Non-Markovian quantum dynamics: local versus non-local

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We analyze non-Markovian evolution of open quantum systems. It is shown that any dynamical map representing evolution of such a system may be described either by non-local master equation with memory kernel or equivalently by equation which is local in time. These two descriptions are complementary: if one is simple the other is quite involved, or even singular, and vice versa. The price one pays for the local approach is that the corresponding generator keeps the memory about the starting point 't_0'. This is the very essence of non-Markovianity. Interestingly, this generator might be highly singular, nevertheless, the corresponding dynamics is perfectly regular. Remarkably, singularities of generator may lead to interesting physical phenomena like revival of coherence or sudden death and revival of entanglement.

The non-Markovian dynamics of open quantum systems attracts nowadays increasing attention [1]. It is very much connected to the growing interest in controlling quantum systems and applications in modern quantum technologies such as quantum communication, cryptography and computation [2]. It turns out that the popular Markovian approximation which does not take into account memory effects is not sufficient for modern applications and todays technology calls for truly non-Markovian approach. Non-Markovian dynamics was recently studied in [3–15]. Interestingly, several measures of non-Markovianity were proposed during last year [16–19].

The standard approach to the dynamics of open system uses the Nakajima-Zwanzig projection operator technique [20] which shows that under fairly general conditions, the master equation for the reduced density matrix $\rho(t)$ takes the form of the following non-local equation

$$\frac{d}{dt}\,\rho(t) = \int_{t_0}^{t} \mathcal{K}(t-u)\rho(u)\,du \ , \quad \rho(t_0) = \rho_0 \ , \quad (1)$$

in which quantum memory effects are taken into account through the introduction of the memory kernel $\mathcal{K}(t)$: this simply means that the rate of change of the state $\rho(t)$ at time t depends on its history (starting at $t = t_0$). Usually, one takes $t_0 = 0$, however, in this letter we shall keep 't_0' arbitrary. An alternative and technically much simpler scheme is the time-convolutionless (TCL) projection operator technique [1, 21, 22] in which one obtains a first-order differential equation for the reduced density matrix. The advantage of the TCL approach consists in the fact that it yields an equation of motion for the relevant degrees of freedom which is local in time and which is therefore often much easier to deal with than the Nakajima-Zwanzig non-local master equation (1).

An essential step to derive TCL from (1) relies on the existence of certain operator inverse [22]. However, this inverse needs not exist and then the method does not work [1, 22]. Moreover, even if it exists the corresponding local in time TCL generator is usually defined by the perturbation series (see e.g. detailed discussion in [1]) in powers of the coupling strength characterizing the system. However in general the perturbative approach leads to significant problems. For example the dynamical map needs not be completely positive if one takes only finite number of terms from the perturbative expansion.

In the present paper we take a different path. We show that any solution of the non-local equation (1) always satisfies equation which does not involve the integral memory kernel, i.e. it is local in time. However, the corresponding generator is effectively non-local due to the fact that it keeps the memory about the starting point t_0. Moreover, as we shall see, this generator may be singular, nevertheless, it always leads to perfectly regular dynamics.

Let us start with the standard Markovian master equation

$$\frac{d\rho(t)}{dt} = \mathcal{L}\,\rho(t) \ , \quad \rho(t_0) = \rho_0 \ , \quad (2)$$

where $\mathcal{L}$ is a time-independent generator possessing the following well known representation [23, 24]

$$\mathcal{L}\rho = -i[H,\rho] + \sum_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \frac{1}{2}\{V_\alpha^\dagger V_\alpha, \rho\} \right) \ . \quad (3)$$

The above structure of $\mathcal{L}$ guaranties that dynamical map $\Lambda(t,t_0)$, defined by $\rho(t) = \Lambda(t,t_0)\rho_0$, is completely positive and trace preserving for $t \geq t_0$. Note that $\Lambda(t,t_0)$ itself satisfies Markovian master equation

$$\frac{d}{dt}\Lambda(t,t_0) = \mathcal{L}\,\Lambda(t,t_0) \ , \quad \Lambda(t_0,t_0) = \mathbb{1} \ , \quad (4)$$

and the solution for $\Lambda(t,t_0)$ is given by $\Lambda(t,t_0) = e^{(t-t_0)\mathcal{L}}$, which implies that $\Lambda(t,t_0)$ depends only upon the difference '$t-t_0$' and hence $\Lambda(t) := \Lambda(t,0)$ defines a 1-parameter semigroup satisfying homogeneous composition law

$$\Lambda(t_1)\Lambda(t_2) = \Lambda(t_1+t_2) \ , \quad (5)$$

for $t_1, t_2 \geq 0$. In general the external conditions which influence the dynamics of an open system may very in time. The natural generalization of the Markovian master equation (2) involves time-dependent generator $\mathcal{L}(t)$ which has exactly the same representation as in (3) with time-dependent Hamiltonian $H(t)$ and time-dependent Lindblad operators $V_\alpha(t)$. Therefore one gets the following master equation for the dynamical map $\Lambda(t, t_0)$

$$\frac{d}{dt}\Lambda(t, t_0) = \mathcal{L}(t)\,\Lambda(t, t_0) \ , \quad \Lambda(t_0, t_0) = \mathbb{1} \ , \quad (6)$$

with leads to the following solution

$$\Lambda(t, t_0) = \mathrm{T}\,\exp\left(\int_{t_0}^{t} \mathcal{L}(\tau)d\tau\right) , \quad (7)$$

where T stands for the chronological operator. Clearly, $\Lambda(t, t_0)$ no longer depends upon '$t-t_0$' but it still satisfies inhomogeneous composition law

$$\Lambda(t, s) \cdot \Lambda(s, t_0) = \Lambda(t, t_0) \ , \quad (8)$$

for $t \geq s \geq t_0$. We stress that (6) although time-dependent is perfectly Markovian.

Let us turn to the non-Markovian evolution (1). One obtains the following equation for the corresponding dynamical map

$$\frac{d}{dt}\Lambda(t, t_0) = \int_{t_0}^{t} d\tau\,\mathcal{K}(t-\tau)\,\Lambda(\tau, t_0) \ , \quad \Lambda(t_0, t_0) = \mathbb{1} \ . \ (9)$$

Now comes an essential observation: $\Lambda(t, t_0)$ does depend upon the difference '$t - t_0$' and hence it shares the same feature as the Markovian dynamics with time independent generator (2). The proof is very easy. Observe that any non-Markovian dynamics in $\mathcal{H}$ may be defined as a reduced Markovian dynamics on the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$ ($\mathcal{H}_a$ denotes ancilla Hilbert space). If ω denotes a fixed state of the ancilla, then

$$\Lambda(t, t_0)\rho := \mathrm{Tr}_a[e^{(t-t_0)L}(\rho \otimes \omega)] \ , \quad (10)$$

where we trace out over ancilla degrees of freedom and L denotes the total Markovian generator in $\mathcal{H} \otimes \mathcal{H}_a$. Since the r.h.s of (10) depends on '$t - t_0$' so does the non-Markovian dynamical map $\Lambda(t, t_0)$. Hence, the non-Markovian dynamics is homogeneous (depends on $t - t_0$) but of course does not satisfy the composition law (5). This is the very essence of non-Markovianity and it does provide the evident sign of the memory.

Suppose now that $\Lambda(t, t_0)$ satisfies non-local equation (9). Taking into account that Λ is a function of $\tau = t-t_0$, let us consider its spectral decomposition

$$\Lambda(\tau)\rho = \sum_\mu \lambda_\mu(\tau)\,F_\mu(\tau)\mathrm{Tr}(G_\mu^\dagger(\tau)\rho) \ , \quad (11)$$

where $F_\mu(\tau)$ and $G_\mu(\tau)$ define the damping basis for $\Lambda(\tau)$, that is, $\mathrm{Tr}(F_\mu(\tau)G_\nu^\dagger(\tau)) = \delta_{\mu\nu}$. Clearly, for $\tau = 0$ one has $\lambda_\mu(0) = 1$. Now one defines the formal inverse

$$\Lambda^{-1}(\tau)\rho = \sum_\mu \lambda_\mu^{-1}(\tau)\,F_\mu(\tau)\mathrm{Tr}(G_\mu^\dagger(\tau)\rho) \ , \quad (12)$$

such that $\Lambda(\tau)\Lambda^{-1}(\tau) = \mathbb{1}$ for $\tau \geq 0$. It should be stressed that $\Lambda^{-1}(\tau)$ needs not exist (it does exist if and only if $\lambda_\mu(\tau) \neq 0$). Moreover, the existence of $\Lambda^{-1}(\tau)$ does not mean that the dynamics is invertible. Note, that even if $\Lambda^{-1}(\tau)$ does exist it is in general not completely positive and hence can not describe quantum evolution backwards in time. Actually, $\Lambda^{-1}(\tau)$ is completely positive if and only if $\Lambda(\tau)$ is unitary or anti-unitary. In this case $|\lambda_\mu(\tau)| = 1$ and $\lambda_\mu^{-1}(\tau) = \overline{\lambda_\mu(\tau)}$. It is therefore clear that the non-local equation (9) reduces formally to the following one

$$\frac{d}{dt}\Lambda(t, t_0) = \mathcal{L}(t - t_0)\Lambda(t, t_0) \ , \quad \Lambda(t_0, t_0) = \mathbb{1} \ , \quad (13)$$

where the time-dependent generator $\mathcal{L}(\tau)$ is defined by the following logarithmic derivative of the dynamical map

$$\mathcal{L}(\tau) := \frac{d}{d\tau}\Lambda(\tau) \cdot \Lambda^{-1}(\tau) \ . \quad (14)$$

One easily finds the following formula

$$\mathcal{L}(\tau)\rho = \sum_{\mu\nu} \mathcal{L}_{\mu\nu}(\tau)\mathrm{Tr}(G_\nu^\dagger(\tau)\rho) \ , \quad (15)$$

with

$$\mathcal{L}_{\mu\nu} = \left(\frac{\dot{\lambda}_\mu}{\lambda_\nu}F_\mu + \frac{\lambda_\mu}{\lambda_\nu}\dot{F}_\mu\right)\delta_{\mu\nu} + \frac{\lambda_\mu}{\lambda_\nu}F_\mu\mathrm{Tr}(\dot{G}_\mu^\dagger F_\nu) \ ,$$

where for simplicity we omit the time dependence. In particular, if the damping basis is time-independent, and $\lambda_\mu(\tau) = e^{\gamma_\mu(\tau)}$, then the spectral decomposition of $\mathcal{L}(\tau)$ has a particulary simple form

$$\mathcal{L}(\tau)\rho = \sum_\mu \dot{\gamma}_\mu(\tau)F_\mu\mathrm{Tr}(G_\mu^\dagger\rho) \ . \quad (16)$$

Summarizing, we have shown that each solution $\Lambda(t, t_0)$ to the non-local non-Markovian equation (9) does satisfy the first order differential equation (13). Let us observe that Eq. (13) is local in time but its generator does remember about the starting point 't_0'. This is the most important difference with the time-dependent Markovian equation (6). The appearance of 't_0' in the generator $\mathcal{L}(t - t_0)$ implies that $\mathcal{L}$ is effectively non-local in time, that is, it contains a memory. Therefore, the local equation (13) is non-Markovian contrary to the local equation (6) which does does not keep any memory about t_0. Note, that solution to (13) is given by

$$\Lambda(t, t_0) = \mathrm{T}\,\exp\left(\int_0^{t-t_0} \mathcal{L}(\tau)\,d\tau\right) \ . \quad (17)$$

It shows that $\Lambda(t, t_0)$ is indeed homogeneous in time (depends on '$t - t_0$'). However, contrary to (7), it does not satisfy the composition law. Again, this is a clear sign for the memory effect.

One may ask a natural question: how to construct non-Markovian generator $\mathcal{L}(\tau)$. The general answer is

not known but one may easily propose special constructions. Let $\mathcal{L}$ be a Markovian generator defined by (3) and define $\mathcal{L}(\tau) = \alpha(\tau)\mathcal{L}$. It is clear that if $\int_0^\tau \alpha(u)du \geq 0$ for $\tau \geq 0$, then $\Lambda(\tau) = \exp(\int_0^\tau \alpha(u)du\, \mathcal{L})$ defines completely positive non-Markovian dynamics. This construction may be generalized as follows: consider N mutually commuting Markovian generators $\mathcal{L}_1, \ldots, \mathcal{L}_N$ and N real functions $\alpha_k(\tau)$ satisfying $\int_0^\tau \alpha_k(u)du \geq 0$. Then $\mathcal{L}(\tau) = \alpha_1(\tau)\mathcal{L}_1 + \ldots + \alpha_N(\tau)\mathcal{L}_N$ serves as a generator of non-Markovian evolution. Finally, let us observe that if $\mathcal{L}(t)$ is a time-dependent Markovian generator (i.e. it has the Lindblad form (3) with time-dependent Hamiltonian $H(t)$ and noise operators $V_\alpha(t)$), then $\mathcal{L}(t - t_0)$ generates the non-Markovian dynamics for $t \geq t_0$. We stress that these constructions provide only restricted classes of examples of non-Markovian generators. All of them start with a set of Markovian generators and produce a non-Markovian one. It turns out (see Example 3 below) that one may construct generators which do not fit these classes.

Let us illustrate our analysis with the following simple examples.

Example 1 Consider the dynamical map for a qudit (d-level quantum system) given by

$$\Lambda(\tau) = \left(1 - \int_0^\tau f(u)du\right)\mathbb{1} + \int_0^\tau f(u)du\,\mathcal{P}\ , \quad (18)$$

where $\mathcal{P} : \mathcal{B}(\mathbb{C}^d) \longrightarrow \mathcal{B}(\mathbb{C}^d)$ denotes completely positive trace preserving projection. For example take a fixed qudit state ω and define $\mathcal{P}$ by the following formula $\mathcal{P}\rho = \omega \operatorname{Tr}\rho$. The real function '$f$' satisfies:

$$0 \ \leq\ \int_0^\tau f(u)du \ \leq\ 1\ ,$$

for any $\tau > 0$. Note that $f(u)$ needs not be positive. If $f(u) \geq 0$ ($u \geq 0$), then $\Lambda(\tau)$ defines quantum semi-Markov process and the function $f(u)$ may be interpreted as a waiting time distribution for this process [4, 13]. Clearly, $\Lambda(\tau)$ being a convex combination of $\mathbb{1}$ and $\mathcal{P}$ is completely positive trace preserving map and hence it defines legal quantum dynamics of a qudit. The corresponding memory kernel is well known [4, 13] and it is given

$$\mathcal{K}(\tau) = \kappa(\tau)\mathcal{L}_0\ , \quad (19)$$

where the function $\kappa(\tau)$ is defined in terms of its Laplace transform as follows

$$\widetilde{\kappa}(s) = \frac{s\widetilde{f}(s)}{1 - \widetilde{f}(s)}\ , \quad (20)$$

and $\mathcal{L}_0$ is defined by $\mathcal{L}_0 = \mathcal{P} - \mathbb{1}$. Note, that $\mathcal{L}_0$ has exactly the structure of the Markovian generator (3) with $H = 0$, and the Lindblad operators V_α define Kraus representation of $\mathcal{P}$, that is $\mathcal{P}\rho = \sum_\alpha V_\alpha \rho V_\alpha^\dagger$. One easily finds for the corresponding generator

$$\mathcal{L}(\tau) = \alpha(\tau)\,\mathcal{L}_0\ , \quad (21)$$

where

$$\alpha(\tau) = \frac{f(\tau)}{1 - \int_0^\tau f(u)du}\ . \quad (22)$$

Let us observe that

$$\int_0^\tau \alpha(u)du = -\ln\left(1 - \int_0^\tau f(u)du\right) \geq 0\ ,$$

and hence this example gives rise to $\mathcal{L}(\tau) = \alpha(\tau)\mathcal{L}_0$, with Markovian $\mathcal{L}_0$ and $\alpha(\tau)$ satisfying $\int_0^\tau \alpha(u)du \geq 0$. We stress that $\alpha(\tau)$ needs not be positive. It is positive if and only if $f(\tau)$ corresponds to the waiting time distribution [4, 13]. Note the striking similarity between formulae (20) and (22). It should be stressed that in this case one knows an explicit formula for time-local generator $\mathcal{L}(\tau)$. Note, however, that in general one is not able to invert the Laplace transform of $\widetilde{\kappa}(s)$ from the formula (20) and hence the explicit formula for the memory kernel $\mathcal{K}(t)$ is not known.

Example 2 The previous example may be easily generalized to bipartite systems. Consider for example a 2-qubit system and let $\mathcal{P}$ be a projector onto the diagonal part with respect to the product basis $|m \otimes n\rangle$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Let us take as an initial density matrix so called X-state [25] represented by

$$\rho_0 = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}\ . \quad (23)$$

It is easy to see that $\Lambda(\tau)$ defined by (18) does preserve the structure of X-state, that is, $\rho(\tau)$ has exactly the same form as in (23) with τ-dependent ρ_{mn}. It is clear that the diagonal elements are time independent $\rho_{kk}(\tau) = \rho_{kk}$, and $\rho_{kl}(\tau) = (1 - \int_0^\tau f(u)du)\rho_{kl}$, for $k \neq l$. The entanglement of the 2-qubit X-state $\rho(\tau)$ is uniquely determined by the concurrence $C(\tau) = 2\max\{c_1(\tau), c_2(\tau), 0\}$, where

$$c_1(\tau) = |\rho_{23}(\tau)| - \sqrt{\rho_{11}\rho_{44}}\ , \quad c_2(\tau) = |\rho_{14}(\tau)| - \sqrt{\rho_{22}\rho_{33}}\ ,$$

that is, $\rho(\tau)$ is entangled if and only if $c_1(\tau) > 0$ or $c_2(\tau) > 0$. Let us observe that the function $f(\tau)$ controls the evolution of quantum entanglement. Consider for example $f(\tau) = \varepsilon\gamma e^{-\gamma\tau}$, with $\gamma > 0$ and $\varepsilon \in (0, 1]$. One finds from (22) the following formula $\alpha(\tau) = \varepsilon\gamma[(1-\varepsilon)e^{\gamma\tau}+\varepsilon]^{-1}$. Note, that for $\varepsilon = 1$ it reduces to $\alpha(\tau) = \gamma$, that is, it corresponds to the purely Markovian case. Hence, the parameter '$1 - \varepsilon$' measures the non-Markovianity of the dynamics. Suppose now that ρ_0 is entangled. The entanglement of the asymptotic state is governed by $C(\infty) = 2\max\{c_1(\infty), c_2(\infty), 0\}$, with $c_1(\infty) = (1 - \varepsilon)|\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}$ and $c_2(\infty) = (1 - \varepsilon)|\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}$. It is clear that in the Markovian case ($\varepsilon = 1$) the asymptotic state is always separable ($C(\infty) = 0$). However, for sufficiently small 'ε' (i.e.

sufficiently big non-Markovianity parameter '$1 - \varepsilon$') one may have $c_1(\infty) > 0$ or $c_2(\infty) > 0$, that is, the asymptotic state might be entangled. This example proves the crucial difference between Markovian and non-Markovian dynamics of composed systems. In particular controlling 'ε' we may avoid sudden death of entanglement [25].

Example 3 Consider the pure decoherence model defined by the following Hamiltonian $H = H_R + H_S + H_{SR}$, where H_R is the reservoir Hamiltonian, $H_S = \sum_n \epsilon_n P_n$ ($P_n = |n\rangle\langle n|$) the system Hamiltonian and

$$H_{SR} = \sum_n P_n \otimes B_n \qquad (24)$$

the interaction part, $B_n = B_n^\dagger$ being reservoirs operators. The initial product state $\rho \otimes \omega_R$ evolves according to the unitary evolution $e^{-iHt}(\rho \otimes \omega_R)e^{iHt}$ and by partial tracing with respect to the reservoir degrees of freedom one finds for the evolved system density matrix

$$\rho(t) = \mathrm{Tr}_R[e^{-iHt}(\rho \otimes \omega_R)e^{iHt}] = \sum_{n,m} c_{mn}(t)P_m \rho P_n \ ,$$

where $c_{mn}(t) = \mathrm{Tr}(e^{-iZ_m t}\omega_R e^{iZ_n t})$, $Z_n = \epsilon_n \mathbb{I}_R + H_R + B_n$ being reservoir operators. Note that the matrix $c_{mn}(t)$ is semi-positive definite and hence

$$\Lambda(\tau)\,\rho = \sum_{n,m} c_{mn}(\tau)P_m \rho P_n \ . \qquad (25)$$

defines the Kraus representation of the completely positive map $\Lambda(\tau)$. The solution of the pure decoherence model can therefore be found without explicitly writing down the underlying master equation. Our method, however, enables one to find the corresponding generator $\mathcal{L}(\tau)$. It is given by the following formula

$$\mathcal{L}(\tau)\,\rho = \sum_{n,m} \alpha_{mn}(\tau)P_m \rho P_n \ , \qquad (26)$$

where the functions $\alpha_{mn}(\tau)$ are defined by $\alpha_{mn} = \dot{c}_{mn}/c_{mn}$. It shows that the pure decoherence model may be defined by local in time master equation (13) with the non-Markovian generator (26). It should be stressed that this generator is not of the Lindblad form.

Example 4 Consider the non-Markovian dynamics of a qubit generated by the following singular generator

$$\mathcal{L}(\tau) = \tan\tau\,\mathcal{L}_0 \ , \qquad (27)$$

with $\mathcal{L}_0$ being the pure dephasing generator defined by $\mathcal{L}_0\rho = \sigma_z \rho \sigma_z - \rho$. This generator was analyzed in [17, 18] in the context of quantifying non-Markovianity of quantum dynamics. Note that $\mathcal{L}(\tau)$ has an infinite number of singular points $\tau_n = (n + \frac{1}{2})\pi$. One easily finds the following perfectly regular solution for the dynamical map $\Lambda(\tau) = \frac{1}{2}(1 + \cos\tau)\mathbb{1} + \frac{1}{2}(1 - \cos\tau)(\mathcal{L}_0 + \mathbb{1})$, that is, the density matrix evolves as follows

$$\rho(\tau) = \begin{pmatrix} \rho_{11} & \rho_{12}\cos\tau \\ \rho_{21}\cos\tau & \rho_{22} \end{pmatrix} \ , \qquad (28)$$

and hence it displays oscillations of the qubit coherence $\rho_{12}(\tau)$. Note that $\rho(\tau_n)$ is perfectly decohered, whereas for $\tau = n\pi$ the coherence is perfectly restored. Finally, one finds extremely simple formula for the corresponding memory kernel $\mathcal{K}(t) = \frac{1}{2}\mathcal{L}_0$, for $t \geq t_0$. Hence, one obtains (28) either from the non-local equation with time-independent memory kernel $\mathcal{K}(t) = \frac{1}{2}\mathcal{L}_0$, or from time-local equation with highly singular generator (27).

In conclusion, we have shown that non-Markovian quantum evolution may be described either by the non-local equation (1) or by a time-local equation (13). A similar strategy based on pseudo-inverse maps have been recently applied in [26]. We stress, however, that our approach is different. Clearly, the local approach is technically much simpler, however, the prize we pay for this simplification is that the corresponding generator $\mathcal{L}(t - t_0)$ is no longer local in time but it contains a memory about the starting point 't_0'. Our examples show that these two descriptions are complementary: if $\mathcal{K}(\tau)$ is simple (like $\mathcal{K}(t) = \frac{1}{2}\mathcal{L}_0$), then $\mathcal{L}(\tau)$ is highly singular (like in (27)). Vice-versa in the Markovian case $\mathcal{L}(\tau) = \mathcal{L}_M$ but the memory kernel $\mathcal{K}$ is highly singular and it does involve the Dirac delta-distribution $\mathcal{K}(\tau) = 2\delta(\tau)\mathcal{L}_M$. Remarkably, singularities of $\mathcal{L}$ might provide interesting physical content. Note, that the singularities of '$\tan\tau$' in Example 4 imply the interesting features of the dynamical map (28): if we evolve a maximally entangled state P^+ of two qubits via the channel $\Psi(\tau) := \mathbb{1} \otimes \Lambda(\tau)$, then $\Psi(\tau)P^+$ is separable if and only if $\tau = \tau_n$. It shows that the dynamics $\Psi(\tau)$ gives rise to entanglement sudden death [25] whenever $\mathcal{L}(\tau)$ is singular and then entanglement starts to revive.

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