ for all but finitely many modules X in $\mathcal{C}$, applying arguments dual to those used in the proof of (i).

The fact that all but finitely many modules in $\mathcal{C}$ are sincere follows from (i) (equivalently (ii)), because $\mathcal{P}(B)$ contains all indecomposable projective B-modules and $\mathcal{Q}(B)$ contains all indecomposable injective B-modules. $\square$

A prominent role in our considerations will be played by the following consequence of a result of D. Baer [4] (see [28, Theorem XVIII.5.2]).

THEOREM 3.5. *Let B be a wild concealed algebra, and M, N indecomposable B-modules lying in regular components of Γ_B. Then there exists a positive integer m_0 such that $\mathrm{Hom}_B(M, \tau_B^m N) \neq 0$ for all integers $m \geqslant m_0$.*

LEMMA 3.6. *Let B be a wild concealed algebra and $\mathcal{C}$ a regular component of Γ_B. Then any indecomposable module N in $\mathcal{C}$ is the middle of a short chain in $\mathrm{mod}\, B$.*

Proof. Suppose N is an indecomposable module in $\mathcal{C}$. Obviously $\mathcal{C}$ is of the form $\mathbb{Z}\mathbb{A}_\infty$. Applying Theorem 3.5, we conclude that there is a positive integer m_0 such that $\mathrm{Hom}_B(N, \tau_B^m N) \neq 0$ for all integers $m \geq m_0$. Then we may take an indecomposable module X in $\mathcal{C}$ such that there are a sectional path Ω from X to N and a sectional path Σ from $\tau_B^m N$ to $\tau_B X$ for some integer $m \geq m_0$. Observe that all irreducible homomorphisms corresponding to arrows of Σ are monomorphisms whereas all irreducible homomorphisms corresponding to arrows of Ω are epimorphisms. Hence there are a monomorphism $f : \tau_B^m N \to \tau_B X$ and an epimorphism $g : X \to N$. Since $\mathrm{Hom}_B(N, \tau_B^m N) \neq 0$, we conclude that $\mathrm{Hom}_B(N, \tau_B X) \neq 0$. Therefore, we obtain a short chain $X \twoheadrightarrow N \rightarrow \tau_B X$. $\square$

4. Proofs of Theorem 1.1 and Corollary 1.2

Let A be an algebra and M a module in $\mathrm{mod}\, A$ which is not the middle of a short chain. By $\mathrm{ann}_A(M)$ we shall denote the annihilator of M in A, that is, the ideal $\{a \in A | Ma = 0\}$. Then M is a sincere module over the algebra $B = A/\mathrm{ann}_A(M)$. Moreover, by [23, Proposition 2.3], M is not the middle of a short chain in $\mathrm{mod}\, B$, since M is not the middle of a short chain in $\mathrm{mod}\, A$. Let $B = B_1 \times \ldots \times B_m$ be a decomposition of B into a product of indecomposable algebras and $M = M_1 \oplus \ldots \oplus M_m$ the associated decomposition of M in $\mathrm{mod}\, B$ with M_i a module in $\mathrm{mod}\, B_i$ for any $i \in \{1, \ldots, m\}$. Observe that, for each $i \in \{1, \ldots, m\}$, $B_i = A/\mathrm{ann}_A(M_i)$, M_i is a sincere B_i-module which is not the middle of a short chain in $\mathrm{mod}\, B_i$, and hence B_i is a tilted algebra, by Theorem 3.2. Therefore, we may assume that B is an indecomposable algebra.

We will start our considerations by showing that for a tilted algebra B and a sincere B-module M which is not the middle of a short chain, all indecomposable direct summands of M belong to the same component, which is in fact a connecting component of Γ_B. According to a result of C. M. Ringel [26, p.46] Γ_B admits at most two components containing sincere sections (slices), and exactly two if and only if B is a concealed algebra. We shall discuss this case in the following proposition.

PROPOSITION 4.1. *Let B be a concealed algebra and M a sincere B-module which is not the middle of a short chain. Then $M \in \text{add}(\mathcal{C})$ for a connecting component $\mathcal{C}$ of Γ_B.*

Proof. Observe that M has no indecomposable direct summands in $\mathcal{R}(B)$, by Lemmas 2.4 and 3.6. Hence we may assume that $M = M_P \oplus M_Q$, where M_P is a direct summand of M contained in $\text{add}(\mathcal{P}(B))$, whereas M_Q is a direct summand of M which belongs to $\text{add}(\mathcal{Q}(B))$. We claim that $M_P = 0$ or $M_Q = 0$. Suppose $M_P \neq 0$ and $M_Q \neq 0$. Let M' be an indecomposable direct summand of M_P and M'' an indecomposable direct summand of M_Q.

Consider the case when B is a concealed algebra of Euclidean type, that is, $\mathcal{R}(B)$ is a family of stable tubes. Then it follows from Lemma 2.3 that there is a module Z in $\mathcal{R}(B)$ such that $\text{Hom}_B(M', \tau_B Z) \neq 0$ and $\text{Hom}_B(Z, M'') \neq 0$. This contradicts the assumption that M is not the middle of a short chain. Hence $M_P = 0$ or $M_Q = 0$.

Assume now that B is a wild concealed algebra. Fix a regular component $\mathcal{D}$ of Γ_B. Invoking Proposition 3.4 we conclude that there exists a module $X \in \mathcal{D}$ such that $\text{Hom}_B(M', \tau_B X) \neq 0$ and $\text{Hom}_B(X, M'') \neq 0$. Thus M is the middle of a short chain $X \to M \to \tau_B X$ in $\text{mod } B$. Hence, we get that $M_P = 0$ or $M_Q = 0$.

Therefore, we obtain that M belongs to $\text{add}(\mathcal{C})$ for a connecting component $\mathcal{C} = \mathcal{P}(B)$ or $\mathcal{C} = \mathcal{Q}(B)$. $\qquad\square$

We shall now be concerned with the situation of exactly one connecting component in the Auslander-Reiten quiver of a tilted algebra. Let H be an indecomposable hereditary algebra of infinite representation type, T a tilting module in $\text{mod } H$ and $B = \text{End}_H(T)$ the associated tilted algebra. By $\mathcal{C}_T$ we denote the connecting component in Γ_B determined by T. We keep these notations to formulate and prove the following statement.

PROPOSITION 4.2. *Let $B = \text{End}_H(T)$ be an indecomposable tilted algebra which is not concealed. If M is a sincere B-module which is not the middle of a short chain in $\text{mod } B$, then $M \in \text{add}(\mathcal{C}_T)$.*

Proof. We start with the general view on the module category $\text{mod } B$ due to results established in [14], [15], [16], [19], [36]. Let $\Delta = \Delta_T$ be the canonical section of the connecting component $\mathcal{C}_T$ of Γ_B determined by T. Hence, $\Delta = Q^{\text{op}}$ for $Q = Q_H$. Then $\mathcal{C}_T$ admits a finite (possibly empty) family of pairwise disjoint full translation (valued) subquivers

$$\mathcal{D}_1^{(l)}, ..., \mathcal{D}_m^{(l)}, \mathcal{D}_1^{(r)}, ..., \mathcal{D}_n^{(r)}$$

such that the following statements hold:

(a) for each $i \in \{1, ..., m\}$, there is an isomorphism of translation quivers $\mathcal{D}_i^{(l)} \cong \mathbb{N}\Delta_i^{(l)}$, where $\Delta_i^{(l)}$ is a connected full valued subquiver of Δ, and $\mathcal{D}_i^{(l)}$ is closed under predecessors in $\mathcal{C}_T$;

(b) for each $j \in \{1, ..., n\}$, there is an isomorphism of translation quivers $\mathcal{D}_j^{(r)} \cong (-\mathbb{N})\Delta_j^{(r)}$, where $\Delta_j^{(r)}$ is a connected full valued subquiver of Δ, and $\mathcal{D}_j^{(r)}$ is closed under successors in $\mathcal{C}_T$;

(c) all but finitely many indecomposable modules of $\mathcal{C}_T$ lie in
$$\mathcal{D}_1^{(l)} \cup ... \cup \mathcal{D}_m^{(l)} \cup \mathcal{D}_1^{(r)} \cup ... \cup \mathcal{D}_n^{(r)};$$

(d) for each $i \in \{1, ..., m\}$, there exists a tilted algebra $B_i^{(l)} = \mathrm{End}_{H_i^{(l)}}(T_i^{(l)})$, where $H_i^{(l)}$ is a hereditary algebra of type $(\Delta_i^{(l)})^{\mathrm{op}}$ and $T_i^{(l)}$ is a tilting $H_i^{(l)}$-module without preinjective indecomposable direct summands such that

- $B_i^{(l)}$ is a quotient algebra of B, and hence there is a fully faithful embedding $\mathrm{mod}\, B_i^{(l)} \hookrightarrow \mathrm{mod}\, B$,
- $\mathcal{D}_i^{(l)}$ coincides with the torsion-free part $\mathcal{Y}(T_i^{(l)}) \cap \mathcal{C}_{T_i^{(l)}}$ of the connecting component $\mathcal{C}_{T_i^{(l)}}$ of $\Gamma_{B_i^{(l)}}$ determined by $T_i^{(l)}$;

(e) for each $j \in \{1, ..., n\}$, there exists a tilted algebra $B_j^{(r)} = \mathrm{End}_{H_j^{(r)}}(T_j^{(r)})$, where $H_j^{(r)}$ is a hereditary algebra of type $(\Delta_j^{(r)})^{\mathrm{op}}$ and $T_j^{(r)}$ is a tilting $H_j^{(r)}$-module without preprojective indecomposable direct summands such that

- $B_j^{(r)}$ is a quotient algebra of B, and hence there is a fully faithful embedding $\mathrm{mod}\, B_j^{(r)} \hookrightarrow \mathrm{mod}\, B$,
- $\mathcal{D}_j^{(r)}$ coincides with the torsion part $\mathcal{X}(T_j^{(r)}) \cap \mathcal{C}_{T_j^{(r)}}$ of the connecting component $\mathcal{C}_{T_j^{(r)}}$ of $\Gamma_{B_j^{(r)}}$ determined by $T_j^{(r)}$;

(f) $\mathcal{Y}(T) = \mathrm{add}(\mathcal{Y}(T_1^{(l)}) \cup ... \cup \mathcal{Y}(T_m^{(l)}) \cup (\mathcal{Y}(T) \cap \mathcal{C}_T))$;

(g) $\mathcal{X}(T) = \mathrm{add}((\mathcal{X}(T) \cap \mathcal{C}_T) \cup \mathcal{X}(T_1^{(r)}) \cup ... \cup \mathcal{X}(T_n^{(r)}))$;

(h) the Auslander-Reiten quiver Γ_B has the disjoint union form
$$\Gamma_B = (\bigcup_{i=1}^{m} \mathcal{Y}\Gamma_{B_i^{(l)}}) \cup \mathcal{C}_T \cup (\bigcup_{j=1}^{n} \mathcal{X}\Gamma_{B_j^{(r)}}),$$

where

- for each $i \in \{1, ..., m\}$, $\mathcal{Y}\Gamma_{B_i^{(l)}}$ is the union of all components of $\Gamma_{B_i^{(l)}}$ contained entirely in $\mathcal{Y}(T_i^{(l)})$,
- for each $j \in \{1, ..., n\}$, $\mathcal{X}\Gamma_{B_j^{(r)}}$ is the union of all components of $\Gamma_{B_j^{(r)}}$ contained entirely in $\mathcal{X}(T_j^{(r)})$.

Moreover, we have the following description of the components of Γ_B contained in the parts $\mathcal{Y}\Gamma_{B_i^{(l)}}$ and $\mathcal{X}\Gamma_{B_j^{(r)}}$.

(1) If $\Delta_i^{(l)}$ is a Euclidean quiver, then $\mathcal{Y}\Gamma_{B_i^{(l)}}$ consists of a unique preprojective component $\mathcal{P}(B_i^{(l)})$ of $\Gamma_{B_i^{(l)}}$ and an infinite family $\mathcal{T}^{B_i^{(l)}}$ of pairwise orthogonal generalized standard ray tubes. Further, $\mathcal{P}(B_i^{(l)})$ coincides with the preprojective component $\mathcal{P}(C_i^{(l)})$ of a tame concealed quotient algebra $C_i^{(l)}$ of $B_i^{(l)}$.

(2) If $\Delta_i^{(l)}$ is a wild quiver, then $\mathcal{Y}\Gamma_{B_i^{(l)}}$ consists of a unique preprojective component $\mathcal{P}(B_i^{(l)})$ of $\Gamma_{B_i^{(l)}}$ and an infinite family of components obtained from the components of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of ray insertions. Further, $\mathcal{P}(B_i^{(l)})$ coincides with the preprojective component $\mathcal{P}(C_i^{(l)})$ of a wild concealed quotient algebra $C_i^{(l)}$ of $B_i^{(l)}$.

(3) If $\Delta_j^{(r)}$ is a Euclidean quiver, then $\mathcal{X}\Gamma_{B_j^{(r)}}$ consists of a unique preinjective component $\mathcal{Q}(B_j^{(r)})$ of $\Gamma_{B_j^{(r)}}$ and an infinite family $\mathcal{T}^{B_j^{(r)}}$ of pairwise orthogonal generalized standard coray tubes. Further, $\mathcal{Q}(B_j^{(r)})$ coincides with the preinjective component $\mathcal{Q}(C_j^{(r)})$ of a tame concealed quotient algebra $C_j^{(r)}$ of $B_j^{(r)}$.

(4) If $\Delta_j^{(r)}$ is a wild quiver, then $\mathcal{X}\Gamma_{B_j^{(r)}}$ consists of a unique preinjective component $\mathcal{Q}(B_j^{(r)})$ of $\Gamma_{B_j^{(r)}}$ and an infinite family of components obtained from the components of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of coray insertions. Further, $\mathcal{Q}(B_j^{(r)})$ coincides with the preinjective component $\mathcal{Q}(C_j^{(r)})$ of a wild concealed quotient algebra $C_j^{(r)}$ of $B_j^{(r)}$.

Observe that each indecomposable B-module belongs either to $\mathcal{Y}(T)$ or $\mathcal{X}(T)$ (T is a splitting tilting module). Let M' be an indecomposable direct summand of M, which is contained in $\mathcal{Y}(T)$. We claim that then M' belongs to $\mathcal{Y}(T) \cap \mathcal{C}_T$. Conversely, assume that $M' \in \mathcal{Y}(T) \backslash \mathcal{C}_T$. Then there exists $i \in \{1, ..., m\}$ such that $M' \in \mathcal{Y}\Gamma_{B_i^{(l)}} \backslash \mathcal{C}_T$, equivalently $M' \in \mathcal{Y}\Gamma_{B_i^{(l)}} \backslash \mathcal{C}_{T_i^{(l)}}$. Without loss of generality we may assume that $i = 1$. Since $T_1^{(l)}$ does not contain indecomposable preprojective direct summands, we may distinguish two cases.

Assume first that $T_1^{(l)}$ contains an indecomposable direct summand from $\mathcal{R}(H_1^{(l)})$. This implies that there is a projective module P in $\mathcal{Y}\Gamma_{B_1^{(l)}}$ which does not belong to $\mathcal{P}(B_1^{(l)})$. If $B_1^{(l)}$ is a tilted algebra of Euclidean type, then P is a module from some ray tube $\mathcal{T}$. Then, according to Lemma 2.2, $\mathrm{Hom}_{B_1^{(l)}}(M', X) = 0$ for any $X \in \mathcal{T}$, which leads to conclusion that $M' \notin \mathcal{P}(B_1^{(l)})$, because $\mathcal{T}$ belongs to the family $\mathcal{T}^{B_i^{(l)}}$ of ray tubes separating $\mathcal{P}(B_i^{(l)})$ from the preinjective component $\mathcal{Q}(B_i^{(l)})$ of $\Gamma_{B_i^{(l)}}$. Since M' does not belong to an infinite family of ray tubes (Lemmas 2.2 and 2.4), by (1) we conclude that $M' = 0$, a contradiction. This shows that any indecomposable direct summand of M from $\mathcal{Y}(T)$ is contained in $\mathcal{Y}(T) \cap \mathcal{C}_T$. If $B_1^{(l)}$ is a tilted algebra of wild type, then P belongs to a component obtained from a component of type $\mathbb{Z}\mathbb{A}_\infty$, say $\mathcal{D}$, by a positive number of ray insertions. Then there is a left cone $(\rightarrow N)$ in $\mathcal{D}$ which consists only of $C_1^{(l)}$-modules [15, Theorem 1]. Moreover, $\tau_{C_1^{(l)}} V = \tau_{B_1^{(l)}} V$ for any module $V \in (\rightarrow N)$ and there is an indecomposable module $Y \in \mathcal{R}(H_1^{(l)})$ such that $N = \mathrm{Hom}_{H_1^{(l)}}(T_1^{(l)}, Y)$. Because $M' \in \mathcal{Y}\Gamma_{B_1^{(l)}} \backslash \mathcal{C}_{T_1^{(l)}}$, we have also that $M' = \mathrm{Hom}_{H_1^{(l)}}(T_1^{(l)}, X)$ for some $X \in \mathcal{P}(H_1^{(l)}) \cup \mathcal{R}(H_1^{(l)})$.

Suppose that $X \in \mathcal{R}(H_1^{(l)})$. Invoking Theorem 3.5, we have that there exists a positive integer t such that $\mathrm{Hom}_{H_1^{(l)}}(X, \tau_{H_1^{(l)}}^p Y) \neq 0$ for all integers $p \geq t$. This implies that also $\mathrm{Hom}_{B_1^{(l)}}(M', \tau_{B_1^{(l)}}^p N) \neq 0$ for all integers $p \geq t$. Moreover,

if $X \in \mathcal{P}(H_1^{(l)})$, then $M' = \operatorname{Hom}_{H_1^{(l)}}(T_1^{(l)}, X)$ belongs to $\mathcal{P}(B_1^{(l)})$, which is equal to $\mathcal{P}(C_1^{(l)})$. From Proposition 3.4 we obtain now that there exists a positive integer t such that $\operatorname{Hom}_{B_1^{(l)}}(M', \tau_{B_1^{(l)}}^p N) \neq 0$ for all integers $p \geq t$. Thus we have, independently on the position of X in $\mathcal{P}(H_1^{(l)}) \cup \mathcal{R}(H_1^{(l)})$, a nonzero homomorphism $g : M' \to \tau_{B_1^{(l)}}^p N$, for any integer $p \geq t$. Observe also that M is a faithful B-module because M is sincere and not the middle of a short chain (see [23, Corollary 3.2]). Hence there is a monomorphism $B_B \to M^r$, for some positive integer r, so we have a monomorphism $P \to M^r$, because P is a direct summand of B_B. Further, since $\mathcal{D}$ contains a finite number of projective modules, we may assume, without loss of generality, that P is the one whose radical has an indecomposable direct summand L such that $\tau_{B_1^{(l)}}^s L \neq 0$ for any integer $s \geq 1$. Consider the infinite sectional path Σ in $\mathcal{D}$ which terminates at L. Then there exists an integer $p \geq t$ such that the infinite sectional path Ω which starts at $\tau_{B_1^{(l)}}^p N$ contains a module $\tau_{B_1^{(l)}} Z$ with Z lying on Σ. Then, $\operatorname{Hom}_{B_1^{(l)}}(Z, L) \neq 0$, by Lemma 2.5, and hence $\operatorname{Hom}_{B_1^{(l)}}(Z, M) \neq 0$, since there are monomorphisms $L \to P$ and $P \to M^r$ for some integer $r \geq 1$. Similarly, we obtain $\operatorname{Hom}_{B_1^{(l)}}(M', \tau_{B_1^{(l)}} Z) \neq 0$, because there are a nonzero homomorphism $g : M' \to \tau_{B_1^{(l)}}^p N$ and a monomorphism from $\tau_{B_1^{(l)}}^p N$ to $\tau_{B_1^{(l)}} Z$ being a composition of irreducible monomorphisms. Finally, we get a short chain $Z \to M \to \tau_{B_1^{(l)}} Z$ in $\operatorname{mod} B$, which contradicts the assumption imposed on M.

Assume now that $T_1^{(l)}$ belongs to $\operatorname{add}(\mathcal{P}(H_1^{(l)}))$. Then $B_1^{(l)}$ is a concealed algebra and $B \neq B_1^{(l)}$, since B is not concealed by the assumption. Therefore, since B is indecomposable, there exists a module $R \in \mathcal{Q}(B_1^{(l)})$, more precisely, a module $R \in \mathcal{Q}(B_1^{(l)}) \cap \mathcal{C}_T$ such that W is a direct summand of $\operatorname{rad} P$ of some projective B-module P. Moreover, by Lemmas 2.4 and 3.6, we obtain that $M' \in \mathcal{Y}\Gamma_{B_1^{(l)}} \setminus \mathcal{C}_{T_1^{(l)}}$ implies $M' \in \mathcal{P}(B_1^{(l)})$. We claim that there exists $Z \in \mathcal{R}(B_1^{(l)})$ such that $\operatorname{Hom}_{B_1^{(l)}}(Z, W) \neq 0$ and $\operatorname{Hom}_{B_1^{(l)}}(M', \tau_{B_1^{(l)}} Z) \neq 0$.

If $B_1^{(l)}$ is of Euclidean type then the claims follows from Lemma 2.3. Suppose now that $B_1^{(l)}$ is of wild type. Let $\mathcal{D}$ be a fixed component in $\mathcal{R}(B_1^{(l)})$. From Proposition 3.4 we know that there are nonzero homomorphisms $M' \to U$ for almost all $U \in \mathcal{D}$. Also by this proposition, for almost all $U \in \mathcal{D}$, there is a nonzero homomorphism $U \to W$. Thus, we conclude that there exists a regular $B_1^{(l)}$-module Z such that $\operatorname{Hom}_{B_1^{(l)}}(Z, W) \neq 0$ and $\operatorname{Hom}_{B_1^{(l)}}(M', \tau_{B_1^{(l)}} Z) \neq 0$. Combining now a nonzero homomorphism from Z to W with the composition of monomorphisms $W \to P$ and $P \to M^r$, for some integer $r \geq 1$, we obtain that $\operatorname{Hom}_{B_1^{(l)}}(Z, M) \neq 0$. Consequently, there is a short chain $Z \to M \to \tau_{B_1^{(l)}} Z$ in $\operatorname{mod} B$, a contradiction.

We use dual arguments to show that any indecomposable direct summand M'' of M, which is contained in $\mathcal{X}(T)$, belongs in fact to $\mathcal{X}(T) \cap \mathcal{C}_T$. $\qquad \square$

PROPOSITION 4.3. *Let $B = \operatorname{End}_H(T)$ be an indecomposable tilted algebra, $\mathcal{C}_T$ the connecting component of Γ_B determined by T, and M a sincere module in $\operatorname{add}(\mathcal{C}_T)$ which is not the middle of a short chain. Then there is a section Δ in $\mathcal{C}_T$ such that every indecomposable direct summand of M belongs to Δ.*

Proof. We divide the proof into several steps.

(1) Let M' be an indecomposable direct summand of M and R be an immediate predecessor of some projective module P in $\mathcal{C}_T$ (if $\mathcal{C}_T$ contains a projective module). We prove that, if M' is a predecessor of R in $\mathcal{C}_T$, then M' belongs to $\mathcal{S}_R$. Assume that M' is a predecessor of R in $\mathcal{C}_T$ and M' does not belong to $\mathcal{S}_R$. Since R has no injective nonsectional predecessors in $\mathcal{C}_T$, we have from Proposition 2.6 (i) that $\operatorname{Hom}_B(M', \tau_B U) \neq 0$ for some module $U \in \mathcal{S}_R$. Moreover, $\operatorname{Hom}_B(U, R) \neq 0$, because there is a sectional path from U to R in $\mathcal{C}_T$. Since M is faithful, there is a monomorphism $B_B \to M^r$, for some positive integer r, so we have a monomorphism $P \to M^r$, because P is a direct summand of B_B. Combining now a nonzero homomorphism from U to R with the composition of monomorphisms $R \to P$ and $P \to M^r$, we obtain $\operatorname{Hom}_B(U, M^r) \neq 0$, and hence $\operatorname{Hom}_B(U, M) \neq 0$. Summing up, we have in $\operatorname{mod} B$ a short chain $U \to M \to \tau_B U$, a contradiction.

Dually, using Proposition 2.6(ii) we show that, if an indecomposable direct summand M'' of M is a successor of an immediate successor J of some injective module I in $\mathcal{C}_T$, then M'' belongs to $\mathcal{S}_J^*$.

(2) Let M' and M'' be nonisomorphic indecomposable direct summands of M such that M' is a predecessor of M'' in $\mathcal{C}_T$. We show that every path from M' to M'' in $\mathcal{C}_T$ is sectional. Assume for the contrary that there exists a nonsectional path from M' to M'' in $\mathcal{C}_T$. For each nonsectional path σ in $\mathcal{C}_T$ from M' to M'', we denote by $n(\sigma)$ the length of the maximal sectional subpath of σ ending in M''. Among the nonsectional paths in $\mathcal{C}_T$ from M' to M'' we may choose a path γ with maximal $n(\gamma)$. Let $Y_0 \to Y_1 \to \cdots \to Y_{n-1} \to Y_n = M''$ be the maximal sectional subpath of γ ending in M''. Observe that then γ admits a subpath of the form $\tau_B Y_1 \to Y_0 \to Y_1$, and so Y_1 is not projective.

We show first that there is no sectional path in $\mathcal{C}_T$ from M' to Y_0. Note that there is no sectional path in $\mathcal{C}_T$ from M' to $\tau_B Y_1$. Indeed, otherwise $\operatorname{Hom}_B(M', \tau_B Y_1) \neq 0$ and clearly $\operatorname{Hom}_B(Y_1, M'') \neq 0$, since there is a sectional path from Y_1 to $Y_n = M''$, and consequently M is the middle of a short chain $Y_1 \to M \to \tau_B Y_1$, a contradiction. Moreover, applying (1), we conclude that Y_0 and $\tau_B Y_1$ are not projective. We claim that $\tau_B Y_1$ is a unique immediate predecessor of Y_0 in $\mathcal{C}_T$. Suppose that Y_0 admits an immediate predecessor L in $\mathcal{C}_T$ different from $\tau_B Y_1$. Since there is no sectional path in $\mathcal{C}_T$ from M' to $\tau_B Y_1$, we conclude that γ contains a subpath of the form

$$M' = N_0 \to N_1 \to \cdots \to N_s = \tau_B Z_1 \to Z_0 \to Z_1 \to \cdots \to Z_{t-1} \to Z_t = \tau_B Y_1.$$

Assume first that all modules $Z_2, \ldots, Z_{t-1}$ are nonprojective. Then there is in $\mathcal{C}_T$ a nonsectional path β from M' to M'' of the form

$$M' = N_0 \to N_1 \to \cdots \to \tau_B Z_1 \to \tau_B Z_2 \to \cdots \to \tau_B Z_t \to \tau_B Y_0 \to L \to$$

$$\to Y_0 \to Y_1 \to \cdots \to Y_n = M''$$

with $n(\beta) = n(\gamma) + 1$, a contradiction with the choice of γ. Assume now that one of the modules $Z_2, \ldots, Z_{t-1}$ is projective. Choose $k \in \{2, \ldots, t-1\}$ such that Z_k is projective but $Z_{k+1}, \ldots, Z_{t-1}, Z_t$ are nonprojective. Then $\tau_B Z_{k+1}$ is an immediate predecessor of Z_k in $\mathcal{C}_T$ and hence, applying (1), we infer that there is a sectional path in $\mathcal{C}_T$ from M' to $\tau_B Z_{k+1}$. We obtain then a nonsectional path α in $\mathcal{C}_T$ of the form

$$M' \to \cdots \to \tau_B Z_{k+1} \to \cdots \to \tau_B Z_t \to \tau_B Y_0 \to L \to Y_0 \to Y_1 \to \cdots \to Y_n = M''$$

with $n(\alpha) = n(\gamma) + 1$, again a contradiction with the choice of γ. Summing up, we proved that Y_0, Y_1 are nonprojective and $\tau_B Y_1$ is a unique immediate predecessor

of Y_0 in $\mathcal{C}_T$. Hence every sectional path in $\mathcal{C}_T$ from M' to Y_0 passes through $\tau_B Y_1$. This proves our claim, because there is no sectional path in $\mathcal{C}_T$ from M' to $\tau_B Y_1$.

Observe that $\operatorname{Hom}_B(Y_0, M) \neq 0$, since we have a sectional path in $\mathcal{C}_T$ from Y_0 to the direct summand M'' of M. Denote by f a nonzero homomorphism in $\operatorname{mod} B$ from Y_0 to M and consider a projective cover $g : P_B(Y_0) \to Y_0$ of Y_0 in $\operatorname{mod} B$. Then $fg \neq 0$ and hence there exist an indecomposable projective B-module P and nonzero homomorphism $h : P \to Y_0$ such that $fh \neq 0$. Applying (1) and Proposition 2.6 (ii), we conclude that h factorizes through a module in $\operatorname{add}(\tau_B^- \mathcal{S}_{M'}^*)$. Then there exists a module U in $\mathcal{S}_{M'}^*$ and a nonzero homomorphism $j : \tau_B^- U \to Y_0$ such that $fj \neq 0$. Moreover, $\operatorname{Hom}_B(M', U) \neq 0$ because there is a sectional path from M' to U in $\mathcal{C}_T$. Therefore, M is the middle of a short chain $\tau_B^- U \to M \to U$, with $U = \tau_B(\tau_B^- U)$, a contradiction.

(3) Let M' be an indecomposable direct summand of M which is a predecessor of an indecomposable projective module P in $\mathcal{C}_T$. Then every path in $\mathcal{C}_T$ from M' to P is sectional. Indeed, since M is a faithful module in $\operatorname{mod} B$, there is a monomorphism $B_B \to M^r$ in $\operatorname{mod} B$ for some positive integer r, and hence $\operatorname{Hom}_B(P, M'') \neq 0$ for an indecomposable direct summand M'' of M. Since $\mathcal{C}_T$ is a generalized standard component, we infer that then there is in $\mathcal{C}_T$ a path from P to M''. Therefore, any path in $\mathcal{C}_T$ from M' to P is a subpath of a path in $\mathcal{C}_T$ from M' to M'', and so is sectional, by (2).

(4) Let M'' be an indecomposable direct summand of M which is a successor of an indecomposable injective module I in $\mathcal{C}_T$. Then every path in $\mathcal{C}_T$ from I to M'' is sectional. This follows by arguments dual to those applied in (3).

We denote by Δ_T the section of $\mathcal{C}_T$ given by the images of a complete set of pairwise nonisomorphic indecomposable injective H-modules via the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$.

(5) Let $M_1, M_2, \ldots, M_t$ be a complete set of pairwise nonisomorphic indecomposable direct summands of M. We know that for a given module N in $\mathcal{C}_T$ there exists a unique integer r such that $\tau_B^r N \in \Delta_T$. Let $r_1, r_2, \ldots, r_t$ be the unique integers such that $\tau_B^{r_i} M_i \in \Delta_T$, for any $i \in \{1, \ldots, t\}$. Observe that the modules $\tau_B^{r_1} M_1, \tau_B^{r_2} M_2, \ldots, \tau_B^{r_t} M_t$ are pairwise different because, by (2), every path in $\mathcal{C}_T$ from M_i to M_j, with $i \neq j$ in $\{1, \ldots, t\}$, is sectional. We shall prove our claim by induction on the number $s(\Delta_T) = \sum_{i=1}^t |r_i|$.

Assume $s(\Delta_T) = 0$. Then, for any $i \in \{1, \ldots, t\}$, $M_i \in \Delta_T$ and there is nothing to show.

Assume $s(\Delta_T) \geq 1$. Fix $i \in \{1, \ldots, t\}$ with $|r_i| \neq 0$. Assume that $r_i > 0$, or equivalently, $M_i \in \mathcal{C}_T \cap \mathcal{X}(T)$. Denote by $\Sigma_T^{(i)}$ the set of all modules X in Δ_T such that there is a path in $\mathcal{C}_T$ of length greater than or equal to zero from X to $\tau_B^{r_i} M_i$. We note that every path from a module X in $\Sigma_T^{(i)}$ to $\tau_B^{r_i} M_i$ is sectional, because Δ_T is convex in $\mathcal{C}_T$ and intersects every τ_B-orbit in $\mathcal{C}_T$ exactly once. Further, by (2) and (4), no module in $\Sigma_T^{(i)}$ is a successor of a module M_j with $j \in \{1, \ldots, t\} \setminus \{i\}$ nor an indecomposable injective module, because there is a nonsectional path in $\mathcal{C}_T$ from $\tau_B^{r_i} M_i$ to M_i. Consider now the full subquiver $\Delta_T^{(i)}$ of $\mathcal{C}_T$ given by the modules from $\tau_B^-(\Sigma_T^{(i)})$ and $\Delta_T \setminus \Sigma_T^{(i)}$. Then $\Delta_T^{(i)}$ is a section of $\mathcal{C}_T$ and $s(\Delta_T^{(i)}) \leq s(\Delta_T) - 1$.

Assume now $r_i < 0$, or equivalently, $M_i \in \mathcal{C}_T \cap \mathcal{Y}(T)$. Denote by $\Omega_T^{(i)}$ the set

of all modules Y in Δ_T such that there is a path in $\mathcal{C}_T$ of length greater than or equal to zero from $\tau_B^{r_i} M_i$ to Y. It follows from (2) and (3) that no module in $\Omega_T^{(i)}$ is a predecessor of a module M_j with $j \in \{1, \ldots, t\} \setminus \{i\}$ nor an indecomposable projective module, because there is a nonsectional path in $\mathcal{C}_T$ from M_i to $\tau_B^{r_i} M_i$. Consider now the full subquiver $\Delta_T^{(i)}$ of $\mathcal{C}_T$ given by the modules from $\tau_B(\Omega_T^{(i)})$ and $\Delta_T \setminus \Omega_T^{(i)}$. Then $\Delta_T^{(i)}$ is a section of $\mathcal{C}_T$ and $s(\Delta_T^{(i)}) \leq s(\Delta_T) - 1$.

Summing up, we obtain that there is a section Δ in $\mathcal{C}_T$ containing all modules $M_1, M_2, \ldots, M_t$. $\qquad\square$

We complete now the proof of Theorem 1.1.

Let B be an indecomposable tilted algebra and M a sincere module in $\operatorname{mod} B$ which is not the middle of a short chain in $\operatorname{mod} B$. Applying Propositions 4.1 and 4.2, we conclude that there exists a hereditary algebra $\overline{H}$ and a tilting module $\overline{T}$ in $\operatorname{mod} \overline{H}$ such that $B = \operatorname{End}_{\overline{H}}(\overline{T})$ and M is isomorphic to a B-module $M_1^{n_1} \oplus \ldots \oplus M_t^{n_t}$ with $M_1, \ldots, M_t$ indecomposable modules in $\mathcal{C}_{\overline{T}}$, for some positive integers $n_1, \ldots, n_t$. Further, it follows from Proposition 4.3 that there is a section Δ in $\mathcal{C}_{\overline{T}}$ containing the modules $M_1, \ldots, M_t$. Denote by T_Δ^* the direct sum of all indecomposable B-modules lying on Δ. Then it follows from Theorem 3.3 that $H_\Delta = \operatorname{End}_B(T_\Delta^*)$ is an indecomposable hereditary algebra, $T_\Delta = D(T_\Delta^*)$ is a tilting module in $\operatorname{mod} H_\Delta$, and the tilted algebra $B_\Delta = \operatorname{End}_{H_\Delta}(T_\Delta)$ is the basic algebra of B. Let $H = H_\Delta$. Then there exists a tilting module T in the additive category $\operatorname{add}(T_\Delta)$ of T_Δ in $\operatorname{mod} H = \operatorname{mod} H_\Delta$ such that $B = \operatorname{End}_H(T)$, $\mathcal{C}_{\overline{T}}$ is the connecting component $\mathcal{C}_T$ of Γ_B determined by T, and Δ is the section Δ_T of $\mathcal{C}_T$ given by the images of a complete set of pairwise nonisomorphic indecomposable injective H-modules via the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$. Since $M_1, \ldots, M_t$ lie on $\Delta = \Delta_T$, we conclude that there is an injective module I in $\operatorname{mod} H$ such that the right B-modules $M = M_1^{n_1} \oplus \ldots \oplus M_t^{n_t}$ and $\operatorname{Hom}_H(T, I)$ are isomorphic. This finishes the proof of Theorem 1.1.

We provide now the proof of Corollary 1.2.

Let A be an algebra and M a module in $\operatorname{mod} A$ which is not the middle of a short chain. It follows from Theorem 1.1 that there exists a hereditary algebra H and a tilting module T in $\operatorname{mod} H$ such that the tilted algebra $B = \operatorname{End}_H(T)$ is a quotient algebra of A and M is isomorphic to the right B-module $\operatorname{Hom}_B(T, I)$. Further, the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$ induces an equivalence of the torsion part $\mathcal{T}(T)$ of $\operatorname{mod} H$ with the torsion-free part $\mathcal{Y}(T)$ of $\operatorname{mod} B$, and obviously I belongs to $\mathcal{T}(T)$. Then we obtain isomorphisms of algebras

$$\operatorname{End}_A(M) \cong \operatorname{End}_B(M) \cong \operatorname{End}_B(\operatorname{Hom}_H(T, I)) \cong \operatorname{End}_H(I).$$

Thus Corollary 1.2 follows from the following known characterization of hereditary algebras (see [17] for more general results in this direction).

PROPOSITION 4.4. *Let Λ be an algebra. The following conditions are equivalent:*

(i) Λ *is a hereditary algebra;*

(ii) $\operatorname{End}_\Lambda(P)$ *is a hereditary algebra for any projective module P in $\operatorname{mod} \Lambda$;*

(iii) $\operatorname{End}_\Lambda(I)$ *is a hereditary algebra for any injective module I in $\operatorname{mod} \Lambda$.*
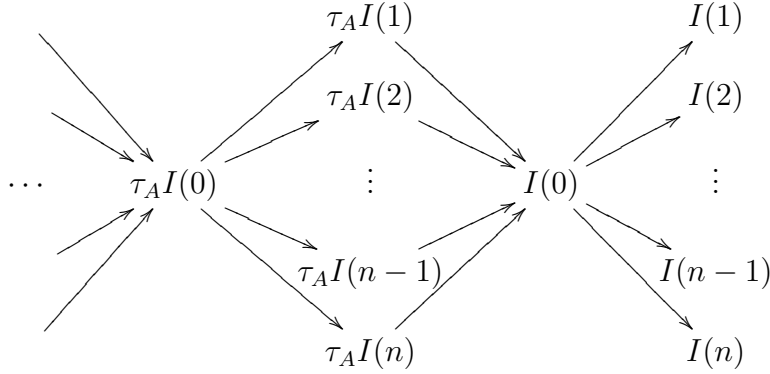
5. Examples

In this section we exhibit examples of modules which are not the middle of short chains, illustrating Theorem 1.1.

EXAMPLE 5.1   Let K be a field, n a positive integer, Q the quiver
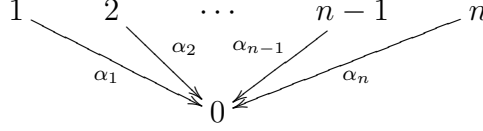
$$1 \quad 2 \quad \cdots \quad n-1 \quad n$$
$$\alpha_2 \quad \alpha_{n-1}$$
$$\alpha_1 \qquad \alpha_n$$
$$0$$

and $A = KQ$ the path algebra of Q over K. Then the Auslander-Reiten quiver Γ_A admits a unique preinjective component $\mathcal{Q}(A)$ whose right part is of the form

$$\tau_A I(1) \qquad I(1)$$
$$\tau_A I(2) \qquad I(2)$$
$$\cdots \quad \tau_A I(0) \qquad \vdots \qquad I(0) \qquad \vdots$$
$$\tau_A I(n-1) \qquad I(n-1)$$
$$\tau_A I(n) \qquad I(n)$$

where $I(0), I(1), I(2) \ldots, I(n-1), I(n)$ are the indecomposable injective right A-modules at the vertices $0, 1, 2, \ldots, n-1, n$, respectively. Consider the semisimple module $M = I(1) \oplus I(2) \oplus \ldots \oplus I(n-1) \oplus I(n)$ in mod A. Then M is not the middle of a short chain in mod A and $B = A/\mathrm{ann}_A(M)$ is the path algebra $K\Delta$ of the subquiver Δ of Q given by the vertices $1, 2, \ldots, n-1, n$, which is isomorphic to the product of n copies of K. Observe also that the injective modules $I(0), I(1), ..., I(n)$ form a section of $\mathcal{Q}(A)$.

EXAMPLE 5.2   Let K be a field and n be a positive integer. For each $i \in \{1, \ldots, n\}$, choose a basic indecomposable finite-dimensional hereditary K-algebra H_i, a multiplicity-free tilting module T_i in mod H_i, and consider the associated tilted algebra $B_i = \mathrm{End}_{H_i}(T_i)$, the connecting component $\mathcal{C}_{T_i}$ of Γ_{B_i} determined by T_i, and the module $M_{T_i} = \mathrm{Hom}_{H_i}(T_i, D(H_i))$ whose indecomposable direct summands form the canonical section Δ_{T_i} of $\mathcal{C}_{T_i}$. It follows from general theory ([14], [36]) that the Auslander-Reiten quiver Γ_{B_i} contains at least one preinjective component. Therefore, we may choose, for any $i \in \{1, \ldots, n\}$, a simple injective right B_i-module S_i lying in a preinjective component $\mathcal{Q}_i$ of Γ_{B_i}. Let $B = B_1 \times \ldots \times B_n$ and $S = S_1 \oplus \ldots \oplus S_n$. Then S is a finite-dimensional K-B-bimodule, and we may consider the one-point extension algebra

$$A = \begin{bmatrix} K & S \\ 0 & B \end{bmatrix} = \left\{ \begin{bmatrix} \lambda & s \\ 0 & b \end{bmatrix} \mid \lambda \in K, \ s \in S, \ b \in B \right\}.$$

Since S is a semisimple injective module in mod B, it follows from general theory (see [25, (2.5)] or [28, (XV.1)]) that, for any indecomposable module X in mod A which is not in mod B, its radical rad X coincides with the largest right B-submodule of

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X and belongs to the additive category $\mathrm{add}(S)$ of S. In particular, for any indecomposable module Z in $\mathrm{mod}\,B$, the almost split sequence in $\mathrm{mod}\,B$ with the right term Z is an almost split sequence in $\mathrm{mod}\,A$. Therefore, the Auslander-Reiten quiver Γ_A of A is obtained from the disjoint union of the Auslander-Reiten quiver $\Gamma_{B_1}, ..., \Gamma_{B_n}$ by glueing of the preinjective components $\mathcal{Q}_1, ..., \mathcal{Q}_n$ into a one component by the new indecomposable projective A-module P with $\mathrm{rad}\,P = S$ (and possibly adding new components). This implies that the right B-module

$$ M = M_{T_1} \oplus \ldots \oplus M_{T_n} $$

is not the middle of a short chain in $\mathrm{mod}\,A$. Moreover, since M_{T_i} is a faithful right B_i-module for any $i \in \{1, \ldots, n\}$, we conclude that $B = A/\mathrm{ann}_A(M)$.

EXAMPLE 5.3 Let K be a field, H be a basic indecomposable finite-dimensional hereditary K-algebra, T a multiplicity-free tilting module in $\mathrm{mod}\,H$, and $B = \mathrm{End}_H(T)$ the associated tilted algebra. For a positive integer $r \geq 2$, consider the r-fold trivial extension algebra

$$ T(B)^{(r)} = \left\{ \begin{bmatrix} b_1 & 0 & 0 & & & & \\ f_2 & b_2 & 0 & & & 0 & \\ 0 & f_3 & b_3 & & & & \\ & & & \ddots & \ddots & & \\ & 0 & & & f_{r-1} & b_{r-1} & 0 \\ & & & & 0 & f_1 & b_1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \\ b_1, \ldots, b_{r-1} \in B, \ f_1, \ldots, f_{r-1} \in D(B) \end{array} \right\} $$

of B. Then $T(B)^{(r)}$ is a basic indecomposable finite-dimensional selfinjective K-algebra which is isomorphic to the orbit algebra $\widehat{B}/(\nu_{\widehat{B}}^r)$ of the repetitive algebra $\widehat{B}$ of B with respect to the infinite cyclic group $(\nu_{\widehat{B}}^r)$ of automorphisms of $\widehat{B}$ generated by the r-th power of the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$. Moreover, we have the canonical Galois covering $F^{(r)} : \widehat{B} \to \widehat{B}/(\nu_{\widehat{B}}^r) = T(B)^{(r)}$ and the associated pushdown functor $F_\lambda^{(r)} : \mathrm{mod}\,\widehat{B} \to \mathrm{mod}\,T(B)^{(r)}$ is dense (see [35, Sections 6 and 7] for more details). We also note that $T(B)^{(r)}$ admits a quotient algebra $B_1 \times B_2 \times \ldots \times B_{r-1}$ with $B_i = B$ for any $i \in \{1, 2, \ldots, r-1\}$.

Fix a positive integer m and consider the selfinjective algebra $A_m = T(B)^{(4(m+1))}$. For each $i \in \{1, 2, \ldots, m\}$, consider the quotient algebra $B_{4i} = B$ of A_m and the right B_{4i}-module $M_{4i} = \mathrm{Hom}_H(T, D(B))$, being the direct sum of all indecomposable modules lying on the canonical section $\Delta_{4i} = \Delta_T$ of the connecting component $\mathcal{C}_{4i} = \mathcal{C}_T$ of $\Gamma_{B_{4i}}$ determined by T. Then, applying arguments as in [24, Section 2], we conclude that

$$ M = \bigoplus_{i=1}^{m} M_{4i} $$

is a module in $\mathrm{mod}\,A_m$ which is not the middle of a short chain and $A_m/\mathrm{ann}_{A_m}(M)$ is isomorphic to the product

$$ \prod_{i=1}^{m} B_{4i} $$

of m copies of the tilted algebra B.

References

1. I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.

2. M. Auslander and I. Reiten, Modules determined by their composition factors, *Illinois J. Math.* **29** (1985), 280-301.

3. M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge University Press, Cambridge Studies Advanced Math., vol. 36, Cambridge University Press, Cambridge, 1995.

4. D. Baer, Wild hereditary artin algebras and linear methods, *Manuscripta Math.* **55** (1986), 69-82.

5. Ø. Bakke, Some characterizations of tilted algebras, *Math. Scand.* **63** (1988), 43–50.

6. R. Bautista and S. O. Smalø, Nonexistent cycles, *Comm. Algebra* **11** (1983), 1755–1767.

7. K. Bongartz, *Tilted algebras*, in: Representations of Algebras, Lecture Notes in Math., vol. 903, Springer-Verlag, Berlin, Heidelberg, New York, 1981, 26–38.

8. S. Brenner and M. C. R. Butler, *Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors*, in: Representation Theory II, Lecture Notes in Math., vol. 832, Springer-Verlag, Berlin, Heidelberg, New York, 1980, 103–169.

9. D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Math. Soc. Lecture Note Series vol. 119, Cambridge University Press, Cambridge, 1988.

10. D. Happel and S. Liu, Module categories without short cycles are of finite type, *Proc. Amer. Math. Soc.* **120** (1994), 371–375.

11. D. Happel and C. M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* **274** (1982), 399–443.

12. D. Happel and C. M. Ringel, Directing projective modules, *Archiv. Math.* (Basel) **60** (1993), 237–246.

13. A. Jaworska, P. Malicki and A. Skowroński, Tilted algebras and short chains of modules, *Math. Z.*, in press.

14. O. Kerner, Tilting wild algebras, *J. London Math. Soc.* **39** (1989), 29–47.

15. O. Kerner, Stable components of wild tilted algebras, *J. Algebra* **142** (1991), 37-57.

16. O. Kerner, Wild tilted algebras revisited, *Colloq. Math.* **73** (1997), 67-81.

17. M. Kleiner, A. Skowroński and D. Zacharia, On endomorphism algebras with small homological dimensions, *J. Math. Soc. Japan* **54** (2002), 621–648.

18. S. Liu, Degrees of irreducible maps and the shapes of the Auslander-Reiten quivers, *J. London Math. Soc.* **45** (1992), 32–54.

19. S. Liu, The connected components of the Auslander-Reiten quiver of a tilted algebra, *J. Algebra* **161** (1993), 505-523.

20. S. Liu, Semi-stable components of an Auslander-Reiten quiver, *J. London Math. Soc.* **47** (1993), 405–416.

21. S. Liu, Tilted algebras and generalized standard Auslander-Reiten components, *Archiv. Math.* (Basel) **61** (1993), 12–19.

22. P. Malicki and A. Skowroński, *Algebras with separating Auslander-Reiten components*, in: Representations of Algebras and Related Topics, European Math. Soc. Series of Congress Reports, European Math. Soc. Publ. House, Zürich, 2011, 251–353.

23. I. Reiten, A. Skowroński, and S. O. Smalø, Short chains and short cycles of modules, *Proc. Amer. Math. Soc.* **117** (1993), 343-354.

24. I. Reiten, A. Skowroński, and S. O. Smalø, Short chains and regular components, *Proc. Amer. Math. Soc.* **117** (1993), 601-612.

25. C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math., vol. 1099, Springer-Verlag, Berlin, Heidelberg, New York, 1984.

26. C. M. Ringel, *Representation theory of finite dimensional algebras*, in: Representations of Algebras, London Math. Soc. Lecture Note Series, vol. 116, Cambridge University Press, Cambridge, 1986, 7-79.

27. D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 2: Tubes and Concealed Algebras of Euclidean Type*, London Mathematical Society Student Texts, vol. 71, Cambridge University Press, Cambridge, 2007.

28. D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 3: Representation-Infinite Tilted Algebras*, London Mathematical Society Student Texts, vol. 72, Cambridge University Press, Cambridge, 2007.

29. A. Skowroński, Generalized standard Auslander-Reiten components without oriented cycles, *Osaka J. Math.* **30** (1993), 515–527.

30. A. Skowroński, Generalized standard Auslander-Reiten components, *J. Math. Soc. Japan* **46** (1994), 517–543.

31. A. Skowroński, On the composition factors of periodic modules, *J. London Math. Soc.* **49** (1994), 477-492.

32. A. Skowroński, Regular Auslander-Reiten components containing directing modules, *Proc. Amer. Math. Soc.* **120** (1994), 19–26.

33. A. Skowroński, Directing modules and double tilted algebras, *Bull. Polish Acad. Sci. Mathematics* **50** (2002), 77–87.

34. A. Skowroński and M. Wenderlich, Artin algebras with directing indecomposable projective modules, *J. Algebra* **165** (1994), 507–530.

35. A. Skowroński and K. Yamagata, *Selfinjective algebras of quasitilted type*, in: Trends in Representation Theory of Algebras and Related Topics, European Math. Soc. Series of Congress Reports, European Math. Soc. Publ. House, Zürich, 2008, 639-708.

36. H. Strauss, On the perpendicular category of a partial tilting module, *J. Algebra* **144** (1991), 43-66.

37. Y. Zhang, The structure of stable components, *Canad. J. Math.* **43** (1991), 652–672.

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