Russell's paradox and the beginnings of mereology

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Introduction

The discovery of the title paradox probably took place around 1901. It was published by Russell in 1903 in his book *Principles of Mathematics*.

Russell's paradox concerns distributive classes (as EXTENSIONS OF CONCEPTS), analyzed by Frege in *Grundgesetze der Arithmetik*. This paradox directly indicated that the theory adopted by Frege was contradictory.

When Frege learned of this from Russell's letter, he stated in the afterword to his book that the logical foundations of arithmetic that he had assumed were "shaken."

Introduction

Russell's paradox inspired Stanisław Leśniewski to create his concept of classes (sets) and their elements. Leśniewski himself writes about it in the first part of "O podstawach matematyki" ("On the foundations of mathematics"), published in 1927 in *Przegląd Filozoficzny*.

Let us quote the words of Prof. Jan Woleński from his article „Tajemnica warszawskiej szkoły logicznej" ("The secret of the Warsaw school of logic"), published in *Wiadomości Matematyczne* in 1985.

Introduction

There is a funny story connected with the origin of mereology. Namely, Leśniewski, once preparing a lecture on the solution of Russell's antinomy a few hours before its delivery, found with horror that the proposed solution was wrong. He began to think about possible improvements, at the same time nervously eating chocolate. Furthermore, suddenly, it occurred to him that it would be best to replace the traditional (distributive) concept of class with the concept of *class in the collective sense*—this is how mereology and a certain solution of Russell's paradox came into being.

Z powstaniem mereologii wiąże się zabawna historyjka. Otóż Leśniewski, przygotowując kiedyś odczyt na temat rozwiązania antynomii Russella, na kilka godzin przed jego wygłoszeniem z przerażeniem stwierdził, iż projektowane rozwiązanie jest błędne. Zaczął rozmyślać o możliwości poprawek, równocześnie nerwowo jedząc czekoladę. I nagle mu wpadło do głowy, że najlepiej, gdy tradycyjne (dystrybutywne) pojęcie klasy zastąpi pojęciem *klasy w sensie kolektywnym* — tak powstała mereologia i pewne rozwiązanie paradoksu Russella.

Introduction

Work plan:

1. Frege's approach to classes and the relation of *being an element*.
2. Russell's paradox in the context of Frege's theory.
3. Leśniewski's solution related to Frege's theory.

We will not present Russell's paradox in a modern approach.

First, it is widely known.

Second, as Quine has shown, in the modern approach, the paradox is not concerned with the notions of *being a class* and *being a member* but is concerned exclusively with quantifier logic. Namely, the following formulas are theses of quantifier logic:

$$\neg\exists x\forall y(yRx \leftrightarrow \neg yRy)$$

$$\neg\exists x\forall y(xRy \leftrightarrow \neg yRy)$$

For example: $R := \in$ or $R :=$ shaves.

Frege's approach to classes

Frege distinguished concepts from objects. Concepts refer to objects but not to other concepts.

In his theory of classes, Frege assumed that for every set of objects that fall under a concept, there corresponds to a certain object that is an extension of that concept.

For Frege, extensions of concepts were objects. Therefore, as objects, extensions of concepts can fall under other concepts.

Frege treated (distributive) classes as extensions of concepts.

Frege's approach to classes

Every concept has exactly one extension.

For simplicity, let us assume the convention that for any concept C:

- εC – denotes the extension of C.

According to the extensionality principle, two concepts have a common extension if and only if they refer to the same objects. Formally, for any concepts C_1 and C_2:

$$\begin{aligned} \varepsilon C_1 = \varepsilon C_2 &\iff \text{exactly the same objects fall under } C_1 \text{ and } C_2 \\ &\iff \forall x (x \text{ falls under } C_1 \Leftrightarrow x \text{ falls under } C_2) \qquad \text{(ext)} \end{aligned}$$

Let us introduce the following paradoxical concept:

- P := *object that is the extension of some concept but does not fall under it.*

That is, for any object x:

- x falls under P $:\iff \exists C(x = \varepsilon C \wedge x$ does not fall under $C)$.

Russell's paradox – the first take

Therefore, for $x = \varepsilon\mathrm{P}$, we have:

$$\varepsilon\mathrm{P} \text{ falls under } \mathrm{P} \;:\Longleftrightarrow\; \exists C(\varepsilon\mathrm{P} = \varepsilon C \wedge \varepsilon\mathrm{P} \text{ does not fall under } C).$$

We have a contradiction because

$$\exists C(\varepsilon\mathrm{P} = \varepsilon C \wedge \varepsilon\mathrm{P} \text{ does not fall under } C) \iff \varepsilon\mathrm{P} \text{ does not fall under } \mathrm{P}.$$

($\Rightarrow$) Suppose thet for some C_0: $\varepsilon\mathrm{P} = \varepsilon C_0$ and $\varepsilon\mathrm{P}$ does not fall under C_0. Then $\varepsilon\mathrm{P}$ does not fall under P since, by (ext), exactly the same objects fall under P and C_0. ($\Leftarrow$) By logic ($C = \mathrm{P}$).

The relation of *being an element*

Since the concept P says of objects that are extensions of concepts, it refers to classes. We can, therefore, replace this concept with another one, in which the notion of *being a class* and the relation of *being an element* (which was defined by Frege) will already appear. With this new concept we will also obtain a contradiction. It will have some connection with the analysis made by Leśniewski.

For any object x and any class K, the following sentence phrase:

- x is an element of K

is meant to mean:

- for some concept C, K is the extension of C and x falls under C.

In symbol:

$$x \in K \;:\Longleftrightarrow\; \exists C(K = \varepsilon C \;\wedge\; x \text{ falls under } C).$$

Russell's paradox – the second take

Let us introduce a concept related to the relation *being an element*, which being a synonym for the concept P:

$$\mathrm{R} := \textit{class that is not an element of itself}$$

Therefore, by the definition of $\in$, we have:

$$\varepsilon\mathrm{R} \in \varepsilon\mathrm{R} \;:\Longleftrightarrow\; \exists C(\varepsilon\mathrm{R} = \varepsilon C \;\wedge\; \varepsilon\mathrm{R} \text{ falls under } C).$$

Moreover, directly from the definition of R, we get:

$$\varepsilon\mathrm{R} \text{ falls under } \mathrm{R} \iff \varepsilon\mathrm{R} \notin \varepsilon\mathrm{R}.$$

Thus, again, we get a contradiction because:

$$\exists C(\varepsilon\mathrm{R} = \varepsilon C \wedge \varepsilon\mathrm{R} \text{ falls under } C) \iff \varepsilon\mathrm{R} \text{ falls under } \mathrm{R}.$$

($\Rightarrow$) Suppose that for some C_0: $\varepsilon\mathrm{R} = \varepsilon C_0$ and $\varepsilon\mathrm{R}$ falls under C_0. Then $\varepsilon\mathrm{R}$ falls under R since, by (ext), exactly the same objects fall under R and C_0. ($\Leftarrow$) By logic ($C = \mathrm{R}$).

The synonymousness of the concepts of P and R

As we might expect, the two defined concepts P and R are synonymous, i.e., they are extensional by their definitions and logic.

Notice that for any object x, we have:

$$x \text{ falls under P} \iff x \text{ falls under R},$$

i.e., $\varepsilon\text{P} = \varepsilon\text{R}$.

Indeed, on the one hand, directly from the definition of P, we have:

- x falls under P $:\!\iff \exists C(x = \varepsilon C \wedge x$ does not fall under $C)$.

On the other hand, by the definitions of R and $\in$, using logic, we have:

- x falls under R $:\!\iff$
  x is a class that is not an element of itself $:\!\iff$
  $\exists C\ x = \varepsilon C \ \wedge\ x \notin x \ :\!\iff$
  $\exists C\ x = \varepsilon C \ \wedge\ \neg\exists C(x = \varepsilon C \ \wedge\ x$ falls under $C) \iff$
  $\exists C\ x = \varepsilon C \ \wedge\ \forall C(x = \varepsilon C \ \Rightarrow\ x$ does not fall under $C) \iff$
  $\exists C(x = \varepsilon C \ \wedge\ x$ does not fall under $C)$

Let us now proceed to the solution of Russell's paradox given by Leśniewski.

From historical facts, it can be assumed that Leśniewski believed that the concepts of *distributive class* and the relation *being an element* were «responsible» for Russell's paradox. He concluded that there are no distributive classes at all. He believed that only collective classes existed. Today, both types of classes are used.

Let us quote Leśniewski from "O podstawach matematyki" ("On the foundations of mathematics", 1927, pp. 204–205).

Motivations

I don't know what Russell and Whitehead understand in the commentaries on their system by class. The fact that, on their position, "class" is supposed to be the same as "extension" does not help me in the slightest, as I don't know what these authors mean by extension. [...] Sensing in the "classes" of Whitehead and Russell, in a similar fashion as with the "extensions of concepts" of Frege, the scent of mythical paradigms from a rich gallery of "invented" objects, I cannot for my part divest myself of the inclination to sympathise "on credit" with the doubts of the authors on the matter of whether objects that are such "classes" exist in the world.

Nie wiem, co pp. Whitehead i Russell rozumieją w komentarzach do swego systemu przez klasę. Okoliczność, iż „klasa" ma być zgodnie ze stanowiskiem autorów ten samem, co „zakres", nie pomaga mi tu ani trochę, nie wiem bowiem również, co autorowie rozumieją przez zakres. [...] Czując w „klasach" pp. Whiteheada i Russella, podobnie jak w „zakresach pojęć" Fregego, zapach mitycznych okazów z obfitej galerji przedmiotów „wymyślonych", nie mogę się ze swej strony pozbyć skłonności do solidaryzowania się „na kredyt" z wątpliwościami autorów, by przedmioty, będące takimi „klasami", istniały na świecie.

Classes in the sense of Leśniewski

In contrast to the theory of distributive classes, in Leśniewski's theory, the concept of *being a class* is defined.

Leśniewski understood the word 'part' as a synonym for the words 'fragment' and 'piece'. So, parts are proper parts!

To simplify the definition of the concept of *being a class*, Leśniewski introduced the concept of *being an ingrediens*.

An *ingrediens* of a given object is itself and any of its parts.

We use the following scheme for any name S:

x is a class of Ss if and only if the following two conditions are met:

1. every S is an ingrediens of x,
2. every ingrediens of x has a common ingrediens with some S.

Since each object is an ingrediens of itself, the second condition requires that:

- If there is a class of Ss, then there is some S.

The axioms of Leśniewski's mereology

Leśniewski adopted the following axioms:

1. The concept of *being a part* is asymmetric and transitive.
2. There is at most one class of Ss.
3. If there is some S, then there is a class of Ss.

Therefore, the concept of *being a part* is also irreflexive, and the concept of *being an ingrediens* is reflexive, transitive, and antisymmetric. Furthermore, every non-empty name designates exactly one class.

Leśniewski's theory is consistent.

Need a fact

Since the relation *being an ingrediens* is reflexive, we get that

- Every object is a class of its ingredienses.

So, for each x, we have:

$$x = \text{the class of ingredienses of } x.$$

Indeed, we put

$S :=$ ingrediens of x.

Then we get tautological conditions in the definition of the notion of *being a class*:

1. every ingrediens of x is an ingrediens of x,
2. every ingrediens of x has a common ingrediens with some ingrediens of x (namely with itself).

Being an element in the sense of Leśniewski

Since Leśniewski rejected the existence of distributive classes, as well as the existence of extensions of concepts, he made a small correction to Frege's definition of the concept of *being an element*..

Recall that Frege adopted the following definition:

$$x \in K \;:\Longleftrightarrow\; \exists C(K = \varepsilon C \;\wedge\; x \text{ falls under } C).$$

For Leśniewski, the phrase:

- an object x is an element of an object y

means:

- with a certain meaning of S: y is a class of Ss and x is an S.

element = ingrediens

Leśniewski shows that for all objects x and y:

$$x \text{ is an element of } y \iff x \text{ is an ingrediens of } y.$$

($\Rightarrow$) Suppose that x is an element of y, i.e., for a certain meaning of S: y is a class of Ss and x is an S. Then, x is an ingrediens of y, by the first conditions of the definition of a class of Ss.
($\Leftarrow$) Suppose that x is an ingrediens of y. But:

$$y = \text{the class of ingredienses of } y.$$

So, with a certain meaning of S (as 'ingrediens of y'), y is a class of Ss and x is an S. Hence, x is an element of y.

The avoidance of paradox

Since *being an ingredient* is reflexive and for all objects x and y:

$$x \text{ is an element of } y \iff x \text{ is an ingrediens of } y,$$

we have:

$$x \text{ is an element of } x.$$

Therefore,

- Every class are an element of itself.

So we see that the name R ('class that is not an element of itself') is empty in Leśniewski's theory. Thus:

- There is no class of Rs.
- There is no class of classes that are not elements of themselves.

Summary

Russell's original paradox concerned distributive classes. Leśniewski's considerations concerned collective classes.

For Leśniewski, only collective classes existed. Therefore, for him, Russell's paradox in its original formulation did not concern anything. Leśniewski wanted to show that in his theory, there is no paradox of the class of classes that are not elements of themselves.