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ON NONDEFINABILITY OF INTERIOR-CONNECTEDNESS VIA THE CONTACT RELATION

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ABSTRACT. This short paper is a small contribution to the field of Boolean Contact Algebras. We analyze the non-definability of the property of *interior-connectedness*, and we prove certain minimality conditions for algebras and spaces that can be used in demonstrating that the aforementioned property cannot be expressed by means of contact within regular closed algebras.

Keywords: Boolean Contact Algebras, binary relations, non-definability, Padoa's method, interior-connectedness, internal connectedness, spatial logic

MSC 2020: 03G05, 03C07

1. INTRODUCTION

Boolean contact algebras (BCAs) are an algebraic framework for formalizing spatial relations between objects that are often thought of as models of natural phenomena, such as bodies or regions, understood as chunks of space. It has been proved that they are essentially expansions of algebras of regular closed (or open) subsets of topological spaces (see [4]; Düntsch and Winter [5]) and that their language is strong enough to express various topological properties such as connectedness, weak regularity, or clopenness (see Bennett and Düntsch [1]).

Recall that a BCA is a pair $\mathfrak{B} \coloneqq \langle B, \mathsf{C}\rangle$, where B is a Boolean algebra and C is a binary *contact* relation on B that satisfies the following five axioms (where $+$ is the join operation, $\leq$ is the standard Boolean order, and $\not\mathsf{C}$ is the complement of C):

(C0) $$(\forall x \in B)\, \mathbf{0} \not\mathsf{C}\, x,$$

(C1) $$(\forall x \in B)\, (x \neq \mathbf{0} \rightarrow x \mathrel{\mathsf{C}} x),$$

(C2) $$(\forall x, y \in B)\, (x \mathrel{\mathsf{C}} y \rightarrow y \mathrel{\mathsf{C}} x),$$

(C3) $$(\forall x, y, z \in B)\, (x \mathrel{\mathsf{C}} y \wedge y \leq z \rightarrow x \mathrel{\mathsf{C}} z),$$

(C4) $$(\forall x, y, z \in B)\, (x \mathrel{\mathsf{C}} y + x \rightarrow x \mathrel{\mathsf{C}} y \vee x \mathrel{\mathsf{C}} z).$$

These form the basic axioms for BCAs that are often extended with additional constraints.

Let $\langle X, \tau\rangle$ be a topological space, where τ is a family of open sets in X. For $A \subseteq X$, $\overline{A}$ is its closure, and $\operatorname{Int} A$ is its interior. A is a *regular closed* subset of X if it is a fixed point of the composition of closure and interior operations: $A = \overline{\operatorname{Int} A}$. Accordingly, it is *regular open* if it is a fixed point of the composition of interior and closure: $A = \operatorname{Int} \overline{A}$. Both compositions are called *regularization* operations. $\mathrm{RC}(\tau)$ (resp. $\mathrm{RO}(\tau)$) is the complete algebra of regular closed (resp. regular open) subsets of X. The meet and complement of $\mathrm{RC}(\tau)$ are defined as, respectively, the regularization of the intersection and the closure of the complement in the power set algebra 2^X

$$A \cdot D \coloneqq \overline{\operatorname{Int}(A \cap D)} \qquad -A \coloneqq \overline{X \setminus A},$$

while the join is the standard set-theoretical union.

The canonical interpretation of BCAs treats $\mathfrak{B}$ as a subalgebra of $\mathrm{RC}(\tau)$, with the contact relation interpreted in the following way:

$$A \mathrel{\mathsf{C}_\tau} D \ :\longleftrightarrow\ A \cap D \neq \emptyset.$$

On the right, we have the set-theoretical intersection, which—in general—is different from the meet operation of regular closed algebras. As is well known, the contact operation lets us differentiate between two situations: when two regions are disjoint and «touch» each other, and when two regions are disjoint and «separated», the distinction which is beyond the expressive power of pure Boolean algebras. Thus, regular closed sets A and B are in contact when they share at least one point, though they may be disjoint in the sense of the meet operation of $\mathrm{RC}(\tau)$.

Every nondegenerate (i.e., having at least two elements) Boolean algebra B carries two extreme contact relations, the *minimal* that is just the overlap relation

$$x \mathrel{\mathsf{C}} y \ :\longleftrightarrow\ x \cdot y \neq \mathbf{0},$$

and the *maximal*, where any two nonzero regions are in contact

$$x \mathrel{\mathsf{C}} y \ :\longleftrightarrow\ x \neq \mathbf{0} \neq y.$$

Two contact algebras $\mathfrak{B}_1 \coloneqq \langle B_1, \mathsf{C}_1 \rangle$ and $\mathfrak{B}_2 \coloneqq \langle B_2, \mathsf{C}_2 \rangle$ are *isomorphic* if there exists a Boolean isomorphism $f\colon B_1 \to B_2$ that preserves and reflects the contact relation

$$x \mathrel{\mathsf{C}_1} y \longleftrightarrow h(x) \mathrel{\mathsf{C}_2} h(y).$$

Such an isomorphism is called a *BCA-isomorphism*.

The representation theorems from [5] and [4] show that all BCAs are (isomorphic to) subalgebras of regular closed (equivalently: open) subsets of certain topological spaces. Goldblatt and Grice [6] have shown that the category of BCAs with contact reflecting Boolean homomorphisms is dually equivalent to the category of mereotopological spaces and mereomorphisms.

Since the dawn of modern region-based investigations into the features of space, their practitioners have delved into the problem of the expressive power of languages with the binary predicate C (Pratt and Schoop [14]; Bennett and Düntsch [1]; Vakarelov [17]). Parallel and complementary investigations (Cohn et al. [3]; Pratt and Lemon [13]; Kontchakov et al. [11]; Pratt and Schoop [14]; Pratt-Hartmann [15]; Kontchakov et al. [12]) tackled the problem of languages whose signature contains unary predicates c and c° (either the former or both) interpreted topologically in the following way:

$$\begin{aligned} c(x) \quad &\text{iff} \quad x \text{ is a connected space,}\\ c^\circ(x) \quad &\text{iff} \quad \text{the interior of } x \text{ is a connected space.}\end{aligned}$$

The second property bears the name of *interior-connectedness* (or *internal connectedness*, as in the case of Ivanova's paper [9]). Below, we will use the former. Specifically, we say that a subset A of a topological space $\langle X, \tau \rangle$ is *interior-connected* if its interior is a connected subspace of X.

Connectedness can be fairly easily defined by means of the contact in the setting of BCAs via the following constraint

$$c(x) \ :\longleftrightarrow\ \big(\forall y, z \in B \setminus \{\mathbf{0}\}\big)\,(x = y + z \to y \mathrel{\mathsf{C}} z).$$

In effect, in the regular closed algebras, for any topological space $\langle X, \tau \rangle$ and any regular closed subset A of X we obtain that

$$\mathrm{RC}(\tau) \vDash c(A) \qquad \text{iff} \qquad A \text{ is connected.}$$

However, it is natural that the expressive power of the language of BCAs must be topologically limited and that although many topological properties can be captured with the contact relation, there will be such that cannot be expressed in terms of it.

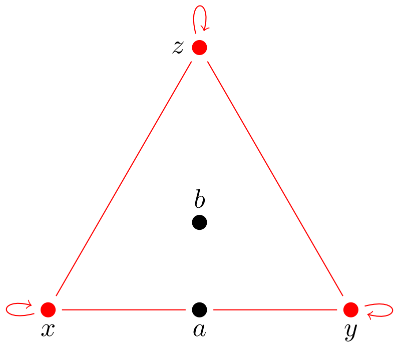

Figure 1. The five-point space.

Interior-connectedness serves as an example, as was proved by Ivanova [9, Proposition 2.1]. In her proof, Ivanova used Padoa's method (see, e.g., Beth [2]) and produced isomorphic contact algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$ such that the first has a regular closed set A whose interior is connected but whose counterpart in $\mathfrak{B}_2$ does not enjoy this property. The construction is based on a topological space with seven points whose topology has certain symmetries. It was our impression that the phenomenon of nondefinability of the property of interior-connectedness can be analyzed in a graph-based setting by means of shapes and colors that enhance topological intuition and understanding of the proof. In effect, we managed to simplify the proof slightly and show that the proof must be minimal in the sense of the cardinalities of both the spaces involved and their regular closed algebras.

Having said that, we want to emphasize that we do not use the graph topology (see Hatcher [8]) in our investigations. The graphs with which we represent spaces are heuristic devices, yet they are not deprived of precise meaning that we of course explain.

In the sequel, we present our findings.

2. The Proof

Let us consider a graph with five elements $T \coloneqq \{x, y, z, a, b\}$, where x, y, and z are distinguished nodes and a and b are non-distinguished nodes, and having seven edges (including three loops), as in Figure 1. The topology τ we are going to consider is generated by the subbasis composed of the three sides of the triangle: $\{x, z\}$, $\{y, z\}$, $\{x, a, y\}$, and of the whole space T. Thus

$$\tau = 2^{\{x,y,z\}} \cup \big\{\{x, a, y\}, \{x, a, y, z\}\big\} \cup \{T\},$$

which, among others, means that there are eleven open sets in τ. Let us call *distinguished* those sets all of whose elements belong to $\{x, y, z\}$. We ask the reader to observe that any open set in T is composed of points that are located on a path that begins with a distinguished node and ends with a distinguished node (possibly the same). Clearly, b is not an element of any open proper subset of T, so the space is connected, and $\emptyset$ and T are its only clopen subsets. The two open proper subsets of T that are not distinguished are connected subspaces of T. T is clearly T_0 (but not T_1, due to b again).

Let us determine the family $\mathrm{RO}(\tau)$ of the regular open subsets of T. Firstly, the operation of closure adds only non-distinguished nodes to distinguished singletons, and so every such singleton is regular open. The closure of $\{x, z\}$ consumes the two non-distinguished nodes, so its interior is again $\{x, y\}$. Symmetrically, $\{z, y\}$ is regular open too. For $\{x, a, y\}$, b is its only limit point beyond the set, so again $\{x, a, y\}$ is regular open. Thus, we have identified six elements of $\mathrm{RO}(\tau)$, and since the algebra cannot have more than eight elements (recall that all in all there are eleven open sets) we have that

$$\mathrm{RO}(\tau) = \big\{\emptyset, \{x\}, \{y\}, \{z\}, \{x, z\}, \{y, z\}, \{x, a, y\}, T\big\}.$$

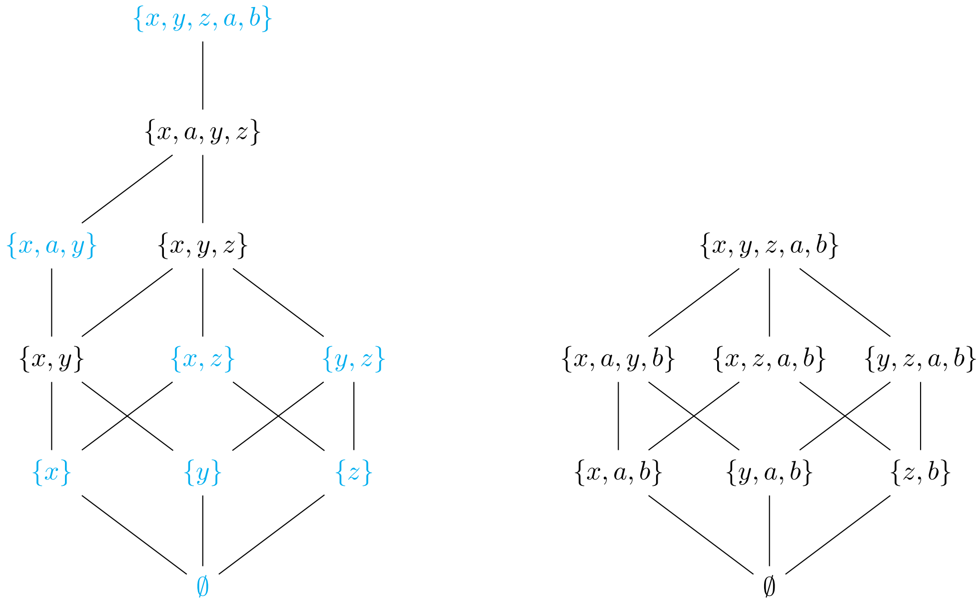

FIGURE 2. On the left, we have the frame of open subsets of T with its regular elements being: the zero and the unity, all the singletons, $\{x, z\}$, $\{y, z\}$, and $\{x, a, y\}$; on the right, we have the Boolean algebra of regular closed subsets of this space.

In consequence

$$\mathrm{RC}(\tau) = \big\{\emptyset, \{x, a, b\}, \{y, a, b\}, \{z, b\}, \{x, z, a, b\}, \{y, z, a, b\}, \{x, a, y, b\}, T\big\}.$$

Observe that any pair of nonempty sets in $\mathrm{RC}(\tau)$ shares the point b, which means that the topological contact on the algebra is its maximal contact relation, and the space T is ultraconnected (see Steen and Seebach [16, p. 29]). Figure 2 presents diagrammatically both the frame of opens[1] of T and the Boolean algebra $\mathrm{RC}(\tau)$.

The regular closed set $R \coloneqq \{x, y, a, b\}$ (see Figure 3) is interior-connected, as its interior $\{x, a, y\}$—being an open set that has a non-distinguished point—is a connected subspace of T.

In the next step, we modify the space by deleting the non-distinguished point a. Let $\langle T \setminus \{a\}, \tau' \rangle$ be a subspace of T.

It is obvious that for any set $S \in \mathrm{RC}(\tau)$, $S \setminus \{a\} \in \mathrm{RC}(\tau')$. Moreover, this transformation does not affect the point b, and so $\mathrm{RC}(\tau')$ has the maximal contact relation. Thus the mapping $f\colon \mathrm{RC}(\tau) \to \mathrm{RC}(\tau')$ such that $f(S) \coloneqq S \setminus \{a\}$ is a BCA-isomorphism. Observe that for R, $f(R) = R \setminus \{a\}$ is not interior-connected. Indeed, $\mathrm{Int}_{\tau'} f(R) = \{x, y\}$, and it is a discrete subspace of $T \setminus \{a\}$ (see Figures 4 and 5).

Since we have the maximal contact relation in both cases, Ivanova proves something a bit stronger, that is, the following theorem.

[1]Recall that in point-free topology, the frame of opens of a topological space is its family of open sets viewed as an algebraic structure. Basically, it is a complete Heyting algebra (see Johnstone [10]).

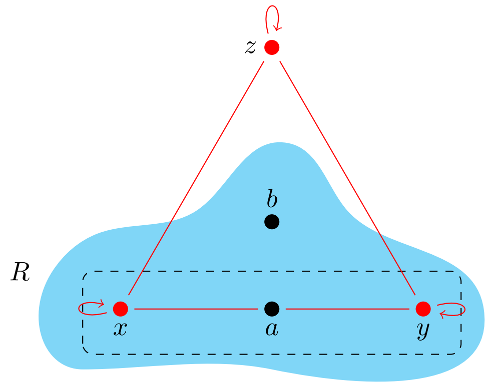

FIGURE 3. R is an interior-connected element of $\mathrm{RC}(\tau)$.

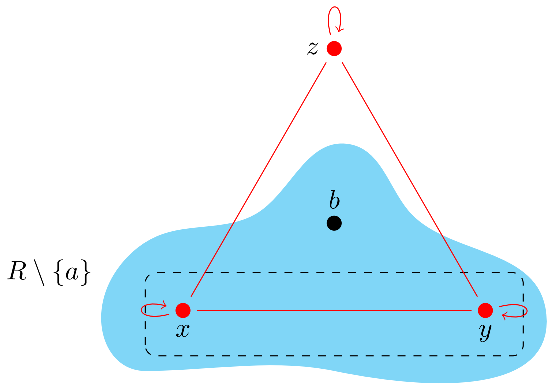

FIGURE 4. $\mathrm{Int}_{\tau'}(R \setminus \{a\})$ is a discrete subspace of $T \setminus \{a\}$, so $R \setminus \{a\}$ cannot be interior-connected.

Theorem 2.1. *The property of* interior-connectedness *is not definable even in the presence of the connectedness axiom for the contact relation*

$$\big(\forall x \in B \setminus \{\mathbf{0}, \mathbf{1}\}\big)\, x \mathrel{\mathsf{C}} -x.$$[2]

Observe as well that since the algebras used in the proof are finite, restricting the class to complete BCAs or even to complete BCAs satisfying the strong version of (C4) from Gruszczyński and Menchón [7]

$$x \mathrel{\mathsf{C}} \bigvee_{i \in I} x_i \rightarrow (\exists i \in I)\, x \mathrel{\mathsf{C}} x_i$$

has no influence on the status of definability of the interior-connectedness property.

3. The Number of Elements of the Algebra

Let $\langle X_1, \tau_1 \rangle$ and $\langle X_2, \tau_2 \rangle$ be topological spaces, and let $f\colon \mathrm{RC}(\tau_1) \to \mathrm{RC}(\tau_2)$ be a BCA-isomorphism. We say that f *preserves interior-connectedness* just in case, for all $b \in \mathrm{RC}(\tau_1)$, b has a connected interior in X_1 if and only if $f(b)$ has a connected interior in X_2.

We will show that the proof of Ivanova is the simplest possible in the following sense.

[2]Any BCA satisfies this axiom if and only if its corresponding topological space is connected, hence its name (see Bennett and Düntsch [1]).

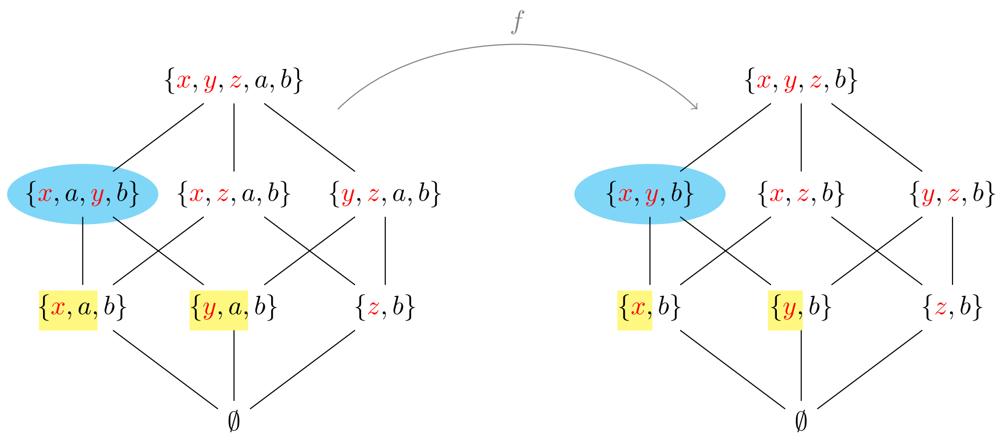

FIGURE 5. A pictorial presentation of the isomorphism $f\colon \mathrm{RC}(\tau) \to \mathrm{RC}(\tau')$. The highlighted rectangular chunks of regions form interiors of the elements enclosed by an ellipse.

Theorem 3.1. *Let $\langle X_1, \tau_1\rangle$ and $\langle X_2, \tau_2\rangle$ be topological spaces. If f is a BCA-isomorphism of the contact algebras $\mathfrak{B}_1 := \langle \mathrm{RC}(\tau_1), \mathsf{C}_{\tau_1}\rangle$ and $\mathfrak{B}_2 := \langle \mathrm{RC}(\tau_2), \mathsf{C}_{\tau_2}\rangle$, but f does not preserve interior-connectedness, then both algebras must have at least eight elements.*

Proof. The algebras cannot have two elements each since, in this case, their unities are the whole topological spaces X_1 and X_2. For the whole space, interior-connectedness coincides with mere connectedness. So, it would have to be the case that X_1 is connected and X_2 is not (or vice versa). But in such a situation, X_2 must be decomposable into two clopen nonempty sets, and so $\mathfrak{B}_2$ must have at least four elements (as every clopen set is regular closed).

Suppose that there exist two isomorphic contact algebras $\langle \mathrm{RC}(\tau_1), \mathsf{C}_{\tau_1}\rangle$ and $\langle \mathrm{RC}(\tau_2), \mathsf{C}_{\tau_2}\rangle$ with only four elements that are isomorphic and for which there is an BCA-isomorphism $f\colon \mathrm{RC}(\tau_1) \to \mathrm{RC}(\tau_2)$ such that there exists a set $A \in \mathrm{RC}(\tau_1)$ which is interior-connected in X_1, but $f(A)$ (the image of A with respect to f) is not interior-connected in X_2, that is, $\mathrm{Int}_{\tau_2} f(A)$ is a disconnected subspace of X_2. Thus, there exist two nonempty open sets D', E' of the subspace $\mathrm{Int}_{\tau_2} f(A)$ such that

$$\mathrm{Int}_{\tau_2} f(A) = D' \cup E' \qquad \text{and} \qquad D' \cap E' = \emptyset.$$

Since $\mathrm{Int}_{\tau_2} f(A)$ is open in X_2, D' and E' are also open in X_2. It follows that

$$f(A) = \overline{\mathrm{Int}_{\tau_2} f(A)} = \overline{D' \cup E'} = \overline{D'} \cup \overline{E'},$$

where the closure operation is the operation of the space X_2. Being the closures of nonempty open sets, $\overline{D'}$ and $\overline{E'}$ are nonempty regular closed sets. Since $D' \cap E' = \emptyset$ and D' and E' are open sets, it follows that $\overline{D'} \neq \overline{E'}$. As $\mathfrak{B}_2$ has only four elements, it must be the case that $f(A) = X_2$, and so (†) $A = X_1$.

Observe that $\overline{D'} \cap \overline{E'} = \emptyset$. Suppose toward a contradiction that there exists $x \in \overline{D'} \cap \overline{E'}$. Since X_2 is the union $D' \cup E'$, x is in one of these sets, and without loss of generality, we can assume that $x \in D'$. Since D' is an open set and $x \in \overline{E'}$, it follows that $D' \cap E' \neq \emptyset$, a contradiction. Therefore, $\overline{D'}$ is not in contact with $\overline{E'}$.

Let $D, E \in \mathrm{RC}(\tau_1)$ be such that $f(D) = \overline{D'}$ and $f(E) = \overline{E'}$. By assumption, D is not in contact with E, that is, $D \cap E = \emptyset$. The algebra $\mathrm{RC}(\tau_1)$ has only four elements, so $X_1 = D \cup E$, which means that X_1 is disconnected, and so its interior is disconnected, too (since this is the whole space). But this contradicts (†). □

4. The Number of Points

In this section, we will prove the following minimality theorem.

Theorem 4.1. *Let $\langle X_1, \tau_1\rangle$ and $\langle X_2, \tau_2\rangle$ be topological spaces. If f is a BCA-isomorphism of the algebras $\mathfrak{B}_1 := \langle \mathrm{RC}(\tau_1), \mathsf{C}_{\tau_1}\rangle$ and $\mathfrak{B}_2 := \langle \mathrm{RC}(\tau_2), \mathsf{C}_{\tau_2}\rangle$, but f does not preserve interior-connectedness, then the minimal number of points of the underlying spaces is five for one of the spaces and four for the other.*

In this way, we will show that Ivanova's construction reduced to a five-point space is the minimal possible construction, not only in the sense of the cardinality of algebras but also in the sense of the cardinality of sets of points of the underlying spaces. The following lemma that stems from Theorem 3.1 will be crucial.

Lemma 4.2. *If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ and f are like in Theorem 4.1, then each of the underlying topological spaces must have at least four points.*

Proof. Assume we have isomorphic BCAs $\mathfrak{B}_1$ and $\mathfrak{B}_2$ that have different properties of interior-connectedness. Let $f\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ be their BCA-isomorphism. Suppose the first is the algebra of regular closed sets of a space $\langle X_1, \tau_1\rangle$ whose domain has precisely three points: $\mathfrak{B}_1 := \langle \mathrm{RC}(\tau_1), \mathsf{C}_1\rangle$. In light of Theorem 3.1, it must be the case that $\mathrm{RC}(\tau_1) = 2^{X_1}$, so all subsets of X_1 are closed, and X_1 is a discrete space. This also means that $\mathrm{RC}(\tau_1)$ has the minimal contact relation. In this case, the only nonempty interior-connected subsets of X_1 are the singletons.

Suppose $\langle X_2, \tau_2\rangle$ is a topological space whose regular closed algebra $\mathfrak{B}_2 := \langle \mathrm{RC}(\tau_2), \mathsf{C}_2\rangle$ is isomorphic to $\mathfrak{B}_1$, that is, $\mathrm{RC}(\tau_2)$ has exactly eight elements. Since f is an isomorphism of contact algebras, we have that

$$A \cap D \neq \emptyset \qquad \text{iff} \qquad f(A)\ \mathsf{C}_2\ f(D).$$

Let $\cdot$ be the operation of meet of $\mathrm{RC}(\tau_2)$. Then we have that $f(A)\ \mathsf{C}_2\ f(D)$ entails that $A \cap D \neq \emptyset$, so $f(A) \cdot f(D) \neq \emptyset$, as f is a Boolean homomorphism. This implies that the contact relation of $\mathrm{RC}(\tau_2)$ must also be minimal. But this means that $\mathrm{RC}(\tau_2)$ coincides with the algebra of clopen subsets of the space X_2. So, its atoms are the only connected components of the space that are also interior-connected. Therefore, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have the same notions of interior-connectedness, which is a contradiction. So, both spaces must have at least four points each. This ends the proof. □

We can see that the constructions so far are almost minimal, as we have a BCA-isomorphism between two eight-element regular closed algebras, where the first one is based on the space with five points, and the second on the space of four points. We will show that we cannot reduce the cardinality of the first space to four points, and in this way we will prove Theorem 4.1.

Firstly, let us observe that in the case we begin with the regular closed algebra of a four-point space $\langle X, \tau\rangle$ we cannot consider the regular contact algebra of sixteen elements. This is because in such a situation we would have that $\mathrm{RC}(\tau) = 2^X$, the space is discrete, and the only nonempty interior-connected sets are the singletons. So any isomorphic algebra of another four-point space would also have to be the whole power set algebra, and would have the same topological properties as $\mathrm{RC}(\tau)$. Thus, we may focus on eight-element algebras of four-point spaces.

If we want to construct a contact relation on the algebra of eight elements, we have three possibilities for the atoms:

1. all three atoms are pairwise in contact,
2. one pair of atoms is in contact,
3. contact is empty on atoms.

We will consider the four-element set $S := \{p, q, r, s\}$ and we analyze the three above-mentioned cases on S as the domain of spaces. Since the algebras we have in mind are regular closed algebras, the contact must be defined by set-theoretical intersection.

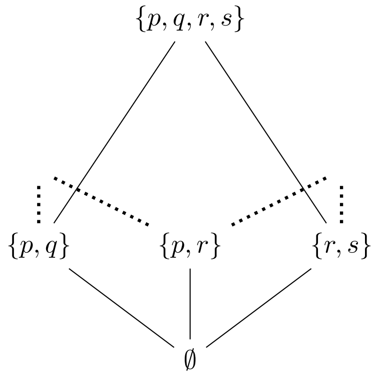

FIGURE 6. An impossible configuration of atoms on the four-point space S: atoms $\{p, q\}$ and $\{r, s\}$ add up to S.

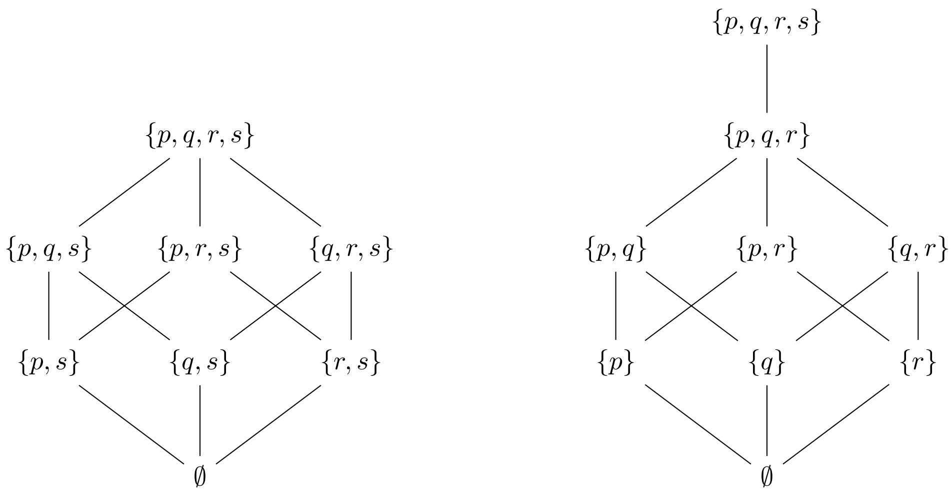

FIGURE 7. The eight-element regular closed algebra with full contact and determined by it the frame of open subsets of S. We can see that S is—up to homeomorphism—the space $T \setminus \{a\}$ in Figure 4.

Observe that it indeed cannot be the case that exactly two pairs of atoms are in contact, since then there would have to be a pair of atoms that add up to the whole space, as in Figure 6, which is impossible.

In the first case, the only possibility for three atoms to be pairwise in topological contact is that they have a point in common; let it be s, as in Figure 7. In the same figure, the diagram on the right represents the frame structure of the topology τ determined by the algebra. From it, we can see that $\langle S, \tau \rangle$ is—up to homeomorphism—precisely the space $T \setminus \{a\}$ in Figure 4, whose only nonempty interior-connected subsets are atoms and the unity. By Lemma 4.2, the isomorphic image of $\mathrm{RC}(\tau)$—if we limit ourselves to only at most four-point spaces—must have the maximal contact, and so the contact must hold among all pairs of atoms. Then, it will again be homeomorphic to $T \setminus \{a\}$, and so will have no different interior-connected regular closed sets, and thus there is no way to map any of the interior-connected sets to a set that does not have the property. In consequence, in the case of all three atoms in contact, at least one of the spaces involved must have at least five points.

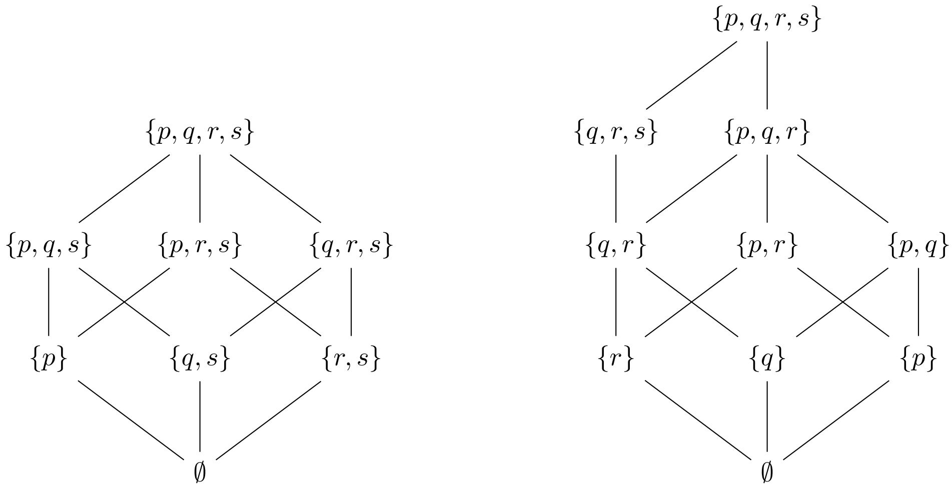

FIGURE 8. The regular closed algebra of S with one pair of atoms in contact, $\{q, s\}$ and $\{r, s\}$; and the frame of opens determined by the algebra. We can see that $\{q, r, s\}$ is an interior-connected clopen set.

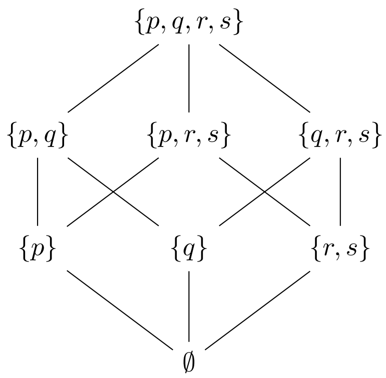

FIGURE 9. The regular closed algebra on the four-point set $\{p, q, r, s\}$ with all atoms separated. Routine verification shows that $\mathrm{RC}(\tau) = \mathrm{RO}(\tau) = \tau$, so this is the algebra of clopen subsets of the space.

In the second case, where only one pair of atoms is in contact, that is, the situation in Figure 8 holds. Here, we have only one interior-connected set that is neither the empty set, nor an atom, nor the whole space, that is, $\{q, r, s\}$ (interestingly, this set is clopen). As—up to isomorphism—this is the only regular contact algebra in which only one pair of atoms is in contact, the only possibility to «spoil» interior-connectedness of $\{q, r, s\}$ is via an automorphism. Since such an automorphism must, in particular, preserve contact on atoms, the only two possibilities are the identity relation or the function f that swaps $\{q, s\}$ for $\{r, s\}$ (and vice versa), and sends $\{p\}$ to $\{p\}$. However, the second possibility cannot hold, since if we put $f(\{q, s\}) \coloneqq \{r, s\}$ and $f(\{r, s\}) \coloneqq \{q, s\}$, in consequence we must have $f(\{p, q, s\}) = \{q, r, s\}$ and $f(\{q, r, s\}) = \{p, q, s\}$. But then such an f fails to be an isomorphism, as $\{p\} \subseteq \{p, r, s\}$ but $f(\{p\}) \nsubseteq f(\{p, r, s\})$. So, the identity mapping is the only option. Therefore, also in this case at least one of the two algebras must have more than four points.

The last situation is when the contact relation is empty on the set of atoms. Since the atoms must add up to the unity, in the case of four-point space and eight-element algebra, the only possibility is that we have two singletons, say $\{p\}$ and $\{q\}$, and one two-element set $\{r, s\}$. In this case, we obtain the regular closed algebra in Figure 9, whose topological space has the same algebra of regular open sets, and the same frame of opens, which is the algebra of clopen subsets of the space. Up to isomorphism, it is the only regular closed eight-element algebra so automorphisms are the only possibility. As no nontrivial set is interior-connected, we will not show nondefinability of interior-connectedness using this algebra.

In this way, we proved Theorem 4.1.

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