Absurd sentences and normal deductions: A case of the logic of demodalised analytic implication with *falsum*

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Abstract

The logic of demodalised Parry analytic implication **DAI** was introduced by J. M. Dunn. In its original formulation, the language of **DAI** consisted of classical negation, classical conjunction, and demodalised analytic implication. Later, R. L. Epstein rediscovered **DAI** as a content-inclusion logic, by providing semantics in terms of set-assignment models. Epstein called it the dependence logic **D**. He also studied **DAI** expanded with the constants *falsum* and *verum* in the context of algebraic analysis. More recently, A. Ledda, F. Paoli, and M. Pra Baldi investigated **DAI** with constants and provided an algebraic semantics in terms of implicative involutive bisemilattices.

In this paper, we study **DAI** expressed in a language without negation but with the constant *falsum*. First, we examine several semantic treatments of *falsum*, each of which gives rise to different definable negations and, consequently, to distinct logics. But we focus only on one logic, namely **DAI** without negation but with *falsum*. Second, we introduce two labelled deductive systems for the resulting logic. We prove for each deductive system the soundness and completeness theorems and establish a Prawitz-style normalisation theorem.

Keywords: demodalised analytic implication; *falsum* constant; deductive systems; normalisation theorem; relating semantics; set-assignment semantics.

1 Introduction

The logic of demodalised analytic implication **DAI** was introduced by J. M. Dunn in 1972 [4] as a modification (demodalisation) of W. T. Parry's logic of analytic (strict) implication ([16, 17], cf. [22]). In its original formulation, the language of **DAI** consisted of classical negation, classical conjunction, and demodalised analytic implication. In 1987, R. L. Epstein [5, 6] rediscovered **DAI** as content-inclusion logic and called it the dependence logic **D**. Dunn presented **DAI** using matrices, whereas Epstein employed set-assignment semantics, which was designed to represent (sentential) content formally.

There is a substantial literature on both the model theory and the proof theory of **DAI**. In what follows, we focus, on the one hand, on algebraic approaches and, on the other, on deductive systems other than axiomatic systems, since these strands connect directly to the questions studied in this paper.

Also in 1987, Epstein [5], considered **DAI** expanded with the constants *falsum* and *verum* and offered the first algebraic treatment of the resulting system. Later, in 2019, A. Ledda, F. Paoli and M. Pra Baldi [13] analysed **DAI** with constants in an algebraic setting and presented the algebraic semantics for it in terms of implicative involutive bisemilattices.[1]

Besides providing algebraic insight into **DAI** with constants, these analyses bring to the fore the question of how to understand absurd sentences (represented by *falsum*) and tautological sentences (represented by *verum*) in terms of content. Epstein and Ledda et al. adopt different conceptions: roughly, should the content of such sentences be the whole universe of content, or the empty set? Epstein's set-assignment semantics offers a natural framework in which to analyse this issue formally. We discuss the problem in detail and indicate several possible resolutions. The issue is not only algebraic; it is also proof-theoretic, in particular for natural-deduction systems.

As for proof theory, in 1987, W. A. Carnielli [1] introduced tableau systems for all of Epstein's logics, including **DAI**. In 2022, Jarmużek and Klonowski [8] proposed a different approach to tableau systems. In 1991, L. Fariñas del Cerro and V. Lugardon [3] developed sequent systems for certain modifications of Epstein's logics (including modification of **DAI**, though not **DAI**). In 1997, they [2] extended logics of this kind to the first-order case, and provided corresponding sequent systems. Other interesting results were presented in 2007 by F. Paoli [15] (cf. [14]), who studied algebraic and proof-theoretic aspects of FDE-fragments of Epstein's logics, including **DAI**.

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[1] N. Zamperlin in 2025 [22] has proposed yet another algebraic approach to **DAI**. Dunn, of course, gave the first approach to algebraic semantics for the considered logic, using matrices.

However, proof-theoretic work on **DAI** still lacks a natural deduction (or sequent) treatment satisfying the standard normalisation (or cut-elimination) property. A first attempt at a natural-deduction-like system was made in [12] but the systems presented there do not admit a normalisation theorem. The present paper aims to fill this gap.

This paper studies **DAI** expressed in a language without negation but with the constant *falsum*. The goals are (i) to analyse different semantic treatments of *falsum* in connection with the content of absurd sentences, and (ii) to present natural deduction systems for **DAI** (with *falsum*) together with a proof of normalisation.

2 Logic DAI$_\bot$: Set-assignment semantics and a content of absurd sentences

In what follows, we consider a propositional language (zeroth-order) with countably many (propositional) variables: $p_1, p_2, p_3, \ldots$, constant: $\bot$ (*falsum*), connectives: $\wedge$ (conjunction), $\rightarrow$ (demodalised (non-modal) analytic implication) and brackets: $), ($. The set of variables is denoted by Var and the set of formulas defined in the standard way in the considered language is denoted by For. For any formula $A \in \mathsf{For}$, the set of its variables is denoted by $\mathsf{var}(A)$. We sometimes use the following abbreviations: $A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$ and $A \looparrowright B = A \rightarrow (B \rightarrow B)$.

2.1 A set-assignment semantics

We define the logic of demodalised analytic implication with *falsum* by *set-assignment semantics* proposed by Epstein [6], based on a function that represents a sentential content.

A *set-assignment model* (a *sa-model*) is a triple $\langle v, s, R\rangle$ such that $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ is a logical value assignment, $s\colon \mathsf{For} \longrightarrow P(U)$ is a sentential content assignment (a *set-assignment function*), where U is a non-empty set and $P(U)$ is the powerset of U, and $R \subseteq s(\mathsf{For}) \times s(\mathsf{For})$ is a *content relating relation.* We consider the following truth conditions:

+ $M \models A$ iff $v(A) = 1$, if $A \in \mathsf{Var}$,
+ $M \not\models \bot$,
+ $M \models A \wedge B$ iff both $M \models A$ and $M \models B$,
+ $M \models A \rightarrow B$ iff either not $M \models A$ or $M \models B$, and $R(s(A), s(B))$.

We sometimes write $M \models X$ meaning that for all $B \in X$, $M \models B$.

As we can see, the truth condition for implication requires for the conditional to be true that its antecedent is false or its consequent is true (a classical extensional condition) and the content of the antecedent and the content of the consequent are somehow related to each other.

Let S be a class of sa-models. We say that A is a *semantic consequence of X in S* ($X \models_S A$) iff for any sa-model $M \in S$, if $M \models X$, then $M \models A$. We say that A is *valid in S* iff $\emptyset \models_S A$. By a logic we mean a consequence relation determined by a given class of models.

In order to describe the logic that interests us, we must determine the sentential content assignment to various complex sentences, taking into account the content of the component sentences, as well as the content relating relation in such a way as to obtain a demodalised analytical implication.

Epstein assumed that logical constants do not affect the content of a sentence because they are syncategorematic. In his work, he introduced two types of set-assignment functions, each representing a type of content assignment:

+ a *union set-assignment* is defined by the following condition:

  (usa) $s(A \star B) = s(A) \cup s(B)$,

  for any $\star \in \{\wedge, \rightarrow\}$,

+ an *intersection set-assignment* is defined by the following condition:

  (isa) $s(A \star B) = s(A) \cap s(B)$,

  for any $\star \in \{\wedge, \rightarrow\}$.

In the original formulation of **DAI** without constants, but with classical negation $\neg$, for each set-assignment function s, the following content neutrality condition was additionally assumed: $s(\neg A) = s(A)$.

The given conditions (usa) and (isa) are related to two definitions of the content relating relation adopted by Epstein in sa-models. In the case of sa-models $\langle v, s, R\rangle$ such that s satisfies (usa), we assume the following definition:

(usaR) $R(s(A), s(B))$ iff $s(B) \subseteq s(A)$.

In the case of sa-models $\langle v, s, R\rangle$ such that s satisfies (isa), we assume the following definition:

(isaR) $R(s(A), s(B))$ iff $s(A) \subseteq s(B)$.

The given conditions (usa) and (isa) do not involve a constant *falsum*. The question therefore arises about the content of the absurd sentence represented by the constant *falsum*. How should the set-assignment function operate on *falsum*? In the literature directly related to demodalised analytic implication, three approaches to this problem are considered. Each has certain advantages and disadvantages from the perspective of the logical analysis of language.

2.2 A content of absurd sentences

Ledda, Paoli and Pra Baldi [13] suggested the following condition for the set-assignment function s with respect to *falsum*:

(LPP$\bot$) $s(\bot) = \emptyset$.

Of course, with this approach, we must assume the existence of certain absurd sentences that have no content—that is, sentences devoid of any sense. With this approach, classical negation is easy to define.

Let us consider sa-models whose set-assignment function satisfies (usa) and (LPP$\bot$). To define the truth condition for $\rightarrow$, we further assume (usaR). We call these models *usa-LPP-models* and denote the class of all such models by **usa-LPP**.

We define negation in a standard way:

(Def$_1\neg$) $\neg A = A \rightarrow\bot$.

Then we receive the following:

$$\begin{aligned} s(\neg A) =& s(A \rightarrow\bot) \\ =& s(A) \cup s(\bot) \\ =& s(A). \end{aligned}$$

Moreover, by (Def$_1\neg$) we define the classical negation:

$$\begin{aligned} M \models \neg A \text{ iff } & M \models A \rightarrow\bot, \\ & \text{iff } (M \not\models A \text{ or } M \models\bot) \text{ and } s(\bot) \subseteq s(A), \\ & \text{iff } M \not\models A. \end{aligned}$$

Before Ledda et al., Epstein [5] proposed the following condition for the set-assignment function with respect to *falsum*:

(E$\bot$) $s(\bot) = U$,

and assumed the (usa) condition for two-argument constants. With this approach, we must assume the existence of certain absurd sentences, in which we can speak of a kind of content overflow—that is, absurd content that contains any content.

However, in Epstein's approach, the definition of negation becomes problematic. Let us assume that we consider sa-models with set-assignment functions satisfying conditions (usa), (E$\perp$), and to specify the truth condition for $\to$, we assume (usaR). We call such models *usa-E-models*. With the presented approach, we cannot define the classical negation in the standard way by (Def$_1\neg$). Since we have:

$$\begin{aligned} M \models \neg A \text{ iff } & M \models A \to\perp, \\ & \text{iff } (M \not\models A \text{ or } M \models\perp) \text{ and } s(\perp) \subseteq s(A), \\ & \text{iff } M \not\models A \text{ and } s(\perp) \subseteq s(A), \\ & \text{iff } M \not\models A \text{ and } s(A) = U. \end{aligned}$$

In the given approach, negation in the sense of the definition (Def$_1\neg$) constitutes some non-classical negation that might be worth investigating, but this is a topic for a separate paper.

This does not mean, however, that assuming the usa-E-model, we cannot define the classical negation. We have the following definition:

(Def$_2\neg$) $\neg A = (A \wedge (\perp\to\perp)) \to\perp$.

Let us note that:

$$\begin{aligned} M \models \neg A \text{ iff } & M \models (A \wedge (\perp\to\perp)) \to\perp, \\ & \text{iff } M \not\models A \wedge (\perp\to\perp) \text{ or } M \models\perp, \text{ and } s(\perp) \subseteq s(A\wedge \perp), \\ & \text{iff } M \not\models A \wedge (\perp\to\perp) \text{ and } s(\perp) \subseteq s(A) \cup s(\perp) = s(\perp), \\ & \text{iff } M \not\models A \text{ or } M \not\models\perp\to\perp, \\ & \text{iff } M \not\models A \text{ or } (M \models\perp \text{ and } M \models\not\perp) \text{ or } s(\perp) \not\subseteq s(\perp), \\ & \text{iff } M \not\models A. \end{aligned}$$

Although (Def$_2\neg$) allows us to define the classical negation, the adopted definition does not allow us to obtain a negation for which the content neutrality condition is satisfied. We will find a usa-E-model and a formula A such that: $s(A) \subset s(\neg A) = U$.

However, the analysis presented above does not exclude the possibility that the formulation of (E$\perp$) could yield a classical negation under (Def$_1\neg$) for which the content neutrality condition is satisfied. Let us consider sa-models with set-assignment functions satisfying conditions (isa) and (E$\perp$), and to specify the truth condition for →, we assume (isaR). We call such models *isa-E-models* and denote the class of these models by **isa-E**. We then have:

$$\begin{aligned} s(\neg A) =& s(A \to \bot),\\ =& s(A) \cap s(\bot),\\ =& s(A). \end{aligned}$$

Moreover, we have:

$$\begin{aligned} M \models A \to \bot \ &\text{iff } (M \not\models A \text{ or } M \models \bot) \text{ and } s(A) \subseteq s(\bot),\\ &\text{iff } M \not\models A \text{ or } M \models \bot),\\ &\text{iff } M \not\models A. \end{aligned}$$

Of course, if instead of (usa) and (usaR) we assume (isa) and (isaR), then the previously considered formulation (LPP⊥) does not allow us to express classical negation using ($\mathrm{Def}_1\neg$), just as the formulation (E⊥) under the conditions (usa) and (usaR) deos not. Consider sa-models with a set-assignment function satisfying conditions (isa) and (LPP⊥), and to specify the truth condition for →, we assume (isaR). Such models are called *isa-LPP-models*. We then have:

$$\begin{aligned} M \models A \to \bot \ &\text{iff } (M \not\models A \text{ or } M \models \bot) \text{ and } s(A) \subseteq s(\bot),\\ &\text{iff } M \not\models A \text{ and } s(A) \subseteq s(\bot),\\ &\text{iff } M \not\models A \text{ and } s(A) = \emptyset. \end{aligned}$$

So, once again, we are dealing with a non-classical negation, which would also be worth investigating. But can we somehow express the classical negation in the case of the isa-LPP models we are considering? Of course, we can; we just need to use the definition ($\mathrm{Def}_2\neg$) again. We then have:

$$\begin{aligned} M \models \neg A \ &\text{iff } M \models (A \wedge (\bot \to \bot)) \to \bot \qquad ,\\ &\text{iff } (M \not\models A \wedge (\bot \to \bot) \text{ and } s(A \wedge (\bot \to \bot)) \subseteq s(\bot),\\ &\text{iff } M \not\models A \text{ or } M \not\models \bot \to \bot \ \text{ and } s(A) \cap \emptyset \subseteq \emptyset,\\ &\text{iff } M \not\models A. \end{aligned}$$

As in the earlier case of (E⊥), ($\mathrm{Def}_2\neg$) does not allow us to obtain a negation that satisfies the content neutrality condition in the case under consideration. We will find an isa-LPP-model $\langle v, s, R\rangle$ and a formula A such that: $\emptyset = s(\neg A) \subset s(A)$.

It is worth noting that there is another way to understand the content of absurd sentences. However, this requires us to consider not one constant $\perp$ but infinitely many *falsum* constants, each indexed by a string of propositional variables. Following Lugardon and del Cerro [3], this kind of solution can be proposed once one is familiar with their sequent systems for some modifications of Epstein's content-relationship logics, including **DAI**. Suppose that we modify the initial language. As before, we have infinitely many variables $p_1, \ldots$ and conjunction $\wedge$ and implication $\rightarrow$. Additionally, for any $n \in \mathbb{N}$, we have a constant $\perp_{A_1 \ldots A_n}$, where $A_1, \ldots, A_n \in \mathsf{Var}$. We assume the following conditions for the set-assignment function with respect to *falsum*:

(LD$\perp$) $s(\perp_{A_1 \ldots A_n}) = s(A_1) \cup \ldots \cup s(A_n)$.

With this approach, we can say that absurd sentences in content reduce to the content of the atomic sentences from which they are composed. This means, however, that in terms of content, we are not dealing with a single absurdity but with an infinite number of extensionally indistinguishable absurdities. With this approach, in the context of the earlier considerations, it is obvious that we can define classical negation by modifying the definition of (Def$_1\neg$):

(Def$_3\neg$) $\neg A = A \rightarrow \perp_{A_1 \ldots A_n}$,

where $\mathsf{var}(A) = \{A_1, \ldots, A_n\}$. (Def$_3\neg$) allows us to obtain classical negation that satisfies the content neutrality condition, for sa-models with set-assignment functions satisfying (usa) and (LD$\perp$) where $\rightarrow$ is interpreted via (usaR), and for sa-models satisfying (isa) and (LD$\perp$) where $\rightarrow$ is interpreted via (isaR). The main disadvantage of the given solution is that it requires assuming an infinite number of *falsum* constants.

In the above, we did not consider the problem of defining the constant *verum*. We focused only on the constant *falsum*. In Ledda et al.'s approach, the values of the set-assignment function on *verum* and *falsum* are the same; in Epstein's approach, the set-assignment function assigns *verum* the complement of the set being assigned to *falsum*. Therefore, taking into account two constants and assuming two types of conditions for the set-assignment function on compound formulas, we can consider 8 combinations given in Table 1.

In the remainder of this paper, we focus on the constant *falsum* and the classes **usa-LPP** and **isa-E**. We leave the consideration on *verum* and other classes of models, as well as the logics they define, as a topic for further research.

	falsum	*verum*
usa	universe	universe
isa	universe	universe
usa	universe	$\emptyset$
isa	universe	$\emptyset$
usa	$\emptyset$	universe
isa	$\emptyset$	universe
usa	$\emptyset$	$\emptyset$
isa	$\emptyset$	$\emptyset$

Table 1: Combinations of set-assignments

2.3 Logic $\mathbf{DAI}_\perp$: The negation-less DAI with *falsum*

Let us now define the logic we are interested in. The negation-less logic of demodalised implication with *falsum* $\mathbf{DAI}_\perp$ can be defined using the class **usa-LPP** or the class **isa-E**; to express negation, we use the definition ($\mathrm{Def}_1\neg$). The following relation holds:

$$\models_{\textbf{usa-LPP}} = \models_{\textbf{isa-E}}.$$

It is easy to see that $\mathbf{DAI}_\perp$ can be defined by both considered classes, if we apply a relating semantics for metalogical studies of $\mathbf{DAI}_\perp$ (see [10, 9, 11]).

A *relating model* is a pair $\langle v, R\rangle$ such that $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ is a *logical value assignment* and $R \subseteq \mathsf{For} \times \mathsf{For}$ is a *relating relation*. We have similar truth conditions as before:

- $+$ $M \models A$ iff $v(A) = 1$, if $A \in \mathsf{Var}$,
- $+$ $M \not\models \perp$,
- $+$ $M \models A \wedge B$ iff both $M \models A$ and $M \models B$,
- $+$ $M \models A \to B$ iff either not $M \models A$ or $M \models B$, and $R(A, B)$.

Let **rmD** be the class of all relating models with R satisfying the following conditions (cf. [11]):

(tran) $\forall_{A,B,C\in\mathsf{For}}((R(A,B)$ and $R(B,C))$ only if $R(A,C))$,

(d1) $\forall_{A\in\mathsf{For}} R(A, \neg A)$,

(d2) $\forall_{A\in\mathsf{For}} R(\neg A, A)$,

(d3) $\forall_{A,B,C\in\mathsf{For}}(R(A, B \wedge C)$ iff $(R(A,B)$ and $R(A,C)))$,

(d4) $\forall_{A,B,C\in\mathsf{For}}(R(A, B \to C)$ iff $(R(A,B)$ and $R(A,C)))$.

Let $\models_{\mathbf{rmD}}$ be defined in the standard way by means of the class **rmD**. Then, modifying [6, pp. 120–123,131–133] (cf. [11, pp. 9–18, 20–25]) we have that:

$$\models_{\mathbf{usa\text{-}LPP}} = \models_{\mathbf{rmD}} = \models_{\mathbf{isa\text{-}E}} .$$

Indeed, if $R \subseteq \mathsf{For} \times \mathsf{For}$ satisfies (tran), (d1)–(d2), then for any $A \in \mathsf{For}$ we put $s_R(A) = \mathsf{For} \backslash t(A)$, where $t(A) = \{B \in \mathsf{For} : R(B, A)\}$ and $s'_R(A) = \{B \in \mathsf{For} : R(B, A)\}$. We can easily show that s_R is a set-assignment function that satisfies (usa) and (LPP$\bot$), and s'_R is a set-assignment function that satisfies (isa) and (E$\bot$). We also have that $R_s \subseteq \mathsf{For} \times \mathsf{For}$ satisfies (tran), (d1)–(d2), if we assume any of the following definitions: $R_s(A, B)$ iff $s(B) \subseteq s(A)$, where s is a set-assignment function satisfying (usa) and (LPP$\bot$); $R_s(A, B)$ iff $s(A) \subseteq s(B)$, where s is a set-assignment function satisfying (isa) and (E$\bot$).

By the relations distinguished above, it is easy to see that $\mathbf{DAI}_\bot$ is deductively equivalent to **DAI**, whose language contains *falsum*. However, $\mathbf{DAI}_\bot$ is not deductively equivalent to the original **DAI**. It is impossible to define *falsum* in the original language of **DAI** in such a way as to satisfy, considering sa-models, one of the conditions (LPP$\bot$) or (E$\bot$).

Based on the considerations presented, we can also easily axiomatise $\mathbf{DAI}_\bot$. It suffices to refer to the expressibility of the relating relation using an appropriate schema (see [6, p. 125], cf. [9]): for any $M = \langle v, R\rangle \in \mathbf{rmD}$, for any $A, B \in \mathsf{For}$, $R(A, B)$ iff $M \models A \looparrowright B$. We can then determine the axiomatic system $\mathbf{DAI}_\bot$ using the following schemas: (see [9, 11]):

(CL1) $\neg(A \wedge \neg(A \wedge A))$,
(CL2) $\neg((A \wedge B) \wedge \neg A)$,
(CL3) $\neg(\neg(A \wedge \neg B) \wedge \neg\neg(\neg(B \wedge C) \wedge \neg\neg(C \wedge A)))$,
(MD) $\dfrac{\neg(A \wedge \neg B) \quad A}{B}$
(DAI) $(A \to B) \leftrightarrow ((\neg(A \wedge \neg B)) \wedge (A \looparrowright B))$,
(DAI1) $((A \looparrowright B) \wedge (B \looparrowright C)) \to (A \looparrowright C)$,
(DAI2) $A \looparrowright \neg A$,
(DAI3) $\neg A \looparrowright A$,
(DAI4) $(A \looparrowright (B \to C)) \leftrightarrow ((A \looparrowright B) \wedge (A \looparrowright C))$,
(DAI5) $(A \looparrowright (B \wedge C)) \leftrightarrow ((A \looparrowright B) \wedge (A \looparrowright C))$.

The proofs of the soundness and completeness theorems for the presented axiomatic system are obvious modifications of the proofs presented in [6, 9].

3 The multi-labelled system of deduction for $\mathbf{DAI}_\perp$

We now present a deductive system for $\mathbf{DAI}_\perp$ by modifying the systems presented in [12]. To formally present our system and to prove the normalisation theorem for it, we modify the approach proposed by D. Prawitz [18] (cf. [7, 19, 20]).

3.1 Expressions in deduction

In our system, we use not only formulas, but also formulas labelled with natural numbers. Thus, a *labelled formula* is a pair $\langle A, n\rangle$ such that $A \in \mathsf{For}$ and $n \in \mathbb{N}$. We denote the set of labelled formulas by labEx. We also introduce a new symbol, $\Rrightarrow$. We define the set of expressions Ex as the smallest set X such that $\mathsf{For}, \mathsf{labEx} \subseteq X$ and $\{\langle A, n\rangle \Rrightarrow\perp : A \in \mathsf{For} \text{ and } n \in \mathbb{N}\} \subseteq X$. We often write A, n instead of $\langle A, n\rangle$ and use the following abbreviation: $-A, n = A, n \Rrightarrow\perp$. The elements of Ex are called *d-expressions.* We use $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \ldots$ as variables ranging over the sets of d-expressions and the d-expressions, respectively. Let us also introduce the following function: $\mathsf{exl}(\mathcal{X}) = \{A, n \in \mathsf{For} \times \mathbb{N} : A, n \in \mathcal{X}\} \cup \{A, n \Rrightarrow\perp \in \mathsf{labEx} : A, n \Rrightarrow\perp \in \mathcal{X}\}$.

The expression A, n can be understood as stating that the element of the universe of content to which n refers is an element of the content of A. In turn, the symbol $\Rrightarrow$ codes a material implication from the metalanguage.

3.2 Rules and deductions

The first type of rules in our system consists of the classical rules.

$$(\mathrm{RAA})\ \frac{\begin{matrix}(\neg A)\\ \perp\end{matrix}}{A} \qquad\qquad (\mathrm{EFQ})\ \frac{\perp}{A}$$

$$(\wedge\mathrm{I})\ \frac{A \quad B}{A \wedge B} \qquad (\wedge\mathrm{E}_1)\ \frac{A \wedge B}{A} \qquad (\wedge\mathrm{E}_2)\ \frac{A \wedge B}{B}$$

In the case of the rule (RAA), when deducing $\perp$ there is no undischarged labelled assumption; moreover, $A \neq\perp$ and $A \neq \neg B$ for any $B \in \mathsf{For}$ (cf. [18]).

The second type of rules in our system consists of auxiliary rules, which we call *lab-rules*. They have the following form:

$$(\bot\,\mathrm{lab})\ \frac{A, n}{\bot, n}$$

$$(\wedge\mathrm{labE}_1)\ \frac{A\wedge B, n}{A, n} \qquad (\wedge\mathrm{labE}_2)\ \frac{A\wedge B, n}{B, n} \qquad (\wedge\mathrm{labI})\ \frac{A, n \quad B, n}{A\wedge B, n}$$

$$(\rightarrow\mathrm{labE}_1)\ \frac{A\rightarrow B, n}{A, n} \qquad (\rightarrow\mathrm{labE}_2)\ \frac{A\rightarrow B, n}{B, n} \qquad (\rightarrow\mathrm{labI})\ \frac{A, n \quad B, n}{A\rightarrow B, n}$$

Below, in this section, we show that $\mathsf{var}(A) \subseteq \mathsf{var}(B)$ iff A, n is deducible from B, n in a deductive system consisting solely of lab-rules.

Let us now proceed with defining the rules for implication.

$$(\rightarrow\mathrm{I})\ \frac{\begin{matrix}(A) & (A, n)\\ B & B, n\end{matrix}}{A\rightarrow B} \qquad (\rightarrow\mathrm{E})\ \frac{A\rightarrow B \quad A}{B} \qquad (\rightarrow\mathrm{E}')\ \frac{A\rightarrow B \quad A, n}{B, n}$$

In the case of the rule $(\rightarrow\mathrm{I})$ the deduction of B, n is from a non-empty set of assumptions and A, n is the only undischarged assumption with n in it.

We also assume the following rules for $\Rightarrow$:

$$(\Rightarrow\mathrm{I})\ \frac{\begin{matrix}(A, n)\\ \bot\end{matrix}}{A, n \Rightarrow\bot} \qquad (\Rightarrow\mathrm{E})\ \frac{A, n \Rightarrow\bot \quad A, n}{\bot}$$

We also adopt the following version of *reductio ad absurdum*:

$$(\mathrm{RAA}')\ \frac{\begin{matrix}(-A, n)\\ \bot\end{matrix}}{A, n}$$

The instances of inference rules (lab$\bot$), ($\star$labE1), ($\star$labE2), ($\star$labI) and

($\to$E), ($\to$E$'$) are specified as follows:

($\bot$lab): $(A, n/\bot, n)$,

($\star$labE1): $(A \star B, n/A, n)$,

($\star$labE2): $(A \star B, n/B, n)$,

($\star$labI): $(A, n, B, n/A \star B, n)$,

($\to$E): $(A \to B, A/B)$

($\to$E$'$): $(A \to B, A, n/B, n)$

($\Rightarrow$E): $(A, n \Rightarrow \bot, A, n/ \bot)$.

The instances of the other inference rules are specified in a similar way.

The instances of the deduction rules ($\to$I), ($\Rightarrow$I), and (RAA$'$) are specified as follows:

($\to$I): $\langle\langle \mathcal{X}, B\rangle, \langle \mathcal{Y}, B, n\rangle, \langle \mathcal{Z}, A \to B\rangle\rangle$, where $\mathcal{Z} = (\mathcal{X} \setminus \{A\}) \cup (\mathcal{Y} \setminus \{A, n\})$, $\mathsf{exl}(\mathcal{Y}) \cap (\mathsf{For} \times \{n\}) = \{A, n\}$ and $\mathsf{exl}(\mathcal{Y}) \cap \{C, n \Rightarrow\bot : C \in \mathsf{For} \text{ and } n \in \mathbb{N}\} = \emptyset$

($\Rightarrow$I): $\langle\langle \mathcal{X}, \bot\rangle, \langle \mathcal{Y}, A, n \Rightarrow\bot\rangle\rangle$, where $\mathcal{Y} = (\mathcal{X} \setminus \{A, n\})$

(RAA$'$): $\langle\langle \mathcal{X}, \bot\rangle, \langle \mathcal{Y}, A, n\rangle\rangle$, where $\mathcal{Y} = \mathcal{X} \setminus \{-A, n\}$.

The instances of the other deduction rules are specified in a similar way.

The definition of deduction is a simple modification of Prawitz's definition of deduction received by replacing variables ranging over sets of formulas and formulas by variables ranging over sets of d-expressions and d-expressions, respectively (see [18]). By a lab-deduction, we mean a deduction received by applications of lab-rules only.

We say that A is *deducible from* X *in* $\mathbf{DAI}_\bot$ ($X \vdash_{\mathbf{DAI}_\bot} A$) iff there is a deduction Π of A depending on X or a subset $Y \subset X$. As usual, we say that A is *deducible in* $\mathbf{DAI}_\bot$ iff $\emptyset \vdash_{\mathbf{DAI}_\bot} A$.

Let $\mathcal{X} \cup \{\mathcal{A}\} \subseteq \mathsf{labEx}$ be a finite non-empty set. We say $\mathcal{A}$ is *lab-deducible from* $\mathcal{X}$ *in* $\mathbf{DAI}_\bot$ ($\mathcal{X} \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\bot} \mathcal{A}$) iff there is a lab-deduction Π of $\mathcal{A}$ depending on $\mathcal{X}$.

The following facts show the sense of lab-rules.

Fact 1. For any $n \in \mathbb{N}$, for any $A, B \in \mathsf{For}$, there is a lab-deduction Π of B, n depending on A, n in $\mathbf{DAI}_\bot$ iff $\mathsf{var}(B) \subseteq \mathsf{var}(A)$.

Proof. ($\Rightarrow$) Let Π be a lab-deduction of B, n depending on A, n. We use an induction on the length of sub-deductions of Π depending on A, n, i.e., the length of sub-trees of Π which are deductions depending on A, n.

Base case. Obvious for sub-deductions of the length 1.

Induction hypothesis. Let $k \geq 1$. For any sub-deduction Π' of Π, if Π' is a lab-deduction of C, n depending on A, n in $\mathbf{DAI}_{\bot}$ of length not greater than k, then $\mathsf{var}(C) \subseteq \mathsf{var}(A)$.

Inductive step. Suppose Π' is a sub-deduction of Π of length $k+1$ and a lab-deduction of C, n depending on A, n in $\mathbf{DAI}_{\bot}$. Thus, Π' has the form $(\Pi_1, \ldots, \Pi_m / C, n)$, where $m \leq 2$ and for any $i \leq m$, Π_i is a deduction of A_i, n depending on A, n in $\mathbf{DAI}_{\bot}$. Hence, there is an instance of an auxiliary rule $(A_1, n, \ldots, A_m, n / C, n)$ such that, for any $i \leq m$, Π_i is a deduction of A_i, n depending on A, n in $\mathbf{DAI}_{\bot}$.

Let $(A_1, n, \ldots, A_m, n / C, n) = (D, n / \bot, n)$. Then, by inductive hypothesis we have: $\mathsf{var}(\bot) \subseteq \mathsf{var}(D) \subseteq \mathsf{var}(A)$.

Let $(A_1, n, \ldots, A_m, n / C, n) = (D \star E, n / D, n)$. Then, by inductive hypothesis we have: $\mathsf{var}(D) \subseteq \mathsf{var}(D) \cup \mathsf{var}(E) = \mathsf{var}(D \star E) \subseteq \mathsf{var}(A)$. Similarly, for $(A_1, n, \ldots, A_m, n / C, n) = (D \star E, n / E, n)$.

Let $(A_1, n, \ldots, A_m, n / C, n) = (D, n, E, n / (D \star E), n)$. Then, by inductive hypothesis $\mathsf{var}(D), \mathsf{var}(E) \subseteq \mathsf{var}(A)$ and so $\mathsf{var}(D \star E) \subseteq \mathsf{var}(D) \cup \mathsf{var}(E) \subseteq \mathsf{var}(A)$.

($\Leftarrow$) Let $A \in \mathsf{For}$. We prove the implication by an induction on the degree of formulas.

Base case. Let $B \in \mathsf{For}$ be a formula of degree 0.

Let $B = \bot$. Suppose $\mathsf{var}(B) \subseteq \mathsf{var}(A)$. And so $\mathsf{var}(B) = \emptyset \subseteq \mathsf{var}(A)$. We have:

$$(\mathrm{lab}\bot)\ \frac{A, n}{\bot, n}$$

Let $B \in \mathsf{Var}$. We use an induction on the degrees of subformulas of A.

+ *Base case.* Let $C \in \mathsf{sub}(A)$ be of degree 0. Suppose $\mathsf{var}(B) \subseteq \mathsf{var}(C)$. Then, of course, $B = C$.

+ *Induction hypothesis.* Let $k \geq 0$. For any $C \in \mathsf{sub}(A)$, if the degree of C is not greater than k and $\mathsf{var}(B) \subseteq \mathsf{var}(C)$, then there is a lab-deduction of B, n depending on C, n in $\mathbf{DAI}_{\bot}$.

+ *Inductive step.* Let $C \in \mathsf{sub}(A)$ be of degree $k+1$. Let $C = D \star E$, for some $D, E \in \mathsf{For}$ and $\star \in \{\wedge, \rightarrow\}$. Suppose $\mathsf{var}(B) \subseteq \mathsf{var}(C) = \mathsf{var}(D) \cup \mathsf{var}(E)$. Then $B \in \mathsf{var}(D) \neq \emptyset$ or $B \in \mathsf{var}(E) \neq \emptyset$. Suppose that the first case holds. (The second case can be considered in a similar way.) By inductive hypothesis, we have a lab-deduction $\mathit{\Pi}$ of B, n depending on D, n. We have:

$$
(\star\mathrm{labE}_1)\,\dfrac{C, n}{\dfrac{(D, n)}{\dfrac{\mathit{\Pi}}{B, n}}}
$$

Induction hypothesis. Let $k \geq 0$. For any $B \in \mathsf{For}$, if the degree of B is not greater than k and $\mathsf{var}(B) \subseteq \mathsf{var}(A)$, then there is a lab-deduction $\mathit{\Pi}$ of B, n depending on A, n in $\mathbf{DAI}_\perp$.

Inductive step. Let $B \in \mathsf{For}$ be of degree $k+1$ and $\mathsf{var}(B) \subseteq \mathsf{var}(A)$. Let $B = D \star E$, for some $D, E \in \mathsf{For}$ and $\star \in \{\wedge, \rightarrow\}$. By inductive hypothesis, we have lab-deductions $\mathit{\Pi}_1$ of D, n depending on A, n and $\mathit{\Pi}_2$ of E, n depending on A, n. Hence, we have:

$$
(\star\mathrm{labI})\,\dfrac{\dfrac{\dfrac{[A, n]}{\mathit{\Pi}_1}}{D, n} \qquad \dfrac{\dfrac{[A, n]}{\mathit{\Pi}_2}}{E, n}}{B, n}
$$

□

By Fact 1 we obtain the following corollaries:

Corollary 2. For any $n \in \mathbb{N}$, for all $A_1, \ldots, A_m, A \in \mathsf{For}$, $\{A_1, n, \ldots, A_m, n\} \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp} A, n$ iff $\mathsf{var}(A) \subseteq \bigcup_{i \leq m} \mathsf{var}(A_i)$.

Proof. We obviously have $\{A_1, n, \ldots, A_m, n\} \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp} A_1 \star (\ldots (A_{m-1} \star A_m) \ldots), n$ and for any $i \leq m$, $A_1 \star (\ldots (A_{m-1} \star A_m) \ldots), n \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp} A_i, n$. By Fact 1, $A_1 \star (\ldots (A_{n-1} \star A_m) \ldots), n \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp} A, n$ iff $\mathsf{var}(A) \subseteq \mathsf{var}(A_1 \star (\ldots (A_{n-1} \star A_m) \ldots)) = \bigcup_{i \leq m} \mathsf{var}(A_i)$. □

Corollary 3. For any $n \in \mathbb{N}$, for all $A_1, \ldots, A_m \in \mathsf{Var}$, for any $A \in \mathsf{For}$:

1. $A, n \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp} A_i, n$, for any $i \leq m$ iff $\{A_1, \ldots, A_m\} \subseteq \mathsf{var}(A)$,
2. $\{A_1, \ldots, A_m\}, n \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp} A, n$ iff $\mathsf{var}(A) = \{A_1, \ldots, A_m\}$.

Proof. For 1. Directly by Fact 1.

For 2. By Corollary 2, $\{A_1, \ldots, A_m\}, n \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp} A$ iff $\mathsf{var}(A) \subseteq \{A_1, \ldots, A_m\}$. If $\{A_1, \ldots, A_m\}, n \vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp} A, n$, then by the definition of $\vdash^{\mathrm{lab}}_{\mathbf{DAI}_\perp}$, since there is a deduction of A, n depending on $A_1, n, \ldots, A_m, n$, $A_1, \ldots, A_m \in \mathsf{var}(A)$. □

To demonstrate that our system allows us to derive appropriately modified rules presented in [12], consider the following counterparts of the rules $(\to\mathrm{I}_2)$, $(\to\mathrm{I}_3)$:

$$(\to\mathrm{I}')\ \dfrac{\begin{array}{c}(A)\\ B\end{array}\quad E\to C\quad C\to D\quad \begin{array}{c}(A,n)\\ E,n\end{array}\quad \begin{array}{c}(D,n)\\ B,n\end{array}}{A\to B}$$

where the deduction of E,n is from a non-empty set of assumptions and A,n is the only undischarged assumption with n in it; the deduction of B,n is from a non-empty set of assumptions and D,n is the only undischarged assumption with n in it.

$$(\to\mathrm{I}'')\ \dfrac{\begin{array}{c}(A)\\ B\end{array}\quad A\to C\quad A\to D\quad \begin{array}{c}(C\star D,n)\\ B,n\end{array}}{A\to B}$$

where the deduction of B,n is from a non-empty set of assumptions and $C\star D,n$ is the only undischarged assumption with n in it.

We have the following fact.

Fact 4. $(\to\mathrm{I}')$ and $(\to\mathrm{I}'')$ are derivable in the deductive system for $\mathbf{DAI}_\perp$.

Proof. For $(\to\mathrm{I}')$:

$$(\to\mathrm{I})\ \dfrac{\begin{array}{c}[A]\\ \varPi_1\\ \hline B\end{array}\qquad \begin{array}{c}(\to\mathrm{E}')\ \dfrac{C\to D\qquad (\to\mathrm{E}')\ \dfrac{E\to C\qquad \begin{array}{c}[A,n]\\ \varPi_2\\ \hline E,n\end{array}}{C,n}}{[D,n]}\\ \varPi_3\\ \hline B,n\end{array}}{A\to B}$$

For $(\to\mathrm{I}'')$:

$$(\to\mathrm{I})\ \dfrac{\begin{array}{c}[A]\\ \varPi_1\\ \hline B\end{array}\qquad \begin{array}{c}(\star\mathrm{labI})\ \dfrac{(\to\mathrm{E}')\ \dfrac{A\to C\qquad A,n}{C,n}\qquad (\to\mathrm{E}')\ \dfrac{A\to D\qquad A,n}{D,n}}{[(C\star D),n]}\\ \varPi_2\\ \hline B,n\end{array}}{A\to B}$$

□

3.3 Soundness and completeness

To prove soundness, we begin by introducing some auxiliary notions and proving a preliminary fact.

Consider a model $M = \langle v, s, \subseteq \rangle \in \mathbf{isa\text{-}E}$ and a function $f\colon \mathbb{N} \longrightarrow U$. For any $\mathcal{A} \in \mathsf{Ex}$, we say that $\mathcal{A}$ is *satisfied in* M *wrt* f ($M, f \mid\!\models \mathcal{A}$) iff the following conditions are satisfied for any $A \in \mathsf{For}$ and for any $n \in \mathbb{N}$:

- $+\ M \models A,\ \text{if } \mathcal{A} = A,$
- $+\ f(n) \in s(A),\ \text{if } \mathcal{A} = A, n,$
- $+\ f(n) \notin s(A),\ \text{if } \mathcal{A} = A, n \Rightarrow \perp.$

A function $f\colon \mathbb{N} \longrightarrow U$, for a given U, we call *c-assignment.*

Consider a model $M = \langle v, s, \subseteq \rangle \in \mathbf{isa\text{-}E}$ and a c-assignment $f\colon \mathbb{N} \longrightarrow U$. We say that M is *sound wrt* f *for a pair* $\langle \mathcal{X}, \mathcal{A} \rangle$ ($M, f \mid\!\models \langle \mathcal{X}, \mathcal{A} \rangle$) iff whenever $M, f \mid\!\models \mathcal{B}$ for every $\mathcal{B} \in \mathcal{X}$, then $M, f \mid\!\models \mathcal{A}$. Obviously, we have the following fact.

Fact 5. For any $X \cup \{A\} \subseteq \mathsf{For}$, $X \models_{\mathbf{isa\text{-}E}} A$ iff for any model $M \in \mathbf{isa\text{-}E}$, for any c-assignment f, $M, f \mid\!\models \langle X, A \rangle$.

Let us now proceed to the formulation and proof of the soundness lemma.

Lemma 6.
1. For any rule (R), if the instance of (R) is of the form $(\mathcal{A}_1, \ldots, \mathcal{A}_m / \mathcal{A})$, then for any model $M \in \mathbf{isa\text{-}E}$, for any c-assignment f, $M, f \mid\!\models \langle \{\mathcal{A}_i\}_{i \leq m}, \mathcal{A} \rangle$.
2. For any rule (R), if the instance of (R) is of the form $\langle \langle \mathcal{X}_1, \mathcal{A}_1 \rangle, \ldots, \langle \mathcal{X}_m, \mathcal{A}_m \rangle, \langle \mathcal{X}, \mathcal{A} \rangle \rangle$ and if for any $i \leq m$, for any model $M \in \mathbf{isa\text{-}E}$, for any c-assignment f, $M, f \mid\!\models \langle \mathcal{X}_i, \mathcal{A}_i \rangle$, then for any model $M \in \mathbf{isa\text{-}E}$, for any c-assignment f, $M, f \mid\!\models \langle \mathcal{X}, \mathcal{A} \rangle$.

Proof. Let us consider only selected rules; for others, we reason similarly.

For ($\perp$lab), ($\star$labE1), ($\star$labE2) and ($\star$labI). Consider the instance of a rule of the form $(A_1, n, \ldots, A_m, n / A, n)$. Obviously, we have $\{A_1, n, \ldots, A_m, n\} \vdash^{\mathrm{lab}}_{\mathbf{DAI}_{\perp}} A, n$. By Corollary 3.2 $\mathsf{var}(A) \subseteq \bigcup_{i \leq m} \mathsf{var}(A_i)$. Thus, for any model $M = \langle v, s, \subseteq \rangle \in \mathbf{isa\text{-}E}$, $\bigcap_{i \leq m} s(A_i) \subseteq s(A)$. If $f(n) \in \bigcap_{i \leq m} s(A_i)$, then for any $B \in \bigcup_{i \leq m} \mathsf{var}(A_i)$, $f(n) \in s(B)$. Hence, for any $B \in \mathsf{var}(A)$, $f(n) \in s(B)$, so $f(n) \in s(A)$.

For ($\to$I). Assume for any model M, for any c-assignment f, $M, f \mid\!\models \langle \mathcal{X}, B \rangle$, and $M, f \mid\!\models \langle \mathcal{Y}, B, n \rangle$. Consider any model $N = \langle v, s, \subseteq \rangle \in \mathbf{isa\text{-}E}$ and any c-assignment g and assume $N, g \mid\!\models \mathcal{Z}$.

Since $\mathcal{X} \setminus \{A\} \subseteq \mathcal{Z}$, so if $N, g \VDash A$, then $N, g \VDash B$.

Assume $x \in s(A)$. We show that $x \in s(B)$. We define a function $h\colon \mathbb{N} \longrightarrow U$ in the following way for any $m \in \mathbb{N}$:

$$h(m) = \begin{cases} x, & \text{if } m = n \\ g(m), & \text{if } m \neq n \end{cases}$$

Thus, $N, h \VDash A, n$. Since $\mathcal{Y} \setminus \{A, n\} \subseteq \mathcal{Z}$, $\mathsf{exl}(\mathcal{Y}) \cap (\mathsf{For} \times \{n\}) = \{A, n\}$ and $\mathsf{exl}(\mathcal{Y}) \cap \{C, n \Rightarrow\bot : C \in \mathsf{For} \text{ and } n \in \mathbb{N}\} = \emptyset$, we also have $N, h \VDash \mathcal{Y}$. Hence, $N, h \VDash B, n$. Therefore, $x = h(n) \in s(B)$.

Combining both obtained facts, we have $N, g \VDash A \to B$.

For ($\Rightarrow$I). Assume that for any model M, for any c-assignment f, $M, f \VDash \langle \mathcal{X}, \bot\rangle$. Consider any model $N = \langle v, s, \subseteq\rangle \in$ **isa-E** and c-assignment g, and assume $N, g \VDash \mathcal{Y}$. If $g(n) \in s(A)$, then since $N, g \VDash A, n$ and $\mathcal{Y} = \mathcal{X} \setminus \{A, n\}$, we also have $N, g \VDash \bot$.

For (RAA$'$). Assume that for any model M, for any c-assignment f, $M, f \VDash \langle \mathcal{X}, \bot\rangle$. Consider any model $N = \langle v, s, \subseteq\rangle \in$ **isa-E** and c-assignment g, and assume $N, g \VDash \mathcal{Y}$. If $g(n) \notin s(A)$, then $N, g \VDash -A, n$. Thus, since $\mathcal{Y} = \mathcal{X} \setminus \{-A, n\}$, we would also have $N, g \VDash \bot$. ☐

By Lemma 6 we obtain the following corollary.

Corollary 7. If there is a deduction Π of $\mathcal{A}$ depending on $\mathcal{X}$ in $\mathbf{DAI}_\bot$, then for any model $M = \langle v, s, \subseteq\rangle \in$ **isa-E**, for any c-assignment f, $M, f \VDash \langle \mathcal{X}, A\rangle$.

We now show that, in the system under consideration, every axiom of $\mathbf{DAI}_\bot$ is deducible. This provides a lemma for the completeness theorem.

Lemma 8.
1. All formulas of the form of (CL1)–(CL3) are deducible in $\mathbf{DAI}_\bot$.
2. All formulas of the form of (DAI), (DAI1)–(DAI5) are deducible in $\mathbf{DAI}_\bot$.

Proof. In what follows, we present only deductions of characteristic axioms.

For (DAI).

($\to$) Let Π_1 be the following deduction:

$$(\to\mathrm{E})\,\dfrac{A \to B \qquad (\wedge\mathrm{E}_1)\,\dfrac{(\mathrm{RAA})\,\dfrac{(\to\mathrm{E})\,\dfrac{\neg\neg(A \wedge \neg B) \qquad \neg(A \wedge \neg B)}{\bot}}{A \wedge \neg B}}{A}}{B}$$

Let $\mathit{\Pi}_2$ be the following deduction:

$$(\to\mathrm{E})\,\dfrac{\neg\neg(A\wedge\neg B)\qquad\qquad \neg(A\wedge\neg B)}{(\mathrm{RAA})\,\dfrac{\bot}{(\wedge\mathrm{E}_1)\,\dfrac{A\wedge\neg B}{\neg B}}}$$

Let $\mathit{\Pi}_3$ be the following deduction:

$$(\to\mathrm{I})\,\dfrac{(\to\mathrm{I})\,\dfrac{B\qquad B,1}{B\to B}\qquad (\to\mathrm{labI})\,\dfrac{(\to\mathrm{E}')\,\dfrac{A\to B\qquad A,1}{B,1}\qquad (\to\mathrm{E}')\,\dfrac{A\to B\qquad A,1}{B,1}}{B\to B,1}}{A\looparrowright B}$$

Let $\mathit{\Pi}_4$ be the following deduction:

$$(\to\mathrm{labI})\,\dfrac{(\wedge\mathrm{labI})\,\dfrac{(\to\mathrm{labE}_1)\,\dfrac{A\to B,1}{A,1}\qquad (\to\mathrm{labI})\,\dfrac{(\to\mathrm{labE}_2)\,\dfrac{A\to B,1}{B,1}\qquad (\bot\mathrm{lab})\,\dfrac{A\to B,1}{\bot,1}}{\neg B,1}}{A\wedge\neg B,1}\qquad (\bot\mathrm{lab})\,\dfrac{A\to B,1}{\bot,1}}{\neg(A\wedge\neg B),1}$$

and let $\mathit{\Pi}_5$ be the following deduction:

$$(\to\mathrm{labI})\,\dfrac{(\to\mathrm{labE}_1)\,\dfrac{A\to B,1}{A,1}\qquad (\to\mathrm{labI})\,\dfrac{(\to\mathrm{labE}_2)\,\dfrac{A\to B,1}{B,1}\qquad (\to\mathrm{labE}_2)\,\dfrac{A\to B,1}{B,1}}{B\to B,1}}{A\looparrowright B,1}$$

Finally, we have the following deduction:

$$(\to\mathrm{I})\,\dfrac{(\wedge\mathrm{I})\,\dfrac{(\mathrm{RAA}_1)\,\dfrac{(\to\mathrm{E})\,\dfrac{\mathit{\Pi}_1\qquad \mathit{\Pi}_2}{\bot}}{\neg(A\wedge\neg B)}\qquad \mathit{\Pi}_3}{\neg(A\wedge\neg B)\wedge(A\looparrowright B)}\qquad (\wedge\mathrm{labI})\,\dfrac{\mathit{\Pi}_4\qquad \mathit{\Pi}_5}{\neg(A\wedge\neg B)\wedge(A\looparrowright B),1}}{(A\to B)\to(\neg(A\wedge\neg B)\wedge(A\looparrowright B))}$$

($\leftarrow$) Let $\mathit{\Pi}_1$ be the following deduction:

$$(\mathrm{RAA}_1)\,\dfrac{(\to\mathrm{E})\,\dfrac{(\wedge\mathrm{E}_1)\,\dfrac{\neg(A\wedge\neg B)\wedge(A\looparrowright B)}{\neg(A\wedge\neg B)}\qquad (\wedge\mathrm{I})\,\dfrac{A\qquad \neg(B)}{A\wedge\neg B}}{\bot}}{B}$$

and let $\mathit{\Pi}_2$ be the following deduction:

$$(\to\mathrm{I})\ \dfrac{\Pi_1 \qquad (\to\mathrm{labE}_1)\ \dfrac{(\to\mathrm{E}')\ \dfrac{(\wedge\mathrm{E}_2)\ \dfrac{\neg(A\wedge\neg B)\wedge(A\looparrowright B)}{A\looparrowright B} \qquad A,1}{B\to B,1}}{B,1}}{A\to B}$$

Finally, we have the following deduction:

$$(\to\mathrm{I})\ \dfrac{\Pi_2 \qquad (\to\mathrm{labI})\ \dfrac{(\to\mathrm{labE}_1)\ \dfrac{(\wedge\mathrm{labE}_2)\ \dfrac{\neg(A\wedge\neg B)\wedge(A\looparrowright B),1}{A\looparrowright B,1}}{A,1} \qquad (\to\mathrm{labE}_2)\ \dfrac{(\to\mathrm{labE}_2)\ \dfrac{(\wedge\mathrm{labE}_2)\ \dfrac{\neg(A\wedge\neg B)\wedge(A\looparrowright B),1}{A\looparrowright B,1}}{B\to B,1}}{B,1}}{A\to B,1}}{(\neg(A\wedge\neg B)\wedge(A\looparrowright B))\to(A\to B)}$$

For (DAI1). Let Π_1 be the following deduction:

$$(\to\mathrm{E}')\ \dfrac{(\wedge\mathrm{E}_2)\ \dfrac{(A\looparrowright B)\wedge(B\looparrowright C)}{B\looparrowright C} \qquad (\to\mathrm{labE}_1)\ \dfrac{(\to\mathrm{E}')\ \dfrac{(\wedge\mathrm{E}_1)\ \dfrac{(A\looparrowright B)\wedge(B\looparrowright C)}{A\looparrowright B} \qquad A,1}{B\to B,1}}{B,1}}{C,1}$$

and let Π_2 be the following deduction:

$$(\to\mathrm{I})\ \dfrac{(\to\mathrm{I})\ \dfrac{C \qquad C,1}{C\to C} \qquad (\to\mathrm{labI})\ \dfrac{\Pi_1 \qquad \Pi_1}{C\to C,1}}{A\looparrowright C}$$

We have the following deduction:

$$(\to\mathrm{I})\ \dfrac{\Pi_2 \qquad (\to\mathrm{labI})\ \dfrac{(\to\mathrm{labE}_1)\ \dfrac{(\wedge\mathrm{labE}_1)\ \dfrac{(A\looparrowright B)\wedge(B\looparrowright C),1}{A\looparrowright B,1}}{A,1} \qquad (\to\mathrm{labE}_2)\ \dfrac{(\wedge\mathrm{labE}_2)\ \dfrac{(A\looparrowright B)\wedge(B\looparrowright C),1}{B\looparrowright C,1}}{C\to C,1}}{A\looparrowright C,1}}{((A\looparrowright B)\wedge(B\looparrowright C))\to(A\looparrowright C)}$$

For (DAI2) and (DAI3) the deductions are similar. For (DAI2) we have the following deduction:

$$(\to\mathrm{I})\ \dfrac{(\to\mathrm{I})\ \dfrac{\neg A \qquad \neg A,1}{\neg A\to\neg A} \qquad (\to\mathrm{labI})\ \dfrac{(\to\mathrm{labI})\ \dfrac{A,1 \qquad (\bot\mathrm{lab})\ \dfrac{A,1}{\bot,1}}{\neg A,1} \qquad (\to\mathrm{labI})\ \dfrac{A,1 \qquad (\bot\mathrm{lab})\ \dfrac{A,1}{\bot,1}}{\neg A,1}}{\neg A\to\neg A,1}}{A\to(\neg A\to\neg A)}$$

For (DAI4) and (DAI5) the deductions are similar.

($\rightarrow$) Let $\varPi_1$ be the following deduction:

$$(\star\mathrm{labE}_1)\,\dfrac{(\rightarrow\mathrm{labE}_1)\,\dfrac{(\rightarrow\mathrm{E}')\,\dfrac{A \looparrowright (B\star C) \qquad A,1}{(B\star C)\rightarrow(B\star C),1}}{B\star C,1}}{B,1}$$

and let $\varPi_2$ be the following one:

$$(\star\mathrm{labE}_2)\,\dfrac{(\rightarrow\mathrm{labE}_1)\,\dfrac{(\rightarrow\mathrm{E}')\,\dfrac{A \looparrowright (B\star C) \qquad A,1}{(B\star C)\rightarrow(B\star C),1}}{B\star C,1}}{C,1}$$

Moreover, let $\varPi_3$ be the following deduction:

$$(\wedge\mathrm{I})\,\dfrac{(\rightarrow\mathrm{I})\,\dfrac{(\rightarrow\mathrm{I})\,\dfrac{B \qquad B,1}{B\rightarrow B} \qquad (\rightarrow\mathrm{labI})\,\dfrac{\varPi_1 \qquad \varPi_1}{B\rightarrow B,1}}{A\looparrowright B} \qquad (\rightarrow\mathrm{I})\,\dfrac{(\rightarrow\mathrm{I})\,\dfrac{C \qquad C,1}{C\rightarrow C} \qquad (\rightarrow\mathrm{I})\,\dfrac{\varPi_2 \qquad \varPi_2}{C\rightarrow C,1}}{A\looparrowright C}}{(A\looparrowright B)\wedge(A\looparrowright C)}$$

Let $\varPi_4$ be the following deduction:

$$(\rightarrow\mathrm{labI})\,\dfrac{(\star\mathrm{E}_1)\,\dfrac{(\rightarrow\mathrm{labE}_2)\,\dfrac{(\rightarrow\mathrm{labE}_2)\,\dfrac{A\looparrowright(B\star C),1}{(B\star C)\rightarrow(B\star C),1}}{B\star C,1}}{B,1} \qquad (\star\mathrm{E}_1)\,\dfrac{(\rightarrow\mathrm{labE}_2)\,\dfrac{(\rightarrow\mathrm{labE}_2)\,\dfrac{A\looparrowright(B\star C),1}{(B\star C)\rightarrow(B\star C),1}}{B\star C,1}}{B,1}}{B\rightarrow B,1}$$

and $\varPi_5$ be the following deduction:

$$(\rightarrow\mathrm{labI})\,\dfrac{(\star\mathrm{labE}_2)\,\dfrac{(\rightarrow\mathrm{labE}_2)\,\dfrac{(\rightarrow\mathrm{labE}_2)\,\dfrac{A\looparrowright(B\star C),1}{(B\star C)\rightarrow(B\star C),1}}{B\star C,1}}{C,1} \qquad (\star\mathrm{labE}_2)\,\dfrac{(\rightarrow\mathrm{labE}_2)\,\dfrac{(\rightarrow\mathrm{labE}_2)\,\dfrac{A\looparrowright(B\star C),1}{(B\star C)\rightarrow(B\star C),1}}{B\star C,1}}{C,1}}{C\rightarrow C,1}$$

Moreover, let $\varPi_6$ be the following deduction:

$$(\wedge\mathrm{labI})\,\dfrac{(\rightarrow\mathrm{labI})\,\dfrac{(\rightarrow\mathrm{labE}_1)\,\dfrac{A\looparrowright(B\star C),1}{A,1} \qquad \varPi_4}{A\looparrowright B,1} \qquad (\rightarrow\mathrm{labI})\,\dfrac{(\rightarrow\mathrm{labE}_1)\,\dfrac{A\looparrowright(B\star C),1}{A,1} \qquad \varPi_5}{A\looparrowright C,1}}{(A\looparrowright B)\wedge(A\looparrowright C),1}$$

Finally we get the following deduction:

$$(\to\mathrm{I})\ \dfrac{\Pi_3 \qquad \Pi_6}{(A \looparrowright (B \star C)) \to ((A \looparrowright B) \wedge (A \looparrowright C))}$$

($\leftarrow$) Let Π_1 be the following deduction:

$$(\to\mathrm{labE}_1)\ \dfrac{(\to\mathrm{E}')\ \dfrac{(\wedge\mathrm{E}_1)\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C)}{A \looparrowright B} \qquad A, 1}{B \to B, 1}}{B, 1}$$

and let Π_2 be the following deduction:

$$(\to\mathrm{labE}_1)\ \dfrac{(\to\mathrm{E}')\ \dfrac{(\wedge\mathrm{E}_2)\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C)}{A \looparrowright C} \qquad A, 1}{C \to C, 1}}{C, 1}$$

then Π_3 is the following deduction

$$(\to\mathrm{I})\ \dfrac{(\to\mathrm{I})\ \dfrac{B \star C \qquad B \star C, 1}{(B \star C) \to (B \star C)} \qquad (\to\mathrm{labI})\ \dfrac{(\star\mathrm{labI})\ \dfrac{\Pi_1 \qquad \Pi_2}{B \star C, 1} \qquad (\star\mathrm{labI})\ \dfrac{\Pi_1 \qquad \Pi_2}{B \star C, 1}}{(B \star C) \to (B \star C), 1}}{A \looparrowright (B \star C)}$$

Let Π_4 be the following deduction:

$$\dfrac{(\to\mathrm{labE}_1)\ \dfrac{(\to\mathrm{labE}_2)\ \dfrac{(\wedge\mathrm{labE}_1)\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C), 1}{A \looparrowright B, 1}}{B \to B, 1}}{B, 1} \qquad (\to\mathrm{labE}_1)\ \dfrac{(\to\mathrm{labE}_2)\ \dfrac{(\wedge\mathrm{labE}_2)\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C), 1}{A \looparrowright C, 1}}{C \to C, 1}}{C, 1}}{B \star C, 1}$$

then Π_5 is the following deduction:

$$(\to\mathrm{labI})\ \dfrac{(\to\mathrm{labE}_1)\ \dfrac{(\wedge\mathrm{labE}_1)\ \dfrac{(A \looparrowright B) \wedge (A \looparrowright C), 1}{A \looparrowright B, 1}}{A, 1} \qquad (\to\mathrm{labI})\ \dfrac{\Pi_4 \qquad \Pi_4}{(B \star C) \to (B \star C), 1}}{A \looparrowright (B \star C), 1}$$

Finally, we receive the following deduction:

$$(\to\mathrm{I})\ \dfrac{\Pi_3 \qquad \Pi_5}{((A \looparrowright B) \wedge (A \looparrowright C)) \to (A \looparrowright (B \star C))}$$

□

By Lemmas 6 and 8, we obtain the following theorem.

Theorem 9. For all $X \cup \{A\} \subseteq \mathsf{For}$, $X \vdash_{\mathbf{DAI}_\bot} A$ iff $X \models_{\textbf{isa-E}} A$.

3.4 Normalisation

In order to prove the normalisation theorem, we first assume the following reductions of deductions at a given d-expression:

- $\star$-lab-reduction:

$$\begin{array}{c}(\star\text{labE}_i)\,\dfrac{(\star\text{labI})\,\dfrac{\dfrac{\Sigma_1}{A_1, n} \qquad \dfrac{\Sigma_2}{A_2, n}}{A_1 \star A_2, n}}{(A_i, n)}\\ \Pi\end{array} \quad \curvearrowright \quad \begin{array}{c}\dfrac{\Sigma_i}{(A_i, n)}\\ \Pi\end{array}$$

- $\wedge$-reduction:

$$\begin{array}{c}(\wedge\text{E}_i)\,\dfrac{(\wedge\text{I})\,\dfrac{\dfrac{\Sigma_1}{A_1} \qquad \dfrac{\Sigma_2}{A_2}}{A_1 \wedge A_2}}{(A_i)}\\ \Pi\end{array} \quad \curvearrowright \quad \begin{array}{c}\dfrac{\Sigma_i}{(A_i)}\\ \Pi\end{array}$$

- $\rightarrow$-reduction-1:

$$\begin{array}{c}(\rightarrow\text{E})\,\dfrac{(\rightarrow\text{I})\,\dfrac{\begin{array}{c}[A]\\ \dfrac{\Sigma_1}{B}\end{array} \qquad \begin{array}{c}[A, n]\\ \dfrac{\Sigma_2}{B, n}\end{array}}{A \rightarrow B} \qquad \dfrac{\Sigma_3}{A}}{(B)}\\ \Pi\end{array} \quad \curvearrowright \quad \begin{array}{c}\dfrac{\Sigma_3}{[A]}\\ \dfrac{\Sigma_1}{(B)}\\ \Pi\end{array}$$

- $\rightarrow$-reduction-2:

$$\begin{array}{c}(\rightarrow\text{E}')\,\dfrac{(\rightarrow\text{I})\,\dfrac{\begin{array}{c}[A]\\ \dfrac{\Sigma_1}{B}\end{array} \qquad \begin{array}{c}[A, n]\\ \dfrac{\Sigma_2}{B, n}\end{array}}{A \rightarrow B} \qquad \dfrac{\Sigma_3}{A, m}}{(B, m)}\\ \Pi\end{array} \quad \curvearrowright \quad \begin{array}{c}\dfrac{\Sigma_3}{[A, m]}\\ \dfrac{\Sigma_2[n/m]}{(B, m)}\\ \Pi\end{array}$$

where we put:

  - if $n \neq m$, then $\Sigma_2[n/m]$ is a result of 1) replacing in Σ_2 the label m in all the labelled formulas by a new number that did not appear in Σ_2 or Σ_3, receiving in sequence Π_2' and 2) replacing in Π_2' the label n in all the labelled formulas by m,
  - if $n = m$, then $\Sigma_2[n/m] = \Sigma_2$.

+ $\Rightarrow$-reduction:

$$\begin{array}{c}(\Rightarrow\mathrm{E})\;\dfrac{(\Rightarrow\mathrm{I})\;\dfrac{\begin{array}{c}[A,n]\\ \Sigma_1\\ \hline \bot\end{array}}{-A,n}\qquad \dfrac{\Sigma_2}{A,n}}{(\bot)}\\ \Pi\end{array}\qquad\curvearrowright\qquad\begin{array}{c}\Sigma_2\\ \hline [A,n]\\ \Sigma_1\\ \hline (\bot)\\ \Pi\end{array}$$

Let us now proceed to show that the applications of the rules (RAA) and (RAA′) within deduction can be restricted to cases where their consequences are a propositional variable and a labelled variable, respectively. First, we consider the restriction of the application of (RAA). Second, focusing on deductions in which the consequences of (RAA) are variables, we consider the restriction of the application of (RAA′).

Fact 10. 1. If there is a deduction in $\mathbf{DAI}_\bot$ of A from X, then there is a deduction in $\mathbf{DAI}_\bot$ of A from X in which the consequence of every application of (RAA) is a variable.

2. If there is a deduction in $\mathbf{DAI}_\bot$ of A from X in which the consequence of every application of (RAA) is a variable, then there is a deduction in $\mathbf{DAI}_\bot$ of A from X in which the consequence of every application of (RAA′) is a labelled variable.

Proof. For 1. Let Π be a deduction of A from X in $\mathbf{DAI}_\bot$. Let n be the highest degree of consequence of application of (RAA$_1$) in Π. Consider $B \in \mathsf{For}$ such that:

+ B is consequence of the application of (RAA) in Π,

+ for any $C \in \mathsf{For}$, if C is above B and C is a consequence of the application of (RAA) in Π, then the degree of C is smaller than n.

Thus, Π has the following form:

$$
\begin{array}{c}
(\mathrm{RAA})\ \dfrac{\dfrac{\begin{matrix}[\neg B]\\ \Sigma\end{matrix}}{\bot}}{(B)}\\
\Pi'
\end{array}
$$

where in Σ there is no undischarged assumption with a label.

Let $B = C \wedge D$, for some $C, D \in \mathsf{For}$. Then, we use the standard method to eliminate the application of (RAA).

Let $B = C \to D$, for some $C, D \in \mathsf{For}$. Then, we modify the standard method to eliminate the application of (RAA) for formulas with implication as the main connective. Let Π_1 be the following deduction:

$$
(\mathrm{RAA})\ \dfrac{\dfrac{\begin{matrix}
(\to\mathrm{I})\ \dfrac{(\to\mathrm{E})\ \dfrac{(\to\mathrm{E})\ \dfrac{C \to D \qquad C}{D} \qquad \neg D}{\bot} \qquad (\bot\mathrm{lab})\ \dfrac{C \to D, 1}{\bot, 1}}{[\neg(C \to D)]}\\
\Sigma
\end{matrix}}{\bot}}{D}
$$

and let Π_2 be the following deduction:

$$
(\mathrm{RAA}')\ \dfrac{\dfrac{\dfrac{(\to\mathrm{I})\ \dfrac{(\Rightarrow\mathrm{E})\ \dfrac{(\to\mathrm{E}')\ \dfrac{C \to D \qquad C, 1}{D, 1} \qquad -D, 1}{\bot} \qquad (\bot\mathrm{lab})\ \dfrac{C \to D, 1}{\bot, 1}}{[\neg(C \to D)]}}{\Sigma}}{\bot}}{D, 1}
$$

Then we have:

$$
\begin{array}{c}
(\to\mathrm{I})\ \dfrac{\Pi_1 \qquad \Pi_2}{(C \to D)}\\
\Pi'
\end{array}
$$

For 2. Let Π be a deduction of A from X in $\mathbf{DAI}_{\bot}$ in which the consequence of every application of (RAA) is a variable. Let i be the highest degree of consequence of application of (RAA$'$) in Π. Consider $B \in \mathsf{For}$ such that:

+ B, n is a consequence of the application of (RAA$'$) in Π,

+ for any $C \in \mathsf{For}$, if C, m is above B, n and C, m is a consequence of the application of (RAA$'$) in Π, then the degree of C is smaller than i.

Then Π has the following form:

$$\begin{array}{c} \text{(RAA}')\ \dfrac{\dfrac{\begin{array}{c}[-B,n]\\ \Sigma\end{array}}{\bot}}{(B,n)} \\ \Pi' \end{array}$$

Let $B = C \star D$, for some $C, D \in \mathsf{For}$. We use the following method to eliminate the application of (RAA′).

$$\begin{array}{c} (\star\text{labI})\ \dfrac{\text{(RAA}')\ \dfrac{\dfrac{\Sigma}{\bot}}{C,n} \quad\quad \text{(RAA}')\ \dfrac{\dfrac{\Sigma}{\bot}}{D,n}}{(C\star D,n)} \\ \Pi' \end{array}$$

where the two occurrences of Σ stand under, respectively,

$$\dfrac{(\Rightarrow\text{I})\ \dfrac{(\Rightarrow\text{E})\ \dfrac{(\star\text{labE}_1)\ \dfrac{C\star D,n}{C,n} \qquad -C,n}{\bot}}{[-C\star D,n]}}{\Sigma} \qquad\qquad \dfrac{(\Rightarrow\text{I})\ \dfrac{(\Rightarrow\text{E})\ \dfrac{(\star\text{labE}_2)\ \dfrac{C\star D,n}{D,n} \qquad -D,n}{\bot}}{[-C\star D,n]}}{\Sigma}$$

□

Thus, we can prove the following theorem in a standard way:

Theorem 11. If $X \vdash_{\mathbf{DAI}_\bot} A$, then there is a normal deduction in $\mathbf{DAI}_\bot$ of A from X.

4 The single-labelled system of deduction for $\mathbf{DAI}_\bot$

When defining the concept of a d-expression in the previous section, we considered formulas labelled with natural numbers. Now we will present a system in which the labelled formula will contain only one label of the form $\bullet$. In addition, we eliminate the symbol $\Rightarrow$. This last modification requires the introduction of new rules that will play the role of (RAA′) if we want to prove the normalisation theorem.

4.1 Expressions in deduction

In the system presented below, deductions consist of formulas labelled with $\bullet$; we again call them d-expressions. Often, instead of $\langle A, \bullet \rangle$ we write just $A^\bullet$. The set of d-expressions is again denoted by Ex. For any $X \subseteq \mathsf{For}$, $X^\bullet = \{A^\bullet \in \mathsf{Ex} : A \in X\}$. As before, we use $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \ldots$ as variables ranging over the sets of d-expressions and the d-expressions, respectively. In addition, we will use the following functions $\mathsf{for}(\mathcal{X}) = \{A \in \mathsf{For} : A \in \mathcal{X}\}$ and $\mathsf{for}^\bullet(\mathcal{X}) = \{A \in \mathsf{For} : A^\bullet \in \mathcal{X}\}$.

4.2 Rules and deductions

The first type of rules in our system are classical rules. We adopt exactly the same rules as in the previous section. In the case of the rule (RAA), we modify the given constraints associated with the labelled formulas accordingly.

The second type of rules in our system are auxiliary rules, which we will call $\bullet$-rules. They are a simple modification of the lab-rules presented in the previous section:

$$(\bot\bullet)\ \frac{A^\bullet}{\bot^\bullet}$$

$$(\wedge\bullet\mathrm{E}_1)\ \frac{(A \wedge B)^\bullet}{A^\bullet} \qquad (\wedge\bullet\mathrm{E}_2)\ \frac{(A \wedge B)^\bullet}{B^\bullet} \qquad (\wedge\bullet\mathrm{I})\ \frac{A^\bullet\ \ B^\bullet}{(A \wedge B)^\bullet}$$

$$(\rightarrow\bullet\mathrm{E}_1)\ \frac{(A \rightarrow B)^\bullet}{A^\bullet} \qquad (\rightarrow\bullet\mathrm{E}_2)\ \frac{(A \rightarrow B)^\bullet}{B^\bullet} \qquad (\rightarrow\bullet\mathrm{I})\ \frac{A^\bullet\ \ B^\bullet}{(A \rightarrow B)^\bullet}$$

Of course, as before, we will be able to show that $\mathsf{var}(A) \subseteq \mathsf{var}(B)$ iff A is deducible from B using $\bullet$-rules only (see Fact 1).

Now let us move on to the rules for implication. We have one rule that introduces implication.

$$(\rightarrow\mathrm{I}^\bullet)\ \frac{\begin{array}{cc}(A) & (A^\bullet)\\ B & B^\bullet\end{array}}{A \rightarrow B}$$

In the case of the rule ($\rightarrow$I$^\bullet$) the deduction of $B^\bullet$ depends on a set containing $A^\bullet$ as the only d-expression with $\bullet$.

Only one implication elimination rule, ($\rightarrow$E), is adopted as primary; the second, ($\rightarrow$E$'$), is used solely in its modified form, given below.

$$(\rightarrow\mathrm{E}^\bullet)\ \frac{A \rightarrow B \quad A^\bullet}{B^\bullet}$$

Instead of (→E$^\bullet$) we adopt the following so-called mix-rules:

$$(\mathrm{Mix}_1)\ \dfrac{\begin{array}{c}(\neg(A\to B))\\ \bot\end{array}\qquad A_1^\bullet\ \ldots\ A_n^\bullet}{C^\bullet}$$

$$\begin{array}{l}\text{where } \mathsf{var}(A)=\{A_1,\ldots,A_n\}\\ \text{and } C\in\mathsf{var}(B)\end{array}$$

$$(\mathrm{Mix}_2)\ \dfrac{\begin{array}{c}(\neg(\bot\to B))\\ \bot\end{array}\qquad \bot^\bullet}{C^\bullet}$$

$$\text{where } C\in\mathsf{var}(B)$$

In what follows, we will sometimes refer to one of the two mix-rules using the notation (Mix). Given the context, this reference will not be ambiguous.

However, a valid question arises: why don't we adopt (→E$^\bullet$) in our system instead of mix-rules? To obtain a complete system, we could use the more natural rule (→E$^\bullet$) instead of mix-rules. The problem, however, is the possibility of a non-eliminable maximal formula appearing. Consider the following deduction $r^\bullet$ depending on $p, \neg(p\wedge\neg(q\to r)), q^\bullet$:

$$(\to\mathrm{E}^\bullet)\ \dfrac{(\mathrm{RAA})\ \dfrac{(\to\mathrm{E})\ \dfrac{\neg(p\wedge\neg(q\to r))\qquad (\wedge\mathrm{I})\ \dfrac{p\qquad \neg(q\to r)}{p\wedge\neg(q\to r)}}{\bot}}{q\to r}\qquad q^\bullet}{r^\bullet}$$

Let us also note that instead of (Mix$_1$) and (Mix$_2$) we could introduce one rule which is a kind of general form of mix-rules:

$$(\mathrm{Mix}+)\ \dfrac{\begin{array}{c}(\neg(A\to B))\\ \bot\end{array}\qquad A^\bullet}{B^\bullet}$$

The approach using two mix-rules appears more natural when addressing the normalisation problem. This is because the premise appearing in place of $A^\bullet$ in (Mix+), after being introduced by the $\bullet$-rules and then, in a sense, eliminated using (Mix+), has the character of a maximal formula.

Let us again adopt the formal definitions of the analysed concepts such as rules, deduction, etc. following Prawitz, but instead of taking formulas into account, we take d-expressions (see [18]).

We will now define the form of an instance of the different rules. As before, we will focus only on the selected rules.

The instances of inference rules ($\bot\bullet$), ($\star\bullet$E1), ($\star\bullet$E2), ($\star\bullet$I) and ($\to$E) are specified as follows:

($\bot\bullet$): $(A^\bullet/\ \bot^\bullet)$,

($\star\bullet$E1): $((A \star B)^\bullet/A^\bullet)$,

($\star\bullet$E2): $((A \star B)^\bullet/B^\bullet)$,

($\star\bullet$I): $(A^\bullet, B^\bullet/(A \star B)^\bullet)$,

($\to$E): $((A \to B), A/B)$.

The instances of the other inference rules are specified in a similar way.

The instances of the deduction rules ($\to$I$^\bullet$), (Mix$_1$), and (Mix$_2$) are specified as follows:

($\to$I$^\bullet$): $\langle\langle\mathcal{X}, B\rangle, \langle\mathcal{Y}, B^\bullet\rangle, \langle\mathcal{Z}, (A \to B)\rangle\rangle$, where $\mathcal{Z} = (\mathcal{X} \setminus \{A\}) \cup (\mathcal{Y} \setminus \{A^\bullet\})$ and $\mathsf{for}^\bullet(\mathcal{Y}) = \{A\}$,

(Mix$_1$): $\langle\langle\mathcal{X}, \bot\rangle, \langle\mathcal{Y}_1, A_1^\bullet\rangle, \ldots, \langle\mathcal{Y}_n, A_n^\bullet\rangle, \langle\mathcal{Z}, C^\bullet\rangle\rangle$, where $\mathcal{Z} = (\mathcal{X} \setminus \{\neg(A \to B)\}) \cup \bigcup_{i\leq n} \mathcal{Y}_i$, $\mathsf{var}(A) = \{A_1, \ldots, A_n\}$, and $C \in \mathsf{var}(B)$,

(Mix$_2$): $\langle\langle\mathcal{X}, \bot\rangle, \langle\mathcal{Y}, \bot^\bullet\rangle, \langle\mathcal{Z}, C^\bullet\rangle\rangle$, where $\mathcal{Z} = (\mathcal{X} \setminus \{\neg(\bot\to B)\}) \cup \mathcal{Y}$ and $C \in \mathsf{var}(B)$.

The instances of the other deduction rules are specified in a similar way.

As before, the definition of deduction is a simple modification of Prawitz's definition of deduction received by replacing variables ranging over sets of formulas and formulas by variables ranging over sets of d-expressions and d-expressions, respectively (see [18]). By a $\bullet$-deduction we mean a deduction received by applications of $\bullet$-rules only.

We define the relation $\vdash_{\mathbf{DAI}_\bot}$ as before. Moreover, we say that A is $\bullet$-*deducible from* X *in* $\mathbf{DAI}_\bot$ ($X \vdash^\bullet_{\mathbf{DAI}_\bot} A$) iff there is a $\bullet$-deduction Π of $A^\bullet$ depending on $X^\bullet$, for any finite non-empty X.

Let us note that we can easily receive the following counterparts of the Fact 1 and its corollaries:

F1. For any $A, B \in \mathsf{For}$, there is a $\bullet$-deduction Π of $B^\bullet$ depending on $A^\bullet$ in $\mathbf{DAI}_\bot$ iff $\mathsf{var}(B) \subseteq \mathsf{var}(A)$.

F2. For all $A_1, \dots, A_n, A \in \mathsf{For}$, $\{A_1, \dots, A_n\} \vdash^{\bullet}_{\mathbf{DAI}_\bot} A$ iff $\mathsf{var}(A) \subseteq \bigcup_{i \leq n} \mathsf{var}(A_i)$.

F3. For all $A_1, \dots, A_n \in \mathsf{Var}$, for any $A \in \mathsf{For}$:

1. $A \vdash^{\bullet}_{\mathbf{DAI}_\bot} A_i$, for any $i \leq n$ iff $\{A_1, \dots, A_n\} \subseteq \mathsf{var}(A)$,
2. $\{A_1, \dots, A_n\} \vdash^{\bullet}_{\mathbf{DAI}_\bot} A$ iff $\mathsf{var}(A) = \{A_1, \dots, A_n\}$.

In order to show that the rule ($\to$E$^\bullet$) is derivable in our system, we will show that the rule (Mix+) is derivable.

Fact 12. (Mix+) is derivable in the deductive system for $\mathbf{DAI}_\bot$.

Proof. In the case where $B = \bot$. We apply the rule ($\bot\bullet$).

Assume $B \neq \bot$. If $\mathsf{var}(A) = \{A_1, \dots, A_n\}$, then by F3.1, let Π_i be $\bullet$ deduction of $A_i^\bullet$ depending on $A^\bullet$. Let Σ be the sequence of the following form:

$$\dfrac{\begin{matrix}[\neg(A \to B)] \\ \Pi_0\end{matrix}}{\bot} \qquad \dfrac{\begin{matrix}[A^\bullet] \\ \Pi_1\end{matrix}}{A_1^\bullet} \qquad \dots \qquad \dfrac{\begin{matrix}[A^\bullet] \\ \Pi_n\end{matrix}}{A_n^\bullet}$$

If $\mathsf{var}(A) = \emptyset$, i.e., $A = \bot$, then Σ has the following form:

$$\dfrac{\begin{matrix}[\neg(\bot \to B)] \\ \Pi_0\end{matrix}}{\bot} \qquad \bot^\bullet$$

By F3.2, let Π be $\bullet$ deduction of $B^\bullet$ depending on $\mathsf{var}(B)^\bullet$. Then, the deduction Π has one of the following forms:

$$C_1^\bullet \qquad\qquad \dfrac{C_1^\bullet \quad C_2^\bullet}{D_1^\bullet} \qquad\qquad \dfrac{\begin{matrix}\dfrac{\dfrac{C_1^\bullet \quad C_2^\bullet}{D_1^\bullet} \quad C_3^\bullet}{} \\ \vdots \\ D_{k-1}^\bullet \quad C_{k+1}^\bullet\end{matrix}}{D_k^\bullet}$$

where $k \geq 2$, for any $i \leq k+1$, $C_i \in \mathsf{var}(B)$, $C_1, C_2 \vdash^\bullet D_1$, for any $1 < i \leq k$, $D_{i-1}, C_{i+1} \vdash^\bullet D_i$.

We then have, respectively:

$$(\mathrm{Mix})\,\dfrac{\Sigma}{(C_1^\bullet)} \qquad\qquad \dfrac{(\mathrm{Mix})\,\dfrac{\Sigma}{(C_1^\bullet)} \quad (\mathrm{Mix})\,\dfrac{\Sigma}{(C_2^\bullet)}}{D_1^\bullet}$$

$$\cfrac{\begin{array}{c}\cfrac{\text{(Mix)}\,\cfrac{\Sigma}{(C_1^\bullet)}\quad\text{(Mix)}\,\cfrac{\Sigma}{(C_2^\bullet)}}{D_1^\bullet}\quad\text{(Mix)}\,\cfrac{\Sigma}{C_3^\bullet}\\ \hline \vdots\\ D_{k-1}^\bullet\end{array}\qquad\text{(Mix)}\,\cfrac{\Sigma}{(C_{k+1}^\bullet)}}{D_k^\bullet}$$

□

We obtain the following corollary.

Corollary 13. ($\to$E$^\bullet$) is derivable in the deductive system for $\mathbf{DAI}_\perp$.

Proof. We use Fact 12.

$$\text{(Mix+)}\,\cfrac{(\to\text{E})\,\cfrac{\neg(A\to B)\quad\cfrac{\mathit{\Pi}_1}{(A\to B)}}{\perp}\qquad\cfrac{\mathit{\Pi}_2}{A^\bullet}}{B^\bullet}$$

□

In the rest of the article, we will sometimes use derivable rules to shorten the proofs.

4.3 Soundness and completeness

In what follows, we modify the notion of a model being sound for a pair consisting of a set of d-expressions and a single d-expression. We say that the model $M=\langle v,s,\subseteq\rangle\in\textbf{isa-E}$ is *sound for a pair* $\langle\mathcal{X},\mathcal{A}\rangle$ ($M\Vdash\langle\mathcal{X},\mathcal{A}\rangle$) iff the following cases hold:

+ $M\not\models\mathsf{for}(\mathcal{X})\cup\{\neg A\}$, if $\mathcal{A}=A$, for some $A\in\mathsf{For}$,

+ $M\models\mathsf{for}(\mathcal{X})$ only if $\bigcap\{s(B):B\in\mathsf{for}^\bullet(\mathcal{X})\}\subseteq s(A)$, if $\mathcal{A}=A^\bullet$, for some $A\in\mathsf{For}$.

Let us now proceed to the formulation and proof of the soundness lemma.

Lemma 14. 1. For any rule (R), if the instance of (R) is of the form $(\mathcal{A}_1,\ldots,\mathcal{A}_n/\mathcal{A})$, then for any model $M\in\textbf{isa-E}$, $M\Vdash\langle\{\mathcal{A}_i\}_{i\leq n},\mathcal{A}\rangle$.

2. For any rule (R), if the instance of (R) is of the form $\langle\langle\mathcal{X}_1,\mathcal{A}_1\rangle,\ldots,\langle\mathcal{X}_n,\mathcal{A}_n\rangle,\langle\mathcal{X},\mathcal{A}\rangle\rangle$ and for any $i\leq n$, for any $M\in\textbf{isa-E}$, $M\Vdash\langle\mathcal{X}_i,\mathcal{A}_i\rangle$, then for any $M\in\textbf{isa-E}$, $M\Vdash\langle\mathcal{X},\mathcal{A}\rangle$.

Proof. Let us consider only selected rules; for others we reason similarly. For ($\bot\bullet$), ($\star\bullet$E1), ($\star\bullet$E2) and ($\star\bullet$I) we reason as in the proof of Lemma 6.

For ($\to$ I$^\bullet$). Assume that for any model $M = \langle v, s, \subseteq\rangle \in$ **isa-E**, $M \not\models \mathsf{for}(\mathcal{X}) \cup \{\neg B\}$, and $M \models \mathsf{for}(\mathcal{Y})$ only if $\bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Y})\} \subseteq s(B)$. Consider any model $M = \langle v, s, \subseteq\rangle \in$ **isa-E** and assume $M \models \mathsf{for}(\mathcal{Z}) \cup \{A\}$. Since $\mathcal{Z} = (\mathcal{X}\setminus\{A\}) \cup (\mathcal{Y}\setminus\{A^\bullet\})$, then $M \models B$ and $\bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Y})\} \subseteq s(B)$. Since $\mathsf{for}^\bullet(\mathcal{Y}) = \{A\}$, we have $s(A) \subseteq s(B)$. Therefore, $M \models A \to B$.

For (Mix$_1$). Assume that for any model $M = \langle v, s, \subseteq\rangle \in$ **isa-E**, $M \not\models \mathsf{for}(\mathcal{X})$ and for any $i \leq n$, $M \models \mathsf{for}(Y_i)$ only if $\bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Y}_i)\} \subseteq s(A_i)$. Consider any model $M = \langle v, s, \subseteq\rangle \in$ **isa-E** and assume $M \models \mathsf{for}(\mathcal{Z})$. Since $\mathcal{Z} = (\mathcal{X}\setminus\{\neg(A \to B)\}) \cup \bigcup_{i\leq n} \mathcal{Y}_i$, then $M \not\models \neg(A \to B)$, so $M \models A \to B$, and for any $i \leq n$, $\bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Y}_i)\} \subseteq s(A_i)$. Since $M \models A \to B$, we have $s(A_1) \cap \ldots \cap s(A_n) = s(A) \subseteq s(B) = s(B_1) \cap \ldots \cap s(B_m)$, where $\mathsf{var}(B) = \{B_1, \ldots, B_m\}$. And since for any $i \leq n$, $\bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Z})\} \subseteq \bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Y}_i)\} \subseteq s(A_i)$, we have for any $j \leq m$, $\bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Z})\} \subseteq s(B_j)$.

For (Mix$_2$). Assume that for any model $M = \langle v, s, \subseteq\rangle \in$ **isa-E**, $M \not\models \mathsf{for}(\mathcal{X})$, and $M \models \mathsf{for}(\mathcal{Y})$ only if $\bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Y})\} \subseteq s(\bot)$. Consider any model $M = \langle v, s, \subseteq\rangle \in$ **isa-E** and assume $M \models \mathsf{for}(\mathcal{Z})$. Since $\mathcal{Z} = (\mathcal{X} \setminus \{\neg(\bot\to B)\}) \cup \mathcal{Y}$, then $M \not\models \neg(\bot\to B)$, so $M \models \bot\to B$. Thus, $s(B) = U = \bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Y})\}$. Since $s(B) = s(B_1) \cap \ldots \cap s(B_m)$, where $\mathsf{var}(B) = \{B_1, \ldots, B_m\}$, we also have for any $j \leq m$, $\bigcap\{s(C) : C \in \mathsf{for}^\bullet(\mathcal{Z})\} \subseteq s(B_j)$. □

As in the previous section, to prove completeness, it is enough to show that, in the system we are analysing, all axioms of logic **DAI**$_\bot$ are deducible.

Lemma 15.

1. All formulas of the form of (CL1)–(CL3) are deducible in **DAI**$_\bot$.
2. All formulas of the form of (DAI), (DAI1)–(DAI5) are deducible in **DAI**$_\bot$.

Proof. The deduction for axioms is a simple modification of the deductions presented in the previous section in the proof of Lemma 8. □

As before, by Lemmas 14 and 15 we receive the following theorem:

Theorem 16. For all $X \cup \{A\} \subseteq \mathsf{For}$, $X \vdash_{\mathbf{DAI}_\bot} A$ iff $X \models_{\textbf{isa-E}} A$.

4.4 Normalisation

To prove the normalisation theorem, we assume $\wedge$-reduction and make obvious modifications to $\to$-reduction-1 and $\star$-lab-reduction presented in the previous section.

Let us also show that the application of rule (RAA), as before, can be restricted to consequence with variables.

Fact 17. If $X \vdash_{\mathbf{DAI}_\bot} A$, then there is a deduction in $\mathbf{DAI}_\bot$ of A from X in which the consequence of every application of (RAA) is variable.

Proof. Let us make standard assumptions. Let Π have the following form:

$$\begin{array}{c} [\neg B] \\ \dfrac{\Sigma}{\bot} \\ \end{array}$$

$$\text{(RAA)}\,\dfrac{\begin{array}{c}[\neg B]\\ \dfrac{\Sigma}{\bot}\end{array}}{(B)} \atop \Pi'$$

Let us consider only the case where $B = C \to D$, for some $C, D \in \mathsf{For}$. Then, we modify the standard method to eliminate the application of (RAA) for formulas with implication as the main connective:

$$\begin{array}{c}
(\to\mathrm{I}^\bullet)\,\dfrac{
\text{(RAA)}\,\dfrac{\begin{array}{c}
(\to\mathrm{I}^\bullet)\,\dfrac{(\to\mathrm{E})\,\dfrac{(\to\mathrm{E})\,\dfrac{C \qquad C \to D}{D} \qquad \neg D}{\bot} \qquad (\bot\bullet)\,\dfrac{(C \to D)^\bullet}{\bot^\bullet}}{[\neg(C \to D)]} \\
\dfrac{\Sigma}{\bot}
\end{array}}{D}
\qquad
(\mathrm{Mix}+)\,\dfrac{\begin{array}{c}[\neg(C \to D)] \\ \dfrac{\Sigma}{\bot}\end{array} \qquad C^\bullet}{D^\bullet}
}{(C \to D)} \\
\Pi'
\end{array}$$

□

Thus, as in the previous case, we can prove the following theorem in the standard way:

Theorem 18. If $X \vdash_{\mathbf{DAI}_\bot} A$, then there is a normal deduction in $\mathbf{DAI}_\bot$ of A from X.

5 Summary

Our article consists of two parts. The first is semantic and philosophical: we discuss how to represent the content of absurd sentences expressed by *falsum*, and how various negations can be defined using demodalised analytic implication together with *falsum*. The second part is proof-theoretic: fixing one approach to *falsum*, we present two labelled deductive systems and prove soundness, completeness, and normalisation theorems for each of them. In both systems, deductions are expressed in a mixed language that contains both formulas and labelled formulas. The second system reduces the first to a language with only single-labelled formulas (in addition to ordinary formulas), but this reduction requires introducing so-called mix rules in place of more natural ones.

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