Spectra of the Spike-Flow Graphs in Geometrically Embedded Neural Networks

Jaroslaw Piersa and $\boxed{\text{Tomasz Schreiber}}$[⋆]

piersaj[at]mat.umk.pl
Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Poland

Abstract. In this work we study a simplified model of a neural activity flow in networks, whose connectivity is based on geometrical embedding, rather than being lattices or fully connected graphs. We present numerical results showing that as the spectrum (set of eigenvalues of adjacency matrix) of the resulting activity-based network develops a scale-free dependency. Moreover it strengthens and becomes valid for a wider segment along with the simulation progress, which implies a highly organised structure of the analysed graph.

Keywords: geometric neural networks, graph spectrum, scale-freeness

1 Introduction

The spectrum of the graph is considered as an important characteristic of the graph. A set of graph features can be easily obtained from the sole spectrum analysis, for instance bi-partitioning, connectivity, a clustering coefficient [6] etc. While the spectrum does not provide a unique description of the graph up to the isomorphism [6] it is graph-invariant.

Throughout this paper, by *graph spectrum* we understand a set of eigenvalues of the graph adjacency matrix A, namely set of $\lambda \in \mathbb{C}$ such that $A \cdot x = \lambda \cdot x$ for some vector x . Since A is symmetric, all eigenvalues are strictly real [6].

In this work we set out to analyse activity graphs of the geometrically embedded neural networks. Clearly, one can distinguish between the *structural* (i.e. underlying) and the *functional* or *spike-flow graphs* (evolved during the dynamics) of the network [3]. Since the structural graph is strictly dependent on selected network model, we shall focus on the latter case, namely a spontaneously emerged subgraph. As it was presented in [10, 11], the resulting graph has a significantly different input degree distribution. Here we provide more insight into differences between predefined and resulting networks.

⋆ Deceased author, 1975-2010, Associate Professor at Faculty of Mathematics and Computer Science, NCU, researcher in the fields of probability, artificial intelligence and physics.

The rest paper is organized as follows: we present the simplified activity flow model and the formal definition of the spike-flow graph in Sec. 2. Then we discuss the results concerning an emergence of *power-law* or *scale-free* dependency (scaling as $\mathbb{P}(X = x) \propto \frac{1}{x^k}$, where X is measured value and $k > 1$ is fixed parameter) in the spectrum of the obtained network in Sec. 3. Finally the work is concluded and potential aims of a future work are pointed in Sec. 4.

2 Simulation Model

We adopt the simplified model of neural activity coined in [10]. We argue, that it is valid for a modelling of activity in neurons and groups of neurons on cortical level. The model might seem to have an abundance of degrees of freedom, but they are mandatory, if the model is to exhibit an energy-driven self-organisation which is a feature of complex systems such as brain, see [5].

Given a two-dimensional sphere S_2 and expected density $\rho \gg 1$ of neurons in a square unit of the surface, we pick neurons from Poisson process on the sphere with intensity $\lambda = |S_2|\rho$. Each neuron is given its Euclidean coordinates $(x_v, y_v, z_v) \in \mathbb{R}^3$ accordingly to the process as well as an *initial activity* or charge σ_v, which is stored in the unit. It should be interpreted as an abstract activity level of the neuron. In the model only non-negative integer values for charge are allowed $\sigma_v \in \mathbb{N}_{\geq 0}$. Starting configuration can be seen in Fig. 1.

For every pair of neurons $\{u, v\}$ a *synaptic connection* $e = \{u, v\}$ is added to the set of network's edges $\mathcal{E}$ independently with probability $\mathbb{P}(e \in \mathcal{E}) = g(e)$, where $g()$ is a connectivity function

$$g(\{u, v\}) = \begin{cases} d(u, v)^{-\alpha} & d(u, v) \geq 1 \\ 1 & \text{otherwise}, \end{cases} \tag{1}$$

where $d()$ is euclidean distance and α is the *decay exponent* approximately equal to the dimension of the embedding space (2 in our case), the formula of the connectivity function was put forward in [7]. The synapse denotes a possibility of direct interaction between connected units. Note, that formula (1) admits self-loops. Since the expected density of neurons ρ is large, with huge probability we obtain a connected network. The geometrical embedding of the network along with formula of $g()$ result in varying lengths of the synapses. The number of short or local ones is much greater, than of the long ones, which provide a connection between the distant areas of the network. On the other hand a scale-free formula of Eq. (1) ensures, that long synapses do not vanish too fast when the radius of the sphere increases, which would result in lattice-like structural network. Such system seems to be more feasible to modelling of a real neural networks, than built on all-to-all connected graphs or regular grids.

Each synapse, which was added to the graph, receives a gaussian *weight* w_{uv}, which indicates its excitatory or inhibitory nature. The weight can be read as an averaging over a number of factors, thus a gaussian distribution seems adequate. Positive weight indicates a tendency favour equal activity levels for connected neurons, while inhibitory synapse results in large differences (activity in first unit keeps the second silent).

Given a network activity configuration $\bar{\sigma} = [\sigma_1, ..., \sigma_N]$ we define an *energy* of the system as follows:

$$E(\bar{\sigma}) = \sum_{(u,v)\in\mathcal{E}} w_{u,v}|\sigma_v - \sigma_u| \tag{2}$$

The formula (2) bears similarity to the stochastic Boltzmann machine [1], although has been adjusted in order to account for any nonnegative value of the σ_v. Nonetheless, it still has the same interpretation of summarized interaction cost in the network and it still can be negative.

The network undergoes its evolution according to the following dynamics:

1. iterate many times:
   (a) randomly pick a pair of units $u, v \in \mathcal{V}$, such that
      – $\{u, v\} \in \mathcal{E}$,
      – $\sigma_u \geq 1$,
   (b) try to transfer one unit of charge from u to v (i.e. $\sigma_u := \sigma_u - 1;\quad \sigma_v := \sigma_v + 1;$),
   (c) if this reduces network energy, then accept the transfer,
   (d) otherwise accept it with probability

$$\mathbb{P}(u \to v) = \exp(-\beta \Delta E), \tag{3}$$

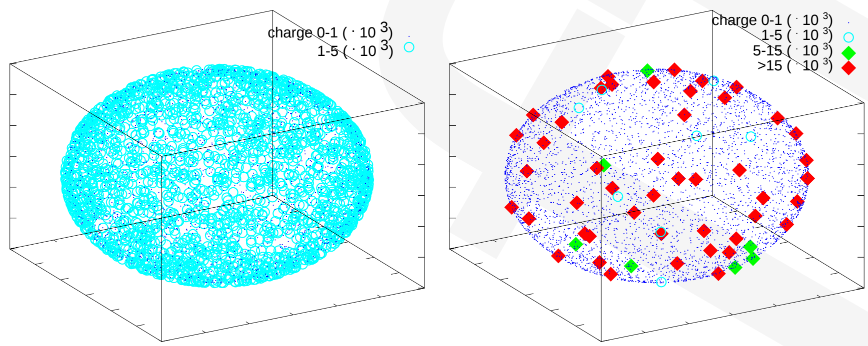
   where ΔE is an energy increase, which would be caused by the transfer, $\beta > 0$ is an inverse temperature and is assumed to be high ($\beta \gg 1$), which results in rejecting most of jumps towards higher energy states.

The transfer from u to v can be interpreted as spending some activity by u in order to cause an excitation in v or (in the case of inhibitory synapse) inhibiting u by increasing activity in v. Note, that this activity-conserving dynamics mimics a *criticality* state of the dynamics, i.e. the total activity in the network neither vanishes nor explodes. As Chialvo suggests, the emergence of scale-freeness is typical for such critical systems [5].

The stop condition can be either fixed number of iterations or the moment, when the network reaches its steady state i.e. when most of charge is stored in small number of neurons and the dynamics freezes, such situation will be referred as a *ground state* and the neurons with remaining charge as an *elite*. Both starting and ending phases are presented in Fig. 1.

During the evolution the amount of activity flowing through an edge $e = (u, v)$ (in both ways) is recorded, we will adopt the notation d_e for this value. We define a *spike-flow graph* $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ as a multi-graph, consisting of all neurons $\mathcal{V}' = \mathcal{V}$ and all the synapses, which are present in $\mathcal{E}$ and the flow through them exceeded the threshold value $\mathcal{E}' = \{e \in \mathcal{E} : d_e \geq \theta\}$, with their multiplicities equal to the charge flowed ($M(e) = d_e$ if $d_e \geq \theta$ and $M(e) = 0$ otherwise), see Fig. 2.

Unless stated otherwise in our simulations θ is assumed to be one. In such case every accepted transfer adds an edge into $\mathcal{E}'$.



(a) Early phase of the simulation. (b) Ending phase.

Fig. 1. A plot of the network of $5k = 5 \cdot 10^3$ neurons at early and late states of the simulation. Left — starting, roughly uniform setup. Right — ending phase, all the charge has stuck in the small number of *elite* units.

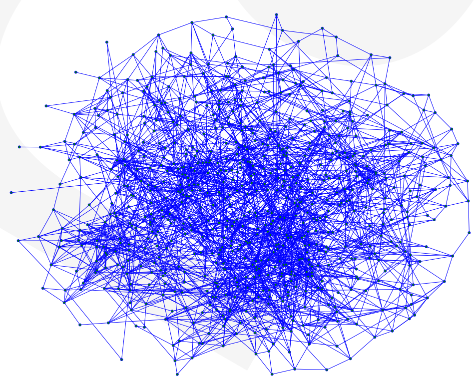
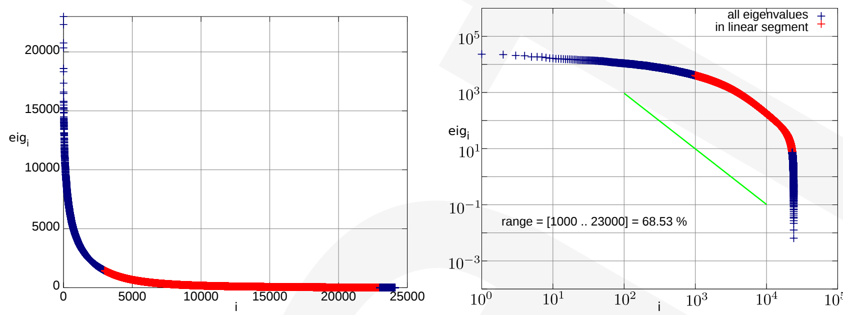


Fig. 2. A plot of resulting spike flow sub-graph obtained at the end phase of the simulation. The resulting graph was limited to around 700 vertices out of over 12000 and was remapped onto a plane.

3 Results

The simulations were carried on networks counting up to 35000 neurons. Figure 3 presents obtained plot of i-th eigenvalue vs index i. The eigenvalues of the resulting spike-flow graph were sorted decreasingly and for log-plot issues negative ones were removed. An interesting feature is the middle part, where locally the plot behaves like a straight line for quite a long segment.



(a) A plot of the graph spectrum. (b) A log-log plot.

Fig. 3. A plot of the spectrum of the spike-flow graph obtained throughout the dynamics of around 30k neurons. The plots present the i-th eigenvalue vs i, eigenvalues are sorted decreasingly. Middle part of the plot covers around 60% of the whole dataset. For reference — a line segment has a slope -2 (a function x^{-2} in log-log plot).

Note, that such behaviour in fully connected networks with winner-takes-all rule dynamics was theoretically predicted in [12] and confirmed in [9]. Despite the facts, that in our model WTA dynamics is only approximated by an inverse temperature β and the structural network is no longer fully connected (actually it is quite sparse), the model produces strikingly similar feature.

The plot failed to indicate clear scale-free dependence for top eigenvalues, see [9]. Instead this dependency arises in its mid-part and covers about 60% of the whole data. We argue, that this feature might be rooted in lack of direct synaptic connections between the elite neurons, so it is not always possible to transfer large amounts of charge directly between them. Somehow reiterated from [10] is an observable exponential truncation of this scale-free dependency, however without rigorous analysis one cannot state whether it is due to finite simulation sample.

Additionally, there is no sign of first outlying eigenvalue, which is typical for Erdős-Réyni random graph [2], recall that Erdős-Réyni random graph with n vertices and $p \in (0, 1)$ is generated by including every edge of the full graph independently with uniform probability p [8]. For comparison, a plot of eigenvalues of the ER model is presented in Fig. 4.

While Fig. 3 presents a spectrum obtained once the simulation was terminated, it turns out even more interesting to see it in various steps of the dynamics.

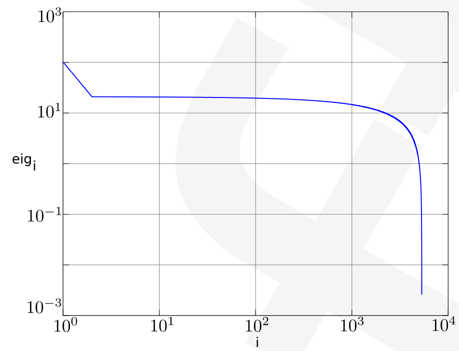


Fig. 4. A log-log plot of the spectrum (the i-th eigenvalue vs i) of the Erdős-Réyni, random graph model with 10k neurons and its average connectivity is the same as in the obtained spike-flow network for the same size. Note the first outlying eigenvalue.

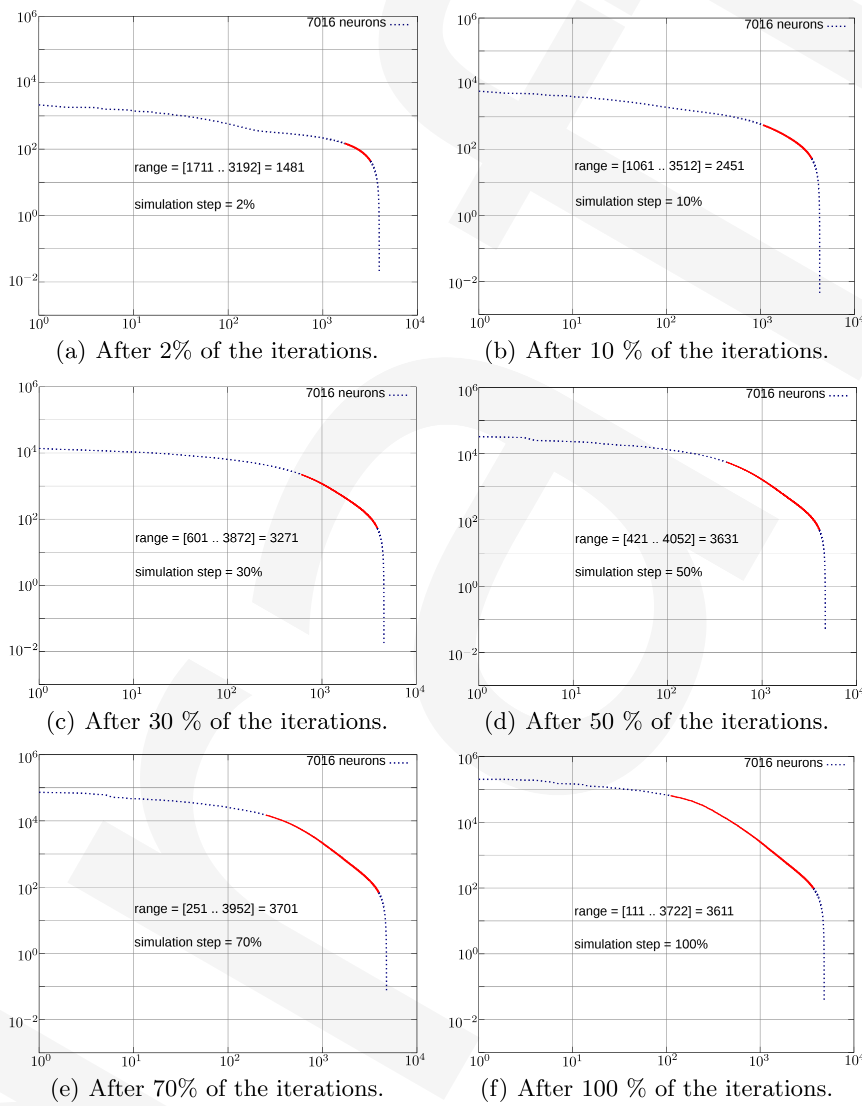
Fig. 5 presents a log-plot of the eigenvalues after computing 2, 10, 30, 50, 70 and 100% of the iterations. It clearly indicates, that the spike-flow graph only begins its self-organization process as a ER random graph. This seems quite natural, as in the dynamics we pick a synaptic edge randomly with uniform distribution and at start its probable that the connected neurons have not been drained from their charge yet. Then, as the dynamics continues, a linear segment emerges and increases in length.

We have adopted following way to estimate the linear segment on the plot. A pair of indices (i_1, i_2) is referred as its borders if:

- $i_1 < i_2$,
- $|\bar{a} - a_1| < E_1$,
- $\frac{1}{(i_2-i_1)} \sum_{i=i_1}^{i_2} (e_i - a_1 \cdot i - a_0)^2 < E_2$,
- $i_2 - i_1$ is maximal of all satisfying above points,

where $y = a_1 x + a_0$ stands for a formula of line approximating the data set $\{(\log i_1, \log e_{i_1})...(\log i_2, \log e_{i_2})\}$, the formula can be obtained using linear regression for instance; e_j is a j-th largest eigenvalue; thresholds E_1 and E_2 were picked arbitrarily depending on the number of neurons in the simulation and desired accuracy and the $\bar{a}$ is an expected slope. If no such pair exists, we conclude that the spectrum does not have any linear segment (such situation did not occurred, until the threshold values were ridiculously strict).

Evolution of both indices i_1, i_2 defined as above and length of the segment (i_2-i_1) is presented in Fig. 6. Somehow unsettling, the growth depends mainly on reducing the beginning index i_1, we also observe fluctuations of the upper bound i_2. This seems be related to the vulnerability linear regression on non-uniformly distributed data. Nevertheless, the i_2-i_1 clearly grows as the network undergoes its dynamics suggesting that i-th eigenvalue of the final spike-flow graph decays as $\frac{c}{i^2}$, even if the structural network was not fully connected.



(a) After 2% of the iterations.

(b) After 10 % of the iterations.

(c) After 30 % of the iterations.

(d) After 50 % of the iterations.

(e) After 70% of the iterations.

(f) After 100 % of the iterations.

Fig. 5. Log-log plots of the spectrum of the spike-flow graph obtained in various steps of the dynamics. Dotted line denotes whole spectrum, solid line — the estimated linear segment. Simulation was carried on about $7 \cdot 10^3$ neurons and $2 \cdot 10^9$ iterations.

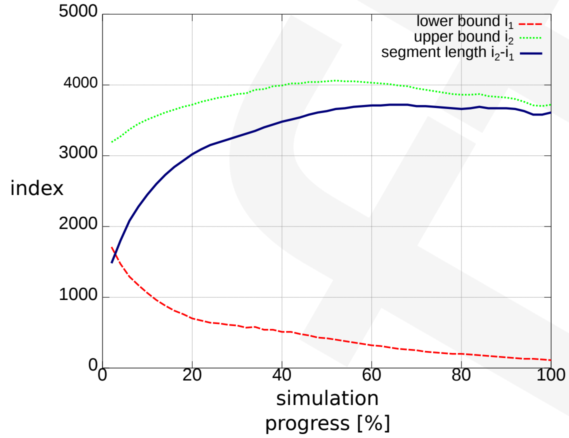


Fig. 6. Estimated length of the linear part of the spectrum throughout the simulation, measured every 2% of the iterations.

4 Conclusion and Future Work

To summarize, we have presented spectra of the spike-flow graphs of the geometrical model of neural network throughout its evolution. As the network approaches its steady state a linear dependency emerges and grows in the spectrum plot, which suggests a (exponentially truncated) power law dependency in graph's eigenvalues. At the early stages the spectrum resembles those of the Erdős-Réyni random graph and only later it evolves towards more sophisticated model.

One of interesting directions, in which this work can be extended, is applying the same spectral analysis to medical data from fMRI scans, as it was discussed concerning a graph degree distribution in [10] or graph diameter [11]. However, to our knowledge, no functional networks obtained from fMRI data have been analysed with spectral methods so far. As a second aim of ongoing research, we point out an analysis of the network resiliency and fault tolerance [2].

Acknowledgements

The author would also like thank to dr Filip Piękniewski for hints and remarks concerning this work.

This research was supported in part by PL-Grid Infrastructure. The author is grateful to PL-Grid Project[1] staff and help-line.

The work has been partially supported by Ministry of Science and Higher Education and grant DEC-2011/01/N/ST6/01931.

[1] http://www.plgrid.pl

References

1. Ackley D. H., Hinton G. E., Sejnowski T. J.: *A Learning Algorithm for Boltzmann Machines*, Cognitive Science, 9(1):147-169, 1985.
2. Albert R., Barabasi A. L.: *Statistical mechanics of complex networks*, Reviews of modern physics, Vol 74, January 2002.
3. Bullmore E., Sporns O.: *Complex brain networks: graph theoretical analysis of structural and functional systems*, Nature Reviews, Neuroscience, vol 10, March 2009.
4. Chung F., Lu L.: *Complex graphs and networks*, American Mathematical Society, 2006.
5. Chialvo D.: *Critical brain networks*, Physica A: Statistical Mechanics and its Applications, Vol. 340, Issue 4, September 2004.
6. Cvetković D., Rowlingson P., Simić S.: *Eigenspaces of graphs*, Cambridge University Press, 1997
7. Eguiluz V., Chialvo D., Cecchi G., Baliki M., Apkarian V.: *Scale-Free Brain Functional Networks*, Physical Review Letters, PRL 94 018102, JAN 2005.
8. Erdős P., Réyni A.: *On random graphs I*, Publ. Math. Debrecen 6, 290–297, 1959.
9. Piekniewski F.: *Spectra of the Spike Flow Graphs of Recurrent Neural Networks*, Artificial Neural Networks - ICANN 2009, 19th International Conference, 2009, Proceedings, Part II.
10. Piersa J., Piekniewski F., Schreiber T.: *Theoretical model for mesoscopic-level scale-free self-organization of functional brain networks*, IEEE Transactions on Neural Networks, vol. 21, no. 11, November 2010.
11. Piersa J.: *Diameter of the spike-flow graphs of geometrical neural networks*, post-proceedings of 9th International Conference on Parallel Processing and Applied Mathematics, serie Lecture Notes on Computer Science, September 2011 (in print).
12. Schreiber T.: *Spectra of winner-take-all stochastic neural networks*, arXiv [available on-line], http://arxiv.org/PS_cache/arxiv/pdf/0810/0810.3193v2.pdf, 3193(0810), October 2008, pp. 1–21.