SIMPLIFIED KRIPKE STYLE SEMANTICS
FOR SOME VERY WEAK MODAL LOGICS

Abstract. In the present paper we examine very weak modal logics \( \text{C1, D1, E1, S0.5, S0.5+}(D), \text{S0.5} \) and some of their versions which are closed under replacement of tautological equivalents (rte-versions). We give semantics for these logics, formulated by means of Kripke style models of the form \( (w, A, V) \), where \( w \) is a «distinguished» world, \( A \) is a set of worlds which are alternatives to \( w \), and \( V \) is a valuation which for formulae and worlds assigns the truth-values such that: (i) for all formulae and all worlds, \( V \) preserves classical conditions for truth-value operators; (ii) for the world \( w \) and any formula \( \varphi \), \( V(\Box \varphi, w) = 1 \) iff \( \forall x \in A \ V(\varphi, x) = 1 \); (iii) for other worlds formula \( \Box \varphi \) has an arbitrary value. Moreover, for rte-versions of considered logics we must add the following condition: (iv) \( V(\Box \chi, w) = V(\Box \chi[\varphi/\psi], w) \), if \( \varphi \) and \( \psi \) are tautological equivalent. Finally, for \( \text{C1, D1} \) and \( \text{E1} \) we must add queer models of the form \( (w, V) \) in which: (i) holds and (ii′) \( V(\Box \varphi, w) = 0 \), for any formula \( \varphi \). We prove that considered logics are determined by some classes of above models.

Keywords: Simplified Kripke style semantics, very weak modal logics.

1. Preliminaries. Some historical notes

Modal formulae are formed in the standard way from the set \( \text{At} \) of propositional letters: ‘\( p’ \’, ‘q’ \’, ‘p_0’ \’, ‘p_1’ \’, ‘p_2’ \’, . . . ; truth-value operators: ‘\( \neg’ \),

\(^1\)This article is the final version of a draft paper [14], mentioned in the references of the papers [13] and [15].
‘∨’, ‘∧’, ‘⊃’, and ‘≡’ (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); the modal operator ‘□’ (necessity; the possibility sign ‘♦’ is the abbreviation of ‘¬□¬’); and brackets. Let For be the set of all modal formulae. For any set \( \Gamma \) of formulae we put 
\[ \Box \Gamma := \{ \Box \varphi \mid \varphi \in \Gamma \}. \]

Let Taut be the set of all classical tautologies (without the modal operator) and—as in [3, 4]—let PL be the set of modal formulae which are instances of classical tautologies.

Let \( \Sigma \) be a set of modal formulae. Also as in [3], \( \Sigma \) is a modal system iff \( \text{PL} \subseteq \Sigma \) and \( \Sigma \) is closed under the following rule of detachment for ‘⊃’ (modus ponens), i.e., for any formulae \( \varphi \) and \( \psi \):

\[
\text{if } \varphi \text{ and } \varphi \supset \psi \text{ are members of } \Sigma, \text{ so is } \psi. \tag{MP}
\]

We say that a modal system is congruential iff it is closed under the following rule of congruence:

\[
\text{if } \varphi \equiv \psi \in \Sigma, \text{ then } \Box \varphi \equiv \Box \psi \in \Sigma. \tag{RE}
\]

Notice that a modal system \( \Sigma \) is congruential iff it is closed under replacement

\[
\text{if } \varphi \equiv \psi \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi[\varphi/\psi] \in \Sigma, \tag{RRE}
\]

or equivalently

\[
\text{if } \varphi \equiv \psi \in \Sigma, \text{ then } \chi[\varphi/\psi] \equiv \chi \in \Sigma, \tag{RRE’}
\]

where \( \chi[\varphi/\psi] \) is any formula that results from \( \chi \) by replacing one or more occurrences of \( \varphi \), in \( \chi \), by \( \psi \).

A modal system \( \Sigma \) is called regular iff it is closed under the following regularity rule:

\[
\text{if } \varphi \supset \chi \in \Sigma, \text{ then } \Box \varphi \supset \Box \chi \in \Sigma. \tag{RR}
\]

A modal system \( \Sigma \) is regular iff it contains all instances of

\[
\Box (p \supset q) \supset (\Box p \supset \Box q) \tag{K}
\]

and is closed under the following monotonic rule

\[
\text{if } \varphi \supset \psi \in \Sigma \text{ then } \Box \varphi \supset \Box \psi \in \Sigma, \tag{RM}
\]

iff it is closed under (RM) and contains all instances of

\[
(\Box p \land \Box q) \supset \Box (p \land q) \tag{C}
\]
iff it is closed under (RE) and contains all instances of
\[ \Box (p \land q) \equiv (\Box p \land \Box q) \]  
(R)

We say that a modal system \( \Sigma \) is *normal* iff it contains all instances of \((K)\) and is closed under the following rule:
\[ \text{if } \varphi \in \Sigma, \text{ then } \top \Box \varphi \top \in \Sigma. \]  
(RN)

A modal system \( \Sigma \) is *normal* iff it is regular and contains the following formula
\[ \Box (p \supset p) \]  
(N)

iff it contains \((N)\) and all instances of \((K)\), and is closed under \((RE)\).

A set \( \Sigma \) of modal formulae is a *logic* iff \( \Sigma \) is a modal system and is closed under the following rule of uniform substitution:
\[ \text{if } \varphi \in \Sigma \text{ then } s \varphi \in \Sigma, \]  
(US)

where \( s \varphi \) is the result of uniform substitution of formulae for propositional letters in \( \varphi \). Of course, the set PL is the smallest modal system and it is a logic.

In [9] Lemmon set out the logic \( S_{0.5} \) and two groups of non-normal modal logics called the “D” and “E” systems.

Firstly, the logic \( S_{0.5} \) is the smallest modal logic which includes \( \Box \text{Taut} \), and contains \((K)\) and the following formula:
\[ \Box p \supset p \]  
(T)

The logic \( S_{0.5}^\circ \) is associated with Lemmon’s \( S_{0.5} \) (for these logics see e.g. [4, 9, 16]). \( S_{0.5}^\circ \) is the smallest logic which includes \( \Box \text{Taut} \) and contains \((K)\). Thus, \( S_{0.5} \) is \( S_{0.5}^\circ \) plus \((T)\). Of course, by \((US)\), \( S_{0.5} \) and \( S_{0.5}^\circ \) include the set \( \Box \text{PL} \), and \( S_{0.5}^\circ \subsetneq S_{0.5} \) (see Fact 4.1).

Secondly, Lemmon “consider a series of Lewis modal systems E1, E2, E3, E4, and E5, which are intended as possible epistemic counterparts to the five systems S0.5, S2, S3, S4, and S5. A distinguishing mark of E-systems is that in none of them is there any thesis of the form \( L\alpha \)” [9, p. 181–182] (in our text \( L\alpha := \Box \varphi \)). All E-systems—just like all S-systems—are logics that contain \((K)\) and \((T)\), include the set Taut, and are closed under the rules: \((MP)\) and \((US)\) (so they include the set PL). Moreover, the logics \( E2-E5 \) are regular. For example, \( E2 \) is the smallest regular modal logic which contains \((T)\). \( E3 \) is the smallest modal logic
which is closed under the rule RM and contains (T) and the following formula:
\[ \Box(p \supset q) \supset \Box(\Box p \supset \Box q) \] (sk)
Thus, by PL, (sK) and (T), the logic E3 contains (K). So it is regular.

The logic E1 is closed neither under (RM) nor under (RR). It is the smallest logic which contains (K) and (T), and includes the following set of formulae:
\[ M_{Taut} := \{ \Box \varphi \supset \Box \psi \uparrow : \varphi \supset \psi \in Taut \} \]
Thus, E1 also includes the following sets of formulae.
\[ M_{PL} := \{ \Box \varphi \supset \Box \psi \uparrow : \varphi \supset \psi \in PL \} \]
\[ R_{PL} := \{ \Box(\Box \varphi \land \Box \psi) \supset \Box \chi \uparrow : \Box(\varphi \land \psi) \supset \chi \in PL \} \]
\[ E_{PL} := \{ \Box \varphi \equiv \Box \psi \uparrow : \varphi \equiv \psi \in PL \} \]
We have \( E1 \subseteq S0.5 \) (see Fact 4.1).

Thirdly, the five D-logics, D1, D2, D3, D4 and D5, were associated with the five E-logics. “The distinguishing feature of D-systems is that axiom (T) of the corresponding E-systems is weakened to (D)” [9, p. 184]
\[ \Box p \supset \neg \Box \neg p \] (D)
Precisely, D1 is the smallest logic which contains (K) and (D), and includes the set \( M_{Taut} \). Thus, the logic D1 also includes the sets \( M_{PL} \), \( R_{PL} \) and \( E_{PL} \). We have \( D1 \subseteq E1 \) (see Fact 4.1). The logics D2–D5 are regular, e.g. D2 is the smallest regular modal logic which contains (D). We have \( D2 \subseteq E2 \).

In [10] the logic C2 is examined. It is E2 without (T) and (D). Precisely, C2 is the smallest regular logic. We have \( C2 \subseteq D2 \).

By analogy to C2, in [16] by ‘C1’ Routley denoted the system E1 without (T) and (D), i.e., C1 is the smallest modal logic which contains (K) and includes the set \( M_{Taut} \). So C1 includes \( M_{PL} \), \( R_{PL} \) and \( E_{PL} \). We have \( C1 \subseteq D1 \) and \( C1 \subseteq S0.5^o \) (see Fact 4.1).

As in [2, 4], we say that a modal system \( \Sigma \) is closed under replacement of tautological equivalents iff for all \( \varphi, \psi, \chi \in \text{For} \):
if \( \Box \varphi \equiv \psi \uparrow \in PL \) and \( \chi \in \Sigma \), then \( \chi[\varphi/\psi] \in \Sigma \). (rte)
or equivalently
if \( \Box \varphi \equiv \psi \uparrow \in PL \), then \( \chi \in \Sigma \) iff \( \chi[\varphi/\psi] \in \Sigma \). (rte’
Thus, by PL, a modal system is closed under (rte) iff it includes the following set of formulae:

\[ \text{RE}_{\text{PL}} := \{ \lceil \chi[\varphi/\psi] \equiv \varphi \equiv \psi \rceil \in \text{PL} \}. \]

In [2] a modal logic is called *classical modal* iff it contains (K) and (N), and is closed under (rte).\(^2\) Notice that

**Lemma 1.1.** If \( \Sigma \) is closed under (rte) and (N) \( \in \Sigma \), then \( \square \text{PL} \subseteq \Sigma \).

**Proof.** For any \( \tau \in \text{PL} \) we have that \( \models (p \supset p) \equiv \tau \in \text{PL} \). Hence \( \models \square \tau \in \Sigma \), by (rte) for \( \chi := (N) \), \( \varphi := 'p \supset p' \) and \( \psi := \tau \).

The non-congruential logics \( S0.9^\circ \), \( S0.9 \), \( S1^\circ \), \( S1 \), \( S2^\circ \), \( S2 \), \( S3 \) and \( S3.5 \) are examples of “classical modal logics” in the sense of [2]. For details concerning these logics see [4, 9] and Appendix A.

### 2. Some very weak systems

#### 2.1. Very weak t-regular systems

Any modal system which includes the set \( R_{\text{PL}} \) we will call *t-regular*. Thus, the set \( R_{\text{PL}} \) replaces the rule (RR) in the formulation of regular systems. Of course, if \( \Sigma \) is a t-regular system and \( \Sigma' \) is a modal system such that \( \Sigma \subseteq \Sigma' \), then \( \Sigma' \) is also a t-regular.

**Lemma 2.1.** All t-regular systems include the sets \( M_{\text{PL}} \) and \( E_{\text{PL}} \).

**Proof.** If \( \models \varphi \supset \psi \in \text{PL} \), then also \( \models (\varphi \land \psi) \supset \psi \in \text{PL} \). So we use \( R_{\text{PL}} \) and PL. Moreover, If \( \models \varphi \equiv \psi \in \text{PL} \), then also \( \models \varphi \supset \psi \in \text{PL} \) and \( \models \psi \supset \varphi \in \text{PL} \). So we use \( M_{\text{PL}} \) and PL.

**Lemma 2.2.** All instances of (K), (C), (R) and

\[ (\square (p \supset q) \land \square (q \supset r)) \supset \square (p \supset r) \]  

are members of all t-regular systems.

**Proof.** Since \( \models ((\varphi \supset \psi) \supset \varphi) \supset \psi \), \( \models (\varphi \land \psi) \supset \varphi \), \( \models (\varphi \land \psi) \supset \psi \), \( \models (\varphi \land \psi) \supset (\varphi \land \psi) \), \( \models (\varphi \land \psi) \supset (\varphi \land \psi) \), and \( \models ((\varphi \supset \psi) \land (\psi \supset \chi)) \supset (\varphi \supset \chi) \) belong to PL and all t-regular systems include \( R_{\text{PL}} \) and \( M_{\text{PL}} \).

\(^2\)In [3, 4] the expression ‘classical modal’ was referred to ‘congruential’.
Lemma 2.3. For any system $\Sigma$ the following conditions are equivalent:

(a) $\Sigma$ is t-regular,
(b) $\Sigma$ contains all instances of (K) and includes the set $M_{PL}$,
(c) $\Sigma$ contains all instances of (C) and includes the set $M_{PL}$,
(d) $\Sigma$ contains all instances of (X) and includes the set $M_{PL}$.

Proof. “(a) $\Rightarrow$ (b)”, “(a) $\Rightarrow$ (c)”, “(a) $\Rightarrow$ (d)” By lemmas 2.1 and 2.2.

“(c) $\Rightarrow$ (a)” If $\Gamma (\varphi \land \psi) \supset \chi \supsetin PL$, then $\Gamma (\Box (\varphi \land \psi) \supset \Box \chi) \supsetin \Sigma$, since $M_{PL} \subseteq \Sigma$. Hence $\Gamma (\Box \varphi \land \Box \psi) \supset \Box \chi \supsetin \Sigma$, by (C) and PL.

“(b) $\Rightarrow$ (a)” If $\Gamma (\varphi \land \psi) \supset \chi \supsetin PL$, then $\Gamma \varphi \supset (\psi \supset \chi) \supsetin PL$, by PL. Hence $\Gamma \Box \varphi \supset \Box (\psi \supset \chi) \supsetin \Sigma$, by $M_{PL} \subseteq \Sigma$. So $\Gamma \Box \varphi \supset (\Box \psi \supset \Box \chi) \supsetin \Sigma$, by (K) and PL.

“(d) $\Rightarrow$ (b)” By (X), $\Gamma (\Box (\tau \supset \varphi) \land \Box (\varphi \supset \psi)) \supset \Box (\tau \supset \psi) \supsetin \Sigma$, for any $\tau \in \text{Taut}$. Since $\Gamma \varphi \equiv (\tau \supset \varphi) \supsetin PL$ and $E_{PL} \subseteq \Sigma$, so $\Gamma \Box \varphi \equiv \Box (\tau \supset \varphi) \supsetin \Sigma$. Similarly for $\psi$. Hence $\Gamma \Box (\varphi \supset \psi) \supset (\Box \varphi \supset \Box \psi) \supsetin \Sigma$, by PL.

All t-regular systems contain all instances of the following formulae:

\[
\begin{align*}
\Diamond p & \equiv \neg \Box \neg p & (\text{df } \Diamond) \\
\Box p & \equiv \neg \Diamond \neg p & (\text{df } \Box) \\
\Diamond (p \lor q) & \equiv (\Diamond p \lor \Diamond q) & (R^\circ) \\
\Diamond (p \supset q) & \equiv (\Box p \supset \Diamond q) & (R^\circ) \\
\end{align*}
\]

The logics $\text{Cl}$, $\text{D1}$ and $\text{E1}$ are t-regular (for these logics see p. 274). The logic $\text{C1}$ is the smallest t-regular system.

Notice that $\text{E1}$ contains the following formula:

\[p \supset \Diamond p\]  

and (D). Moreover, by $(R^\circ)$, $\text{D1}$ contains the following formula:

\[\Diamond (p \supset p)\]  

In this paper by $\text{Cl} + (T_q)$ we denote the smallest t-regular logic which contains the following formula

\[\Box p \supset (p \lor \Box q)\]  

(T_q)
For t-regular logics the formula \((T_q)\) may be replace by

\[-□(q \land \neg q) \supset (∇p \supset p)\]  
\[(T'_q)\]

\[\Diamond(q \supset q) \supset (∇p \supset p)\]  
\[(T''_q)\]

The name ‘\(T_q\)’ is an abbreviation for ‘quasi-\(T\)’, because \((T)\) and \((T_q)\) are valid in all reflexive and quasi-reflexive standard models, respectively.\(^3\)

We have that \(C1 \subsetneq D1 \subsetneq E1\) and \(C1 \subsetneq C1 + (T_q) \subsetneq E1\) (see Fact 4.1).

Notice that the logic \(C1\) plus two axioms \((D)\) and \((T_q)\) equals \(E1\) (i.e. \(E1 = C1 + (D) + (T_q) = D1 + (T_q)\)). Indeed, by \(C1\) and \((D)\) we obtain \((P)\).

Hence we have \((T)\), by \((T''_q)\), \((MP)\) and \((US)\).

In this paper we prove that the logics \(C1\), \(D1\), \(C1 + (T_q)\) and \(E1\) are not closed under \((\text{rte})\). For example, the formula ‘\(\Box \Box p \equiv \Box \Box \neg p\)’ is not a member of these logics (see Remark 3.2 and Fact 4.1).

### 2.2. Very weak t-normal systems

Any modal system which contains all instances of \((K)\) and includes the set \(\Box \text{PL}\) will be called t-normal. Thus, the set \(\Box \text{PL}\) replaces the rule \((\text{RN})\) in the formulation of normal systems. Of course, if \(\Sigma\) is a t-normal system and \(\Sigma'\) is a modal system such that \(\Sigma \subseteq \Sigma'\), then \(\Sigma'\) is also a t-normal.

**Lemma 2.4.** For any system \(\Sigma\) the following conditions are equivalent:

(a) \(\Sigma\) is t-normal,

(b) \(\Sigma\) is t-regular and contains \((N)\).

**Proof.** “(a) \(\Rightarrow\) (b)” \((N)\) \(\in\) \(\Box \text{PL}\). Moreover, if \(\gamma (\varphi \land \psi) \supset \chi \in \text{PL}\), then \(\gamma \varphi \supset (\psi \supset \chi) \in \text{PL}\), by PL and \((\text{MP})\). Hence \(\gamma \Box (\varphi \supset (\psi \supset \chi)) \in \Sigma\), since \(\Box \text{PL} \subseteq \Sigma\). So \(\gamma \Box \varphi \supset (\Box \psi \supset \Box \chi) \in \Sigma\) and \(\gamma (\Box \varphi \land \Box \psi) \supset \Box \chi \in \Sigma\), by \((K)\), PL and \((\text{MP})\).

“(b) \(\Rightarrow\) (a)” By Lemma 2.3, \(\Sigma\) contains all instances of \((K)\) and includes the set \(M_{\text{PL}}\). Let \(\tau \in \text{PL}\). Then \(\gamma (p \supset p) \supset \tau \in \text{PL}\). So \(\gamma (N) \supset \Box \tau \in \Sigma\), since \(M_{\text{PL}} \subseteq \Sigma\). Thus, \(\Box \text{PL} \subseteq \Sigma\).

\(^3\)In any quasi-reflexive standard frame an accessibility relation \(R\) on a set \(W\) of worlds is such that \(\forall x, y \in W (x Ry \Rightarrow x Rx)\). See [3, p. 92, Exercise 3.51], where instead of ‘quasi-reflexive’ the term ‘reverse secondary reflexive’ is used.
The logic $S0.5^\circ$ is the smallest t-normal system; $S0.5$ is the smallest t-normal logic which contains $(T)$ (for these logics see p. 273). Of course, $S0.5$ contains $(T^\circ)$ and $(D)$.

In the present paper by $S0.5^\circ+(D)$ we denote the smallest t-normal logic which contains $(D)$, i.e. $S0.5^\circ$ plus $(D)$. Of course, $S0.5^\circ+(D)$ contains $(P)$. Moreover, by $S0.5^\circ+(T_q)$ we denote the smallest t-normal logic which contains $(T_q)$, i.e. $S0.5^\circ$ plus the axiom $(T_q)$.

We have that $S0.5^\circ \subseteq S0.5^\circ+(D) \subset S0.5$, besides $S0.5^\circ \subset S0.5^\circ+(T_q) \subset S0.5$ and $Cl+(T_q) \subset S0.5^\circ+(T_q)$ (see Fact 4.1).

Notice that the logic $S0.5^\circ$ plus two axioms $(D)$ and $(T_q)$ is equals $S0.5$ (i.e. $S0.5 = S0.5^\circ+(D)+(T_q)$). Indeed, from $S0.5^\circ$ and $(D)$ we obtain $(P)$, and hence $(T)$, by $(T_q''\circ)$, $(MP)$ and $(US)$.

In this paper we prove that $S0.5^\circ, S0.5^\circ+(T_q), S0.5^\circ+(D)$ and $S0.5$ are not closed under $(rte)$. For example, the formula ‘$\square\square p \equiv \square\square \neg p$’ is not a member of these logics (see Remark 3.2 and Fact 4.1).

### 2.3. Very weak t-normal rte-systems

By *rte-systems* we mean modal systems which are closed under $(rte)$. By Lemma 1.1 we have

**Lemma 2.5.** If a rte-system contains $(N)$ and all instances of $(K)$, then it is t-normal.

Let $S0.5^\circ_{rte}, S0.5_{rte}, S0.5^\circ_{rte}+(D)$ and $S0.5^\circ_{rte}+(T_q)$ be, respectively, such versions of the logics $S0.5^\circ, S0.5, S0.5^\circ+(D)$ and $S0.5^\circ+(T_q)$ that are closed under $(rte)$. Thus, $S0.5^\circ_{rte}$ is the smallest t-normal rte-system, and $S0.5_{rte}, S0.5^\circ_{rte}+(D)$ and $S0.5^\circ_{rte}+(T_q)$ are the smallest t-normal rte-logics which contain $(T)$, $(D)$ and $(T_q)$, respectively.\(^4\) We have that $S0.5^\circ_{rte} \subset S0.5^\circ_{rte}+(D) \subset S0.5_{rte}$ and $S0.5^\circ_{rte} \subset S0.5^\circ_{rte}+(T_q) \subset S0.5_{rte}$ (see Fact 4.1).

### 2.4. Very weak t-regular rte-systems

Let $Cl_{rte}, D1_{rte}, E1_{rte}$ and $E1_{rte}+(T_q)$ be, respectively, such versions of the logics $Cl$, $D1$, $E1$ and $Cl+(T_q)$ that are closed under $(rte)$. The

\(^4\)Thus, $S0.5^\circ_{rte}$ is the smallest classical modal logic in the sense of [2], and $S0.5_{rte}, S0.5^\circ_{rte}+(D)$ and $S0.5^\circ_{rte}+(T_q)$ are the smallest classical modal logics (in the sense of [2]) which contain $(T)$, $(D)$ and $(T_q)$, respectively.
logic $\mathbf{C}_{1\text{rte}}$ is the smallest t-regular rte-system. The logics $\mathbf{D}_{1\text{rte}}$, $\mathbf{E}_{1\text{rte}}$ and $\mathbf{E}_{1\text{rte}} + (T_q)$ are smallest t-regular rte-logics which contain $(T)$, $(D)$ and $(T_q)$, respectively. We have that $\mathbf{C}_{1\text{rte}} \subseteq \mathbf{D}_{1\text{rte}} \subseteq \mathbf{E}_{1\text{rte}}$ and $\mathbf{C}_{1\text{rte}} \subseteq \mathbf{E}_{1\text{rte}} + (T_q) \subseteq \mathbf{E}_{1\text{rte}}$ (see Fact 4.1).

3. Semantics for very weak systems

3.1. Models for very weak t-normal and t-regular systems

For very weak t-normal modal systems we are using the following semantics, which consists of “t-normal models”.

A model for very weak t-normal systems (or t-normal model) is any triple $\langle w, A, V \rangle$ in which

1. $w$ is a «distinguished» (normal) world,
2. $A$ is a set of worlds which are alternatives to the world $w$,
3. $V$ is a valuation from $\text{For} \times (\{w\} \cup A)$ to $\{0, 1\}$:

   (i) for all formulae and all worlds, $V$ preserves classical conditions for truth-value operators,
   (ii) for the world $w$ and any $\varphi \in \text{For}$

   \[(V_\Box) \quad V(\Box \varphi, w) = 1 \text{ iff } \forall x \in A \quad V(\varphi, x) = 1,\]
   (iii) for every world from $A \setminus \{w\}$, formulae $\Box \Box \varphi$ have arbitrary values.

A formula $\varphi$ is true in a t-normal model $\langle w, A, V \rangle$ iff $V(\varphi, w) = 1$. We say that a formula is t-normal valid iff it is true in all t-normal models.

We say that a t-normal model $\langle w, A, V \rangle$ is self-associate (resp. empty, non-empty) iff $w \in A$ (resp. $A = \emptyset$, $A \neq \emptyset$). Let $\mathbf{n}_M$ be the class of all t-normal models. Moreover, let $\mathbf{n}_M^{sa}$ (resp. $\mathbf{n}_M^\emptyset$, $\mathbf{n}_M^+$) be the class of t-normal models which are self-associate (resp. empty, non-empty). Of course, $\mathbf{n}_M^{sa} \subseteq \mathbf{n}_M^+$ and $\mathbf{n}_M^\emptyset \cap \mathbf{n}_M^+ = \emptyset$.

Remark 3.1. We may also use the class of models of the form $\langle W, w, A, V \rangle$, where $W$ is a non-empty set of worlds, $w \in W$, $A \subseteq W$, and $w$, $A$ and $V$ are as mentioned above. Of course, the triple $\langle w, A, V \rangle$ may be identified with the quadruple $\langle W, w, A, V \rangle$ such that $W = \{w\} \cup A$. $\dashv$
In the case of very weak t-regular systems we broaden the class of t-normal models by the class of *queer* models of the form \( \langle w, V \rangle \) with only one (queer) world \( w \) and a valuation \( V : \text{For} \times \{ w \} \to \{ 0, 1 \} \) which satisfies classical conditions for truth-value operators and such that

\[(ii')\] for the world \( w \) and any \( \varphi \in \text{For} \)

\[V(\Box \varphi, w) = 0.\]

Of course, a queer model \( \langle w, V \rangle \) may be identified with the valuation \( V : \text{For} \to \{ 0, 1 \} \) such that \( V(\varphi) = V(\varphi, w) \), for any \( \varphi \) from For.

Let \( qM \) be the class of all queer models and we put \( rM := nM \cup qM \), i.e. \( rM \) is the class of models for very weak t-regular systems.

A formula \( \varphi \) is *true* in a queer model \( \langle w, V \rangle \) iff \( V(\varphi, w) = 1 \). We say that a formula is *t-regular valid* iff it is true in all models from \( rM \). We have the following lemmas.

**Lemma 3.1.**
1. If \( \varphi \in \text{PL} \), then \( V(\varphi, x) = 1 \), for any world \( x \) in any model from \( rM \). So all formulae from PL are t-regular valid.
2. All formulae from \( \Box \text{PL} \) are t-normal valid.
3. All formulae from the sets \( M_{\text{PL}} \) \( R_{\text{PL}} \) and \( E_{\text{PL}} \) are t-regular valid.

**Lemma 3.2.**
1. All instances of formulae \( (K) \) and \( (R) \) are t-regular valid.
2. All instances of the formulae \( (T) \) and \( (T_q) \) are true in any model from \( nM^{sa} \cup qM \).
3. All instances of the formula \( (D) \) are true in all models from \( nM^{+} \cup qM \).
4. All instances of the formula \( (T_q) \) are true in all models from \( nM^{\delta} \).

**Fact 3.3.** Let \( \Box \varphi \equiv \psi \equiv \psi^\bot \in \text{PL} \). Then for any classical formula \( \chi \) (without the modal operator) following holds: \( V(\chi, x) = V(\chi[\varphi / \psi], x) \), for any world \( x \) in any model from \( rM \).

**Remark 3.2.** Let \( w \neq a \), \( A := \{ w, a \} \) and \( V \) be an arbitrary valuation such that \( V(\Box p, a) = 1 \) and \( V(\Box \neg p, a) = 0 \). Then \( \langle w, A, V \rangle \) belongs to \( nM^{sa} \) and the formula \( \Box \Box p \equiv \Box \Box \neg p \) is not true in this model. \( \dashv \)
3.2. Models for very weak t-normal and t-regular rte-systems

For very weak t-normal rte-systems we are using t&rte-normal models, where by a t&rte-normal model we mean a t-normal model \( \langle w, A, V \rangle \) which satisfies the following condition:

(iv) for all formulae \( \varphi, \psi \) and \( \chi \): if \( \models \varphi \equiv \psi \in \text{PL} \) and \( \forall x \in A \ V(\chi, x) = 1 \),
then \( \forall x \in A \ V(\chi[\varphi/\psi], x) = 1 \).

Of course, the condition (iv) is equivalent to the following:

(iv') for all formulae \( \varphi, \psi \) and \( \chi \): if \( \models \varphi \equiv \psi \in \text{PL} \), then
\[ \forall x \in A \ V(\chi[\varphi/\psi], x) = 1 \iff \forall x \in A \ V(\chi, x) = 1. \]

Moreover, by \((V\Box)\), the condition (iv) is equivalent to the following one:

(iv'') for all formulae \( \varphi, \psi \) and \( \chi \): if \( \models \varphi \equiv \psi \in \text{PL} \), then
\[ V(\Box \chi, w) = V(\Box \chi[\varphi/\psi], w). \]

Let \( nM_{rte} \) be the class of all t&rte-normal models. Moreover, let \( nM^\equiv_{rte} \) (resp. \( nM^\emptyset_{rte}, nM^+_{rte} \)) be the class of t&rte-normal models which are self-associate (resp. empty, non-empty).

In the case of very weak t-regular rte-systems we broaden the class of t&rte-normal models by queer models. We put \( rM_{rte} := nM_{rte} \cup qM \), i.e. \( rM_{rte} \) is the class of models for very weak t&rte-regular systems.

We say that a formula is t&rte-normal valid (resp. t&rte-regular valid) iff it is true in all models from \( nM_{rte} \) (resp. \( rM_{rte} \)).

We have the following lemma.

**Lemma 3.4.** If \( \models \varphi \equiv \psi \in \text{PL} \), then \( V(\chi, w) = V(\chi[\varphi/\psi], w) \) in all t&rte-normal models and all queer models. So all formulae from RE_{PL} are t&rte-regular valid.

4. Determination theorems

Let \( C \) be any class of considered models. We say that a formula \( \varphi \) is \( C \)-valid (written \( \models_C \varphi \)) iff \( \varphi \) is true in all models from \( C \).

Let \( \Sigma \) be an arbitrary modal system. We say that \( \Sigma \) is sound with respect to \( C \) iff \( \Sigma \subseteq \{ \varphi \in \text{For} : \models_C \varphi \} \). We say that \( \Sigma \) is complete with respect to \( C \) iff \( \Sigma \supseteq \{ \varphi \in \text{For} : \models_C \varphi \} \). We say that \( \Sigma \) is determined by \( C \) iff \( \Sigma = \{ \varphi \in \text{For} : \models_C \varphi \} \), i.e., \( \Sigma \) is sound and complete with respect to \( C \).
4.1. Soundness

By lemmas 3.1, 3.2 and 3.4 we obtain the following facts.

**FACT 4.1.**
1. C1 is sound with respect to the class rM.
2. D1 is sound with respect to the class nM⁺ ∪ qM.
3. E1 is sound with respect to the class nMₛa ∪ qM.
4. C1 ⊕ (Tₜ) is sound with respect to the class nMₛa ∪ nM⁸ ∪ qM.
5. S₀.₅̊ is sound with respect to the class nM.
6. S₀.₅̊ ⊕ (D) is sound with respect to the class nM⁺.
7. S₀.₅ is sound with respect to the class nMₛa.
8. S₀.₅̊ ⊕ (Tₜ) is sound with respect to the class nMₛa ∪ nM⁸.
9. C₁ᵣᵣₑ is sound with respect to the class rMᵣᵣₑ.
10. D₁ᵣᵣₑ is sound with respect to the class nMᵣᵣₑ⁺ ∪ qM.
11. E₁ᵣᵣₑ is sound with respect to the class nMᵣᵣₑ⁺ ∪ qM.
12. E₁ᵣᵣₑ ⊕ (Tₜ) is sound with respect to the class nMᵣᵣₑ⁺ ∪ qM.
13. S₀.₅ᵣᵣₑ is sound with respect to the class nMᵣᵣₑ.
14. S₀.₅ᵣᵣₑ ⊕ (D) is sound with respect to the class nMᵣᵣₑ⁺.
15. S₀.₅ᵣₑ is sound with respect to the class nMᵣᵣₑₛa.
16. S₀.₅ᵣᵣₑ ⊕ (Tₜ) is sound with respect to the class nMᵣᵣₑₛa ∪ nMᵣᵣₑ⁸.

For completeness of considered very weak logics we use canonical models method.

4.2. Notions and facts concerning maximal consistent sets

For the following definitions see, for example, [3, 2.4 and 2.6]. Let Σ and Σ’ be any modal systems, and Γ ⊆ For.

Σ is **consistent** iff Σ ≠ For; equivalently in the light of PL, iff ‘p ∧ ¬ p’ does not belong to Σ. For example, all modal logics from Section 2 are consistent.

A formula ϕ is **deducible** from Γ in Σ (written Γ ⊢ₜ Σ ϕ) iff for some \(\psi_1, \ldots, \psi_n\) ⊆ Γ \((n ≥ 0)\) we have \(Γ (ψ_1 ∧ \cdots ∧ ψ_n) ⊢ Σ \ φ₁\) ∈ Σ. We have \(Γ ⊢ₜ PL ⊆ Γ ⊢ₜ \). Moreover, Σ ⊢ₜ Σ ϕ iff ϕ ∈ Σ iff \(∅ ⊢ₜ Σ \ φ\).

A set Γ is **Σ-consistent** iff for some ϕ ∈ For, Γ ⊬ₜ Σ ϕ; equivalently in the light of PL, iff Γ ⊬ₜ p ∧ ¬ p. We have (see e.g. [3]):
• If $\Gamma$ is $\Sigma$-consistent, then $\Sigma$ is consistent.

• $\Sigma$ is consistent iff $\Sigma$ is $\Sigma$-consistent.

• If $\Gamma$ is $\Sigma$-consistent and $\Sigma' \subseteq \Sigma$, then $\Gamma$ is $\Sigma'$-consistent; so, $\Gamma$ is PL-consistent.

We say that $\Gamma$ is $\Sigma$-maximal iff $\Gamma$ is $\Sigma$-consistent and $\Gamma$ has only $\Sigma$-inconsistent proper extensions. Let $\text{Max}_\Sigma$ be the set of all $\Sigma$-maximal sets.

**Lemma 4.2 ([3]).** Let $\Gamma \in \text{Max}_\Sigma$. Then

1. $\Sigma \subseteq \Gamma$ and $\Gamma$ is a modal system.
2. $\Gamma \vdash_\Sigma \varphi$ iff $\varphi \in \Gamma$.
3. $\neg \varphi \top \in \Gamma$ iff $\varphi \notin \Gamma$.
4. $\varphi \land \psi \top \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.
5. $\varphi \lor \psi \top \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.
6. $\varphi \supset \psi \top \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.
7. $\varphi \equiv \psi \top \in \Gamma$ iff either $\varphi, \psi \in \Gamma$ or $\varphi, \psi \notin \Gamma$.

**Lemma 4.3.** If $\Gamma \in \text{Max}_\Sigma$ and $\Sigma' \subseteq \Sigma$, then $\Gamma \in \text{Max}_{\Sigma'}$. So $\Gamma \in \text{Max}_{\text{PL}}$.

**Proof.** Let $\Gamma \in \text{Max}_\Sigma$ and $\Sigma' \subseteq \Sigma$. Then $\Gamma$ is $\Sigma'$-consistent and PL-consistent. Moreover, suppose that $\Gamma \cup \{\varphi\}$ is $\Sigma'$-consistent. Then $\Gamma \cup \{\varphi\}$ is also PL-consistent. So $\neg \varphi \top \notin \Gamma$. Therefore $\varphi \in \Gamma$, by Lemma 4.2.3. Hence $\Gamma \cup \{\varphi\} = \Gamma$. Thus $\Gamma$ be $\Sigma'$-maximal. ⊣

**Lemma 4.4 ([3]).** 1. $\Gamma \vdash_\Sigma \varphi$ iff $\varphi \in \Delta$, for any $\Delta$ such that $\Delta \in \text{Max}_\Sigma$ and $\Gamma \subseteq \Delta$.

2. $\varphi \in \Sigma$ iff $\varphi \in \Delta$, for any $\Delta \in \text{Max}_\Sigma$.

### 4.3. Canonical models

For completeness of very weak logics we need two following auxiliary lemmas.
Lemma 4.5. Let $\Sigma$ be a t-regular consistent system and let $\Gamma$ be a $\Sigma$-maximal set such that $\Gamma \cap \Box_{\text{For}} \neq \emptyset$, i.e. $\{ \psi \in \text{For} : \Box \Box \psi \in \Gamma \} \neq \emptyset$.\(^5\) Then for every $\varphi \in \text{For}$ the following conditions are equivalent:

(a) $\Box \Box \varphi \in \Gamma$.

(b) $\Gamma \vdash_{\Sigma} \Box \varphi$.

(c) $\{ \psi : \Box \Box \psi \in \Gamma \} \vdash_{\text{PL}} \varphi$.

(d) $\varphi \in \Delta$, for any PL-maximal set $\Delta$ such that $\{ \psi : \Box \Box \psi \in \Gamma \} \subseteq \Delta$.

Proof. “(a) $\iff$ (b)” Lemma 4.2.2.

“(a) $\Rightarrow$ (d)” It is trivial, since for any $\Gamma, \Delta \subseteq \text{For}$, if $\Box \Box \varphi \in \Gamma$ and $\{ \psi \in \text{For} : \Box \Box \psi \in \Gamma \} \subseteq \Delta$, then $\varphi \in \Delta$.

“(d) $\iff$ (c)” By Lemma 4.4.1.

“(c) $\Rightarrow$ (b)” Ether $\varphi \in \text{PL}$ or for some $\psi_1, \ldots, \psi_n \in \{ \psi : \Box \Box \psi \in \Gamma \}$, $n > 0$, we have $\Box (\Box \psi_1 \land \cdots \land \Box \psi_n) \supset \varphi \in \text{PL}$. But the first case entails the second case. Hence $\Box (\Box \psi_1 \land \cdots \land \Box \psi_n) \supset \Box \varphi \in \Sigma$, since $R_{\text{PL}} \subseteq \Sigma$. But $\Gamma$ contains each of $\Box \Box \psi_1, \ldots, \Box \Box \psi_n$, so $\Gamma \vdash_{\Sigma} \Box \varphi$.

Let $\Sigma$ be a t-regular system, $\Gamma \in \text{Max}_\Sigma$ and $\{ \psi : \Box \Box \psi \in \Gamma \} \neq \emptyset$. We say that $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is a canonical model for $\Sigma$ and $\Gamma$ iff it satisfies these conditions:

- $w_\Gamma := \Gamma$,
- $A_\Gamma := \{ \Delta \in \text{Max}_{\text{PL}} : \forall \psi \in \text{For} (\Box \Box \psi \in \Gamma \Rightarrow \psi \in \Delta) \}$,
- $V_\Gamma : \text{For} \times (\{ w_\Gamma \} \cup A_\Gamma) \rightarrow \{ 0, 1 \}$ is the valuation such that for all $\varphi \in \text{For}$ and $\Delta \in \{ w_\Gamma \} \cup A_\Gamma$
  $V_\Gamma(\varphi, \Delta) := \begin{cases} 1 & \text{if } \varphi \in \Delta \\ 0 & \text{otherwise} \end{cases}$

Lemma 4.6. For any t-regular system $\Sigma$ and any $\Gamma \in \text{Max}_\Sigma$ such that $\{ \psi : \Box \Box \psi \in \Gamma \} \neq \emptyset$ it holds that:

(a) $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is a t-normal model.

(b) If $\Sigma$ contains all instances of $(T)$, then $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is self-associate.

\(^5\)Notice that all t-normal systems satisfy these assumptions. Firstly, all t-normal systems are t-regular. Secondly, for any t-normal system $\Sigma$, if $\Gamma$ is $\Sigma$-maximal, then $\{ \psi : \Box \Box \psi \in \Gamma \} \neq \emptyset$, since $\Box_{\text{PL}} \subseteq \Sigma \subseteq \Gamma$, by Lemma 4.2.1.
(c) If $\Sigma$ contains all instances of (D), then $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is non-empty.

(d) If $\Sigma$ contains all instances of (T$_q$), then $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is either empty or self-associate.

(e) If $\Sigma$ is a rte-system, then $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is t&rte-normal model.

**Proof.** (a) Thanks to properties of maximal sets (see Lemma 4.2), for every $\Delta \in \{ w_\Gamma \} \cup A_\Gamma$ the assignment $V_\Gamma(\cdot, \Delta)$ preserves classical conditions for truth-value operators. We prove that for $w_\Gamma$ the assignment $V_\Gamma(\cdot, w_\Gamma)$ satisfies the condition ($V_\Box$).

For any $\varphi \in \text{For}$: $V_\Gamma(\Box \varphi, w_\Gamma) = 1$ if $\square \varphi \wedge \varphi \in \Gamma$ (by definition of $V_\Gamma$) iff for every $\Delta \in \text{Max}_{\text{PL}}$ for which $\{ \psi \in \text{For} : \square \psi \in \Gamma \} \subseteq \Delta$ we have $\varphi \in \Delta$ (by Lemma 4.5) iff for every $\Delta \in A_\Gamma$, $\varphi \in \Delta$ (by definition of $A_\Gamma$) iff for every $\Delta \in A_\Gamma$, $V_\Gamma(\varphi, \Delta) = 1$ (by definition of $V_\Gamma$).

(b) We show that $w_\Gamma \in A_\Gamma$. Firstly, by Lemma 4.3, $\Gamma \in \text{Max}_{\text{PL}}$-maximal. Secondly, for any $\psi \in \text{For}$, $\square \psi \supset \psi \in \Gamma$, by Lemma 4.2.1. So, if $\square \psi \in \Gamma$, then $\psi \in \Gamma$, by Lemma 4.2.6.

(c) For some $\varphi_0$ we have $\square \varphi_0 \in \Gamma$. By Lemma 4.2.1, $\square \varphi_0 \supset \neg \square \neg \varphi_0 \in \Gamma$. Hence, by lemmas 4.2.6 and 4.2.1, $\neg \square \neg \varphi_0 \in \Gamma$ and $\square \neg \varphi_0 \notin \Gamma$. Therefore, by Lemma 4.5, $\neg \varphi_0 \notin \Delta_0$, for some $\Delta_0$ such that $\Delta_0$ is PL-maximal and $\{ \psi : \square \psi \in \Gamma \} \subseteq \Delta_0$. Hence $\Delta_0 \in A_\Gamma$. Thus, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in nM^+$.

(d) We show that $w_\Gamma \in A_\Gamma$ or $A_\Gamma = \emptyset$. Notice that, by lemmas 4.2.1 and 4.2.6, $\neg \square(q \wedge \neg q) \supset (\square \psi \supset \psi \in \Gamma)$, for any formula $\psi$. Suppose that $A_\Gamma \neq \emptyset$. Then $\neg \square(q \wedge \neg q) \notin \Gamma$, by Lemma 4.5, since $\neg (q \wedge \neg q) \notin \Delta$, for any $\Delta$ which is PL-consistent. So, $\neg \square(q \wedge \neg q) \in \Gamma$. Therefore $\square \psi \supset \psi \in \Gamma$. Hence $w_\Gamma \in A_\Gamma$, as in (b).

(e) Suppose that $\varphi \equiv \psi \in \text{PL}$. Then $\square \chi[\varphi / \psi] \equiv \square \chi \in \Sigma$, since $\text{RE}_{\text{PL}} \subseteq \Sigma$. So also $\square \chi[\varphi / \psi] \equiv \square \chi \in \Gamma$, by Lemma 4.2.1. Thus, $V(\square \chi, w) = V(\square \chi[\varphi / \psi], w)$, by definition of $V_\Gamma$.

Let $\Sigma$ be a t-regular system, $\Gamma \in \text{Max}_{\Sigma}$ and $\{ \psi : \square \psi \in \Gamma \} = \emptyset$. We say that $\langle w_\Gamma, V_\Gamma \rangle$ is a canonical model for $\Sigma$ and $\Gamma$ if it satisfies these conditions:

- $w_\Gamma := \Gamma$,
- $V_\Gamma$: For $\times \{ w_\Gamma \} \to \{ 0, 1 \}$ is the valuation such that

$$V_\Gamma(\varphi, w_\Gamma) := \begin{cases} 1 & \text{if } \varphi \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$
Lemma 4.7. For any t-regular system $\Sigma$ and any $\Gamma \in \text{Max}_\Sigma$ such that \{ $\psi: [\Box \psi] \in \Gamma$ \} = $\emptyset$; $\langle w_\Gamma, V_\Gamma \rangle$ is a queer model.

Proof. Thanks to properties of maximal sets in modal systems (see Lemma 4.2), the assignment $V_\Gamma$ preserves classical conditions for truth-value operators. Moreover, for any $\varphi \in \text{For}$ we have: $\Box \varphi \not\in \Gamma$. So, $V_\Gamma(\Box \varphi, w_\Gamma) = 0$.

4.4. Completeness

By lemmas 4.4.2 and 4.6 for very weak t-normal and t-normal rte-systems we obtain

Theorem 4.8. 1. $\text{S0.5}^\circ$ is complete with respect to the class $\text{nM}$.

2. $\text{S0.5}^\circ + (\text{D})$ is complete with respect to the class $\text{nM}^+$.  

3. $\text{S0.5}^\circ + (\text{T}_q)$ is complete with respect to the class $\text{nM}^{sa} \cup \text{nM}^\emptyset$.

4. $\text{S0.5}$ is complete with respect to the class $\text{nM}^{sa}$.

5. $\text{S0.5}_{\text{rte}}$ is complete with respect to the class $\text{nM}_{\text{rte}}$.

6. $\text{S0.5}_{\text{rte}} + (\text{D})$ is complete with respect to the class $\text{nM}^{+}_{\text{rte}}$.

7. $\text{S0.5}_{\text{rte}} + (\text{T}_q)$ is complete with respect to the class $\text{nM}^{sa}_{\text{rte}} \cup \text{nM}^\emptyset_{\text{rte}}$.

8. $\text{S0.5}_{\text{rte}}$ is complete with respect to the class $\text{nM}^{sa}_{\text{rte}}$.

Proof. The logics $\text{S0.5}^\circ$, $\text{S0.5}^\circ + (\text{D})$, $\text{S0.5}^\circ + (\text{T}_q)$ and $\text{S0.5}$ are consistent and t-regular. Moreover, for any t-normal logic $\Lambda$, if $\Gamma \in \text{Max}_\Lambda$, then $\{ \psi: [\Box \psi] \in \Gamma \} \neq \emptyset$, since $\Box \text{PL} \subseteq \Lambda \subseteq \Gamma$.

1. Let $\varphi$ be an arbitrary formula such that $\vdash_{\text{nM}} \varphi$. Let $\Gamma$ be an arbitrary $\text{S0.5}^\circ$-maximal set. By Lemma 4.6a, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \text{nM}$. Thus, $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$, by definitions of $w_\Gamma$ and $V_\Gamma$. So, we have shown that $\varphi$ belongs to all $\text{S0.5}^\circ$-maximal sets. Hence $\varphi \in \text{S0.5}^\circ$, by Lemma 4.4.2.

2. By Lemma 4.6c, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \text{nM}^+$. The rest as in 1.

3. By Lemma 4.6d, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \text{nM}^+ \cup \text{nM}^\emptyset$. The rest as in 1.

4. By Lemma 4.6b, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \text{nM}^{sa}$. The rest as in 1.

5. By Lemma 4.6e, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \text{nM}_{\text{rte}}$. The rest as in 1.

6. By Lemma 4.6ce, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \text{nM}^{+}_{\text{rte}}$. The rest as in 1.

7. By Lemma 4.6de, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \text{nM}^{sa}_{\text{rte}} \cup \text{nM}^\emptyset_{\text{rte}}$. The rest as in 1.

8. By Lemma 4.6be, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \text{nM}^{sa}_{\text{rte}}$. The rest as in 1.  

By lemmas 4.4.2, 4.6 and 4.7 for very weak t-regular and t-regular rte-systems we obtain

**Theorem 4.9.**
1. C1 is complete with respect to the class rM.
2. D1 is complete with respect to the class nM⁺ ∪ qM.
3. C1+(T_q) is complete with respect to the class nMsa ∪ nMδ ∪ qM.
4. E1 is complete with respect to the class nMsa ∪ qM.
5. C1_rte is complete with respect to the class rM_rte.
6. D1_rte is complete with respect to the class nM_rte ∪ qM.
7. E1_rte+(T_q) is complete with respect to nM_rte sa ∪ nM_rte ø ∪ qM.
8. E1_rte is complete with respect to nM_rte ∪ qM.

**Proof.**
1. Let ϕ be an arbitrary formula such that |=rM ϕ. Let Γ be an arbitrary C1-maximal set. In both alternative cases from lemmas 4.6 and 4.7, either ⟨wΓ, AΓ, VΓ⟩ ∈ nM or ⟨wΓ, VΓ⟩ ∈ qM. Thus, in both cases we have VΓ(ϕ, wΓ) = 1. Hence ϕ ∈ Γ, by definitions of wΓ and VΓ. So, we have shown that ϕ belongs to all C1-maximal sets. Hence ϕ ∈ C1, by Lemma 4.4.2.

2. ⟨wΓ, AΓ, VΓ⟩ ∈ nM⁺ or ⟨wΓ, VΓ⟩ ∈ qM. The rest as in 1.
3. ⟨wΓ, AΓ, VΓ⟩ ∈ nMsa ∪ nMδ or ⟨wΓ, VΓ⟩ ∈ qM. The rest as in 1.
4. ⟨wΓ, AΓ, VΓ⟩ ∈ nMsa or ⟨wΓ, VΓ⟩ ∈ qM. The rest as in 1.
5. ⟨wΓ, AΓ, VΓ⟩ ∈ nM_rte or ⟨wΓ, VΓ⟩ ∈ qM. The rest as in 1.
6. ⟨wΓ, AΓ, VΓ⟩ ∈ nM_rte⁺ or ⟨wΓ, VΓ⟩ ∈ qM. The rest as in 1.
7. ⟨wΓ, AΓ, VΓ⟩ ∈ nM_rte sa ∪ nM_rte ø or ⟨wΓ, VΓ⟩ ∈ qM. The rest as in 1.
8. ⟨wΓ, AΓ, VΓ⟩ ∈ nM_rte sa ∪ ⟨wΓ, VΓ⟩ ∈ qM. The rest as in 1.

A. Location of very weak modal logics

A.1. Strict implication and strict equivalence

In original Lewis’ works (see e.g. [12]) the primitive modal operator is the possibility sign ‘◊’. The necessity sign ‘□’ is the abbreviation of ‘¬◊¬’. Moreover, for the connective of strict implication ‘¬¬’ was used □(ϕ ⊃ ψ) as an abbreviation of a formula □¬◊(ϕ ∧ ¬ψ). In this paper—as in [9]—the primitive modal operator is ‘□’ and □(ϕ ⊃ ψ) is an abbreviation of □(ϕ ⊃ ψ). Moreover, in this paper—as in [12] and [9]—a strict equivalence □ϕ ≡ ψ is an abbreviation of □(ϕ ⊃ ψ) ∧ (ψ ⊃ □ϕ).
Lemma A.1. For any modal system $\Sigma$ and any $\varphi, \psi \in \text{For}$:

if $\Box \varphi \rightarrow \Box \psi \in \Sigma$, then $\Box \varphi \rightarrow \Box \psi, \Box \psi \rightarrow \Box \varphi \in \Sigma$.

Proof. Let $\Box \varphi \rightarrow \Box \psi \in \Sigma$, i.e., $\Box (\varphi \rightarrow \psi) \wedge \Box (\psi \rightarrow \varphi) \in \Sigma$. Hence $\Box (\varphi \rightarrow \psi) \in \Sigma$, by PL, i.e., $\Box \varphi \rightarrow \Box \psi, \Box \psi \rightarrow \Box \varphi \in \Sigma$. $\dashv$

Lemma A.2. For any $t$-regular system $\Sigma$ and any $\varphi, \psi \in \text{For}$:

$\Box \varphi \rightarrow \Box \psi \in \Sigma$ iff $\Box (\varphi \equiv \psi) \in \Sigma$.

Proof. If $\Box (\varphi \rightarrow \psi) \wedge \Box (\psi \rightarrow \varphi) \in \Sigma$, then $\Box (\varphi \equiv \psi) \in \Sigma$, by (MP) and since $R_{\text{PL}} \subseteq \Sigma$. If $\Box (\varphi \equiv \psi) \in \Sigma$, then $\Box (\varphi \rightarrow \psi), \Box (\psi \rightarrow \varphi) \in \Sigma$, since $\text{PL}, M_{\text{PL}} \subseteq \Sigma$. So, $\Box (\varphi \equiv \psi) \in \Sigma$, by PL. $\dashv$

Lemma A.3 ([4, 9]). If $\Sigma$ is closed under the following rule

if $\Box \varphi \in \Sigma$, then $\varphi \in \Sigma$, \hspace{1cm} (RN$_*$)

then $\Sigma$ is closed under the strict version of modus ponens

if $\Box \varphi \rightarrow \psi \in \Sigma$ and $\varphi \in \Sigma$, then $\psi \in \Sigma$. \hspace{1cm} (SMP)

Hence, any modal system which contains all instances of (T) is also closed under (RN$_*$) and (SMP).

Lemma A.4 ([4]). Let $\Sigma$ be a $\text{rte}$-system which is closed under (SMP). Then $\Sigma$ is closed under (RN$_*$).

Proof. Let $\Box \varphi \in \Sigma$ and $\tau \in \text{PL} \subseteq \Sigma$. Then $\Box \tau \equiv (\tau \supset \varphi) \in \text{PL}$, so $\Box (\tau \supset \varphi) \in \Sigma$, by (rte). So $\varphi \in \Sigma$, by (SMP). $\dashv$

Lemma A.5. Let $\Sigma$ be any system which is closed under (SMP) and includes $M_{\text{PL}}$. Then $\Sigma$ is closed under (RN$_*$).

Proof. Let $\Box \varphi \in \Sigma$ and $\tau \in \text{PL} \subseteq \Sigma$. Then $\Box \tau \supset (\tau \supset \varphi) \in \text{PL}$, so $\Box (\tau \supset \varphi) \in \Sigma$, since $M_{\text{PL}} \subseteq \Sigma$. Thus, $\Box (\tau \supset \varphi) \in \Sigma$, by (MP), and $\varphi \in \Sigma$, by (SMP). $\dashv$
A.2. Strict classical modal systems

Imitating [4], we say that a modal system \( \Sigma \) is \( \text{strict}_T \) classical ("traditionally strict classical") iff \( \Box \text{PL} \subseteq \Sigma \) and \( \Sigma \) is closed under "traditional replacement rule for strict equivalents":

\[
\text{if } \Gamma \varphi \vDash \psi \neg \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi[\varphi/\psi] \in \Sigma. \quad (\text{RRSE}_T)
\]

Moreover, a modal system \( \Sigma \) is called \( \text{strict classical} \) iff \( \Box \text{PL} \subseteq \Sigma \) and \( \Sigma \) is closed under the following replacement rule:

\[
\text{if } \Gamma \Box(\varphi \equiv \psi) \neg \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi[\varphi/\psi] \in \Sigma. \quad (\text{RRSE})
\]

We obtain that for modal logics which contain (K) and/or (X), the above notions are equivalent (see Lemma A.9).

**Lemma A.6 ([4]).** Let \( \Sigma \) be \( \text{strict}_T \) or \( \text{strict classical} \). Then \( \Sigma \) is also a rte-system.

**Proof.** Suppose that \( \Gamma \varphi \vDash \psi \neg \in \text{PL} \) and \( \chi \in \Sigma \). Since \( \Box \text{PL} \subseteq \Sigma \), so we have that \( \Gamma \Box(\varphi \equiv \psi) \neg \in \Sigma \) and \( \Gamma \Box(\varphi \supset \psi) \land \Box(\psi \supset \varphi) \neg \in \Sigma \), by PL. Hence \( \chi[\varphi/\psi] \in \Sigma \) follows by (RRSE) or by (RRSE\(_T\)), respectively. ⊣

By definitions we have the following lemma.

**Lemma A.7.** Let \( \Sigma \) be \( \text{strict}_T \) or \( \text{strict classical} \) and let \( \Sigma \) contain all instances of (K). Then \( \Sigma \) is t-normal.

Now notice that

**Lemma A.8 ([4, 9]).** Let \( \Sigma \) be \( \text{strict}_T \) or \( \text{strict classical} \) and let \( \Sigma \) contain all instances of (X) (resp. \( \Box(X) \)). Then \( \Sigma \) contains all instances of (K) (resp. \( \Box(K) \)).

**Proof.** Let \( \varphi, \psi \in \text{For} \). Since \( \Box \text{PL} \subseteq \Sigma \) and \( \Gamma \varphi \vDash (\tau \supset \varphi) \neg \in \text{PL} \), for any \( \tau \in \text{Taut} \), so we have \( \Gamma \Box(\varphi \equiv (\tau \supset \varphi)) \neg \in \Sigma \), by PL. Similarly for \( \psi \). Let \( \Sigma \) contain all instances of (X). Then \( \Gamma (\square(\varphi \supset \psi) \land \square(\varphi \supset \psi)) \supset \square(\tau \supset \psi) \neg \in \Sigma \). Hence \( \Gamma (\square(\varphi \supset \psi) \supset (\Box \varphi \supset \Box \psi) \neg \in \Sigma \), by PL and either (RRSE\(_T\)) or (RRSE).

Let \( \Sigma \) contain all instances of \( \Box(X) \). Then \( \Gamma (\Box(\tau \supset \varphi) \land \Box(\varphi \supset \psi)) \supset \Box(\tau \supset \psi) \neg \in \Sigma \). Hence \( \Gamma (\Box(\tau \supset \psi) \supset (\Box \varphi \supset \Box \psi) \neg \in \Sigma \), by PL and either (RRSE\(_T\)) or (RRSE). ⊣
By lemmas 2.4, A.2, A.7 and A.8 we have the following lemma.

**Lemma A.9 ([4]).** For any modal system $\Sigma$ which contains all instances of $(K)$ or $(X)$: $\Sigma$ is strict classical iff $\Sigma$ is strict classical.

Moreover, we obtain

**Lemma A.10 ([4]).**
1. If $\Sigma$ is strict classical, then it is also closed under the following “traditional” rule of congruence for strict equivalence
   \[ \text{if } \Gamma \varphi \vDash \exists \psi \in \Sigma, \text{ then } \Gamma \Box \varphi \vDash \Box \psi \in \Sigma. \quad (\text{RSE}_T) \]

2. If $\Sigma$ is strict classical, then it is also closed under the following rule of congruence for strict equivalence
   \[ \text{if } \Gamma \Box (\varphi \equiv \psi) \in \Sigma, \text{ then } \Gamma \Box (\Box \varphi \equiv \Box \psi) \in \Sigma. \quad (\text{RSE}) \]

**Proof.**
1. Since $\Box \text{PL} \subseteq \Sigma$, we have that $\Gamma \Box \varphi \vDash \Box \Box \varphi \in \Sigma$, by PL. Hence if $\Gamma \varphi \vDash \exists \psi \in \Sigma$, then $\Gamma \Box \varphi \vDash \Box \psi \in \Sigma$, by $(\text{RRSE}_T)$.

2. Since $\Box \text{PL} \subseteq \Sigma$, we have that $\Gamma \Box (\Box \varphi \equiv \Box \varphi) \in \Sigma$. Hence if $\Gamma \Box (\varphi \equiv \psi) \in \Sigma$, then $\Gamma \Box (\Box \varphi \equiv \Box \psi) \in \Sigma$, by $(\text{RRSE})$. \( \square \)

**Lemma A.11 ([4, 9]).** Let $\Sigma$ be a t-normal system which closed under $(\text{RSE}_T)$. Then

1. $\Sigma$ is also closed under the following rule of replacement
   \[ \text{if } \Gamma \varphi \vDash \exists \psi \in \Sigma, \text{ then } \Gamma \chi[\varphi/\psi] \vDash \chi \in \Sigma, \quad (\text{RSE}'_T) \]

2. If $\Sigma$ is also closed under $(\text{SMP})$, then $\Sigma$ is closed under $(\text{RRSE}_T)$.

**Proof.**
1. By induction.

2. Let $\Gamma \varphi \vDash \exists \psi \in \Sigma$ and $\chi \in \Sigma$. Then $\Gamma \chi[\varphi/\psi] \vDash \chi \in \Sigma$, by 1. Hence $\Gamma \chi \vDash \chi[\varphi/\psi] \in \Sigma$, by Lemma A.1. So $\chi[\varphi/\psi] \in \Sigma$, by $(\text{SMP})$. \( \square \)

**A.3. The logics S0.9, S0.9°, S1 and S1°**

In [9] Lemmon provided a simple axiomatization of the Lewis’ logic $\text{S1}$, where it is the smallest strict classical modal logic which contains formulae $\Box (X)$, $(T)$ and $\Box (T)$. Of course, the logic $\text{S1}$ contains also $(X)$ and, by Lemma A.8, the formulae $(K)$ and $\Box (K)$. So $\text{S1}$ is strict classical and it is a t-normal rte-logic (see lemmas A.6, A.7 and A.9).
In [9] Lemmon also introduced the logic S0.9, where it was meant as the smallest modal logic which included □Taut, contained formulae □(K), (T) and □(T), and is closed under (RSE\textsubscript{T}). So S0.9 contains (K) and is t-normal. Hence, contains (X), since S0.9 is also t-regular. Moreover, by lemmas A.8 and A.10.1, we obtain that S0.9 ⊆ S1. In [7] it was proved that S0.9 ≠ S1, since □(X) /∈ S0.9 (see also [4]).

“The other two systems, S1\textdegree and S0.9\textdegree, are often loosely described as S1 and S0.9 minus the schema T” [4, p. 12]. In [4] the Feys’ logic S1\textdegree from [5] is described as the smallest strict\textsubscript{T} classical modal logic which contains the formulae (X) and □(X), and is closed under (SMP). Thus, S1\textdegree contains (K) and □(K), by Lemma A.8. So, it is also a strict classical rte-logic.

Moreover, in [4] the logic S0.9\textdegree is described as the smallest strict\textsubscript{T} classical modal logic which contains the formulae (K) and □(K), and is closed under (SMP).

Thus we have the following axiomatizations (of course, in each case PL, (MP) and (US) are added as default items):

- S0.9: □Taut, □(K), (T), □(T) and (RSE\textsubscript{T}),
- S0.9\textdegree: □Taut, (K), □(K), (RRSE\textsubscript{T}) and (SMP),
- S1: □Taut, □(X), (T), □(T) and (RRSE\textsubscript{T}),
- S1\textdegree: □Taut , (X), □(X), (RRSE\textsubscript{T}) and (SMP).

By Lemma A.10 the logic S0.9\textdegree is also closed under the rules (RSE\textsubscript{T}) and (RSE\textsubscript{T}). So S0.9\textdegree ⊆ S0.9, since S0.9 is also closed under (SMP) and (T), □(T) /∈ S0.9\textdegree. Hence, by Lemma A.8, we have that S0.9\textdegree ⊆ S1\textdegree, since □(X) /∈ S0.9. Moreover, since S1 is also closed under (SMP) and (T), □(T) /∈ S1\textdegree. We have that S1\textdegree ⊆ S1.

By Lemma A.6, the logics S0.9\textdegree, S1 and S1\textdegree are a t-normal rte-logic. Moreover, by lemmas A.3, A.11, A.9 and A.6, we have:

**Corollary A.12 ([4]).** S0.9 is strict\textsubscript{T} and strict classical, and it is a t-normal rte-logic.

Notice that using lemmas given in sections A.1 and A.2 as well as Lemma 1.1 we obtain the following facts.

**Fact A.13 ([4]). 1.** S0.9 is the smallest rte-logic which is closed under (RN\textsubscript{\ast}) and (RRSE) (resp. (RRSE\textsubscript{T})), and contains the formulae (N), □(T) and □(K).
2. $\mathbf{S0.9}^\circ$ is the smallest rte-logic which is closed under $(\text{RN}_*)$ and $(\text{RRSE})$ (resp. $(\text{RRSE}_T)$), and contains the formulae $(\mathcal{N})$ and $\Box (K)$.

3. $\mathbf{S1}$ is the smallest rte-logic which is closed under $(\text{RN}_*)$ and $(\text{RRSE})$ (resp. $(\text{RRSE}_T)$), and contains the formulae $(\mathcal{N})$, $\Box (T)$ and $\Box (X)$.

4. $\mathbf{S1}^\circ$ is the smallest rte-logic which is closed under $(\text{RN}_*)$ and $(\text{RRSE})$ (resp. $(\text{RRSE}_T)$), and contains the formulae $(\mathcal{N})$ and $\Box (X)$.

**FACT A.14.**

1. $\mathbf{S0.9}$ is the smallest strict (resp. strict$_T$) classical logic which is closed under $(\text{RN}_*)$, and contains the formulae $\Box (T)$ and $\Box (K)$.

2. $\mathbf{S0.9}^\circ$ is the smallest strict (resp. strict$_T$) classical logic which is closed under $(\text{RN}_*)$, and contains the formula $\Box (K)$.

3. $\mathbf{S1}$ is the smallest strict (resp. strict$_T$) classical logic which is closed under $(\text{RN}_*)$, and contains the formulae $\Box (T)$ and $\Box (X)$.

4. $\mathbf{S1}^\circ$ is the smallest strict (resp. strict$_T$) classical logic which is closed under $(\text{RN}_*)$, and contains the formula $\Box (X)$.

A.4. The logics $\mathbf{S2}$, $\mathbf{S2}^\circ$, $\mathbf{S3}$, $\mathbf{S3.5}$, $\mathbf{S4}$ and $\mathbf{S5}$

We say the a modal logic $\Lambda$ is closed under Becker’s rule iff

$$\text{if } \Gamma \varphi \rightarrow \psi \models \Lambda, \text{ then } \Gamma \Box \varphi \rightarrow \Box \psi \models \Lambda.$$  \hspace{1cm} (RB)

In [9] (see also [1]) the logic $\mathbf{S2}$ is described as the smallest modal logic which includes $\Box \text{Taut}$, contains the formulae $(T)$, $\Box (T)$, and $\Box (K)$, and is closed under $(\text{RB})$. Of course, $\mathbf{S2}$ includes $\Box \text{PL}$, contains $(K)$ and, by Lemma A.3, it is closed under $(\text{RN}_*)$ and $(\text{SMP})$.

Moreover, in [1] the logic $\mathbf{S2}^\circ$ is described as the smallest modal logic which includes $\Box \text{Taut}$, contains $\Box (K)$, and is closed under $(\text{RB})$ and $(\text{RN}_*)$. Of course, $\mathbf{S2}^\circ$ includes $\Box \text{PL}$, contains $(K)$ and, by Lemma A.3, it is closed under $(\text{SMP})$. So $\mathbf{S2}^\circ \subseteq \mathbf{S2}$. For example $(T), \Box (T) \notin \mathbf{S2}^\circ$.

Moreover, by $(\text{RB})$ and PL, the logics $\mathbf{S2}$ and $\mathbf{S2}^\circ$ are closed under $(\text{RSE}_T)$. Thus, by lemmas A.3, A.11 and A.9, the logics $\mathbf{S2}$ and $\mathbf{S2}^\circ$ are strict$_T$ and strict classical, but they are not congruential.

In [4] the Lewis version $\text{Lew}(\Lambda)$ of a logic $\Lambda$ understood as the smallest modal logic which includes $\Lambda$ and contains the formula $(\mathcal{N})$. We have: $\mathbf{S2}^\circ = \text{Lew}(\mathbf{C2})$ and $\mathbf{S2} = \text{Lew}(\mathbf{E2})$. Moreover, for every $\varphi \in \text{For}$: $\varphi \in \mathbf{C2}$ iff $\Gamma \Box \varphi \models \mathbf{S2}^\circ$; $\varphi \in \mathbf{E2}$ iff $\Gamma \Box \varphi \models \mathbf{S2}$ (see e.g. [4, 8]).
In [9] Lemmon proved that $\Box(X) \in S2$. His proof shows that also $\Box(X) \in S2^\circ$. We have that $S1^\circ \subset S2^\circ$ and $S1 \subset S2$. For example, the formulae ‘$\Box(p \land q) \to (\Box p \land \Box q)$’, ‘$(\Box p \land \Box q) \to \Box(p \land q)$’ and ‘$\Diamond(p \land q) \to \Diamond p$’ belong to $S2^\circ$, but they are not members of $S1$.

In [9] the logic $S3$ is described as the smallest modal logic which includes $\Box$Taut and contains the formulae $(T)$, $\Box(T)$ and $\Box(sK)$. Of course, $S3$ contains $(sK)$ and $(K)$. Moreover, it contains also $\Box(K)$.$^6$ So $S3$ is also closed under $(RB)$, $(RSE_T)$, $(RSE)$, and it is strict $T$ and strict classical. We have $S2 \subset S3$. For example $(sK), \Box(sK) \notin S2$. We have: $S3 = \text{Lew(E3)}$. Moreover, for every $\varphi \in \text{For}$: $\varphi \in E3$ iff $\Box \varphi \in S3$ (see e.g. [8]).

Åqvist’s logic $S3.5$ is obtained by adding

\[ \Diamond p \supset \Box \Diamond p \quad (5) \]

or equivalently

\[ p \supset \Box \Diamond p \quad (B) \]

to Lewis’ logic $S3$ (see e.g. [6, p. 208]). We have that $S3 \subset S3.5$. For example $(5), (B) \notin S3$.

In [9] the logic $S4$ is described as the smallest modal logic which contains the formulae $(T)$ and $(sK)$, and is closed under $(RN)$. Of course, $S4$ contains $(K), \Box(K), (sK)$ and $\Box(sK)$. It is closed under $(RB)$, $(RSE_T)$ and is strict $T$ and strict classical. It is known (see e.g. [9]) that $S4$ is the smallest normal logic which contains the formulae $(T)$ and

\[ \Box p \supset \Box \Box p \quad (4) \]

We have that $S3 \subset S4$. For example $(4) \notin S3$.

Finally, $S5$ is the smallest normal logic which contains $(T)$ and $(5)$. Moreover, $S5$ is the smallest normal logic which contains $(T)$, $(B)$ and $(4)$; resp. $(D)$, $(B)$ and $(4)$; resp. $(D)$, $(B)$ and $(5)$; resp. $(D)$ $(5)$ and $(T_q)$. It is known that $S3.5 \subset S5$ and $S4 \subset S5$. For example $\Box(5) \notin S3.5$ and $(5) \notin S4$. Note that $\Box(5)$ strengthens $S3$ to $S5$ (see e.g. [6, p. 208]).

\[ ^6\text{Notice the formula } '\Box(p \supset \Box q) \supset \Box(p \supset q)' \text{ belongs to } S2 \text{ and } S3. \text{ By the substitution } p/\Box(p \supset q) \text{ and } q/\Box p \supset \Box q \text{ we have } ' \Box(sK) \supset \Box(K)' . \]
A.5. Location

Using semantics result we will to situate the logics $C_1$, $D_1$, $C_1 + (T_q)$, $E_1$, $S_{0.5}^\circ$, $S_{0.5}^\circ + (D)$, $S_{0.5}^\circ + (T_q)$, $S_{0.5}$, $C_{1_{	ext{rte}}}$, $D_{1_{	ext{rte}}}$, $E_{1_{	ext{rte}}} + (T_q)$, $E_{1_{	ext{rte}}}$, $S_{0.5_{	ext{rte}}}^\circ$, $S_{0.5_{	ext{rte}}}^\circ + (D)$, $S_{0.5_{	ext{rte}}}^\circ + (T_q)$ and $S_{0.5_{	ext{rte}}}$ among other logics (see Fig. 1; see also diagrams in [1, p. 3], [3, p. 132], [4, p. 21], [9, p. 186], [10, p. 48] and [11, p. 58]).

Using names of formulae, to simplify notation of normal logics we write the Lemmon code $KA_1 \ldots A_n$ to denote the smallest normal logic containing the formulae $(A_1), \ldots, (A_n)$ (see [2, 3]). Thus, for example, $KT_4$ is the smallest normal modal logic which contains $(T)$ and $(4)$. We standardly put $T := KT$ and $D := KD$. We have $S_4 = KT_4$, $KT = KD_T_q$, $KB_4 = KB_5 = K_5 T_q$ and $S_5 = KT_5 = K T B_4 = K D_4 = K D_5 = K D_5 T_q$ (see e.g. [9, 10, 4]).

References


Figure 1. Some t-regular modal logics


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