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Axiomatization of Boolean Connexive Logics with syncategorematic negation and modalities

Abstract. In the article we investigate three classes of extended Boolean Connexive Logics. Two of them are extensions of Modal and non-Modal Boolean Connexive Logics with a property of closure under an arbitrary number of negations. The remaining one is an extension of Modal Boolean Connexive Logic with a property of closure under the function of demodalization. In our work we provide a formal presentation of mentioned properties and axiom schemata that allow us to incorporate them into the axiomatic systems. The presented axiom systems are provided with proofs of soundness and completeness. The properties of closure under negation and demodalization are motivated by the syncategorematic view on the connective of negation and modalities, which is discussed in the paper.

Keywords: axiomatization, Boolean Connexive Logic, demodalization, Modal Boolean Connexive Logic, multiple negations, relating semantics, syncategorematic connectives

Introduction

In our paper, we aim to present axiomatizations of three classes of Boolean Connexive Logics. Starting from non-Modal Boolean Connexive Logics (in short: **BCL**) and Modal Boolean Connexive Logics (in short: **MBCL**), we introduce: (i) **BCL** closed under generalized negation, (ii) **MBCL** closed under generalized negation, and (iii) **MBCL** closed under the function of demodalization. These results are also a contribution to the fast growing area of relating logics and its applications.

Boolean Connexive Logics were introduced in two main papers of Jarmużek and Malinowski [5], [6], and then accepted as one of the solutions to the problem of connexivity [12]. Mentioned systems were constructed as an application of relating semantics to the subject of connexive logics. This area of research resulted in more papers such as [10], [11], where Lewis Carroll's *Barbershop Paradox* was analyzed by the means of **BCL**, or [9] where in the context of **BCL** connexivity and content-relationship were studied. In [7], for some classes of Boolean connexive logics adequate axiomatizations were provided. In the current paper we will incorporate these recent results.

In the first section we will introduce the philosophical motivation behind Boolean Connexive Logics — *minimal change strategy*. Then, we will provide a short description of the syntax and semantics for **BCL** and the axioms proposed in [7]. Analogously, we will introduce **MBCL**.

In the second section we deal with the issue of syncategorematic connectives, particularly negation and modalities. Based on that, we will define the properties on being closed under multiple negations and under the function of demodalization. This will serve us to define extensions of **BCL** and **MBCL**.

The following section is centered around the topic of closure under multiple negations. The property of being closed under negation that appeared already in the previous works, could be generalized to any number of negations. Hence, we will provide a formal description of the mentioned property and then, provide an axiom schema. Using these axioms, we will construct the axiom systems of **BCL** and **MBCL** closed under particular instances of multiple negations. Finally, soundness and completeness will be proved.

The next section, devoted to the studies of closure under function of demodalization, will have the corresponding structure. With the established idea behind the function of demodalization and formal definition of the function, we will construct the axiom system and prove its soundness and completeness in the same way as it was done in the previous section.

1. An introduction to Boolean Connexive Logics

BCL appeared for the first time in the work of Malinowski and Jarmużek (see [5]) as an example of application of relating semantics to connexive logics. Philosophically, its creation was laid upon the idea of *minimal change strategy*, which is a similar idea to Occam's Razor. In short, the application of this idea is that instead of getting rid of some classical laws, or changing the interpretation of more than one logical connective, it is better to change only the meaning of non-Boolean connectives. According to this approach, a common semantic fundamentals for both **CPL** (Classical Propositional Logic) and connexive logic is provided: relating semantics. This strategy allows us to preserve all **CPL** tautologies and changes the interpretation of only one logical connective, namely implication. While the rest of the symbols preserve their Boolean interpretation, the implication has a relating, non–extensional meaning, therefore this operation is no longer material implication. Minimal change strategy in application to **BCL** could be summed up by the following aphorism:

> "What is Boolean remains Boolean ($\neg, \wedge, \vee$), what is not Boolean ($\rightarrow$) becomes connexive."

Due to the fact that the rest of the connectives are interpreted in the standard way, material implication could be incorporated as a derived symbol, defined by means of other Boolean connectives. Connexive implication in **BCL** is interpreted using extensional truth values with the addition of an intensional factor – the relating relation. This relation is intended to express the connection between two expressions. So, instead of saying that two sentences are related, we can specify the way in which they are related, and that means they are *connected* (in some sense of connexivity). This way in relating semantics the very high level of connexivity is preseved. This was the core idea behind the creation of Boolean Connexive Logic.

The language of **BCL**, $\mathcal{L}_{\mathsf{BCL}}$, consists of the same symbols as the language of **CPL**: variables: $\mathsf{Var} = \{p_n : n \in \mathbb{N}\}$, one unary connective $\neg$, three binary connectives $\wedge, \vee, \rightarrow$ and brackets), (. On the other hand, the language of **MBCL**, $\mathcal{L}_{\mathsf{MBCL}}$, is just an extension of $\mathcal{L}_{\mathsf{BCL}}$ with unary modal operators $\Box, \Diamond$. The set of formulas of $\mathcal{L}_{\mathsf{BCL}}$ is defined in the standard way and is denoted as $\mathsf{For}_{\mathsf{BCL}}$. Analogously, $\mathsf{For}_{\mathsf{MBCL}}$ denotes formulas of $\mathcal{L}_{\mathsf{MBCL}}$. Obviously, $\mathsf{For}_{\mathsf{CPL}} = \mathsf{For}_{\mathsf{BCL}} \subset \mathsf{For}_{\mathsf{MBCL}}$.

The main feature of any connexive logic is the validity of Aristotle's and Boethius' theses. The following four expressions are schemata of those theses in the language of Boolean Connexive Logics.

(A1) $\neg(\neg A \rightarrow A)$

(A2) $\neg(A \rightarrow \neg A)$

(B1) $(A \rightarrow B) \rightarrow \neg(A \rightarrow \neg B)$

(B2) $(A \rightarrow \neg B) \rightarrow \neg(A \rightarrow B)$

Let $A, B \in \mathsf{For}_{\mathsf{BCL}}$ (for **BCL**) or $A, B \in \mathsf{For}_{\mathsf{MBCL}}$ (for **MBCL**). In both languages we will use the following abbreviations for, respectively, material implication and material equivalence:

($\supset$) $A \supset B := \neg A \vee B$

($\equiv$) $A \equiv B := (\neg A \vee B) \wedge (\neg B \vee A)$.

Semantics for **BCL** is pretty simple, due to the lack of interpretation of modal symbols, thus it will be described firstly. A model for $\mathcal{L}_{\mathsf{BCL}}$ is defined as an ordered pair of a valuation function and a binary relation on formulas with the following truth-conditions.

Definition 1.1 (Model for $\mathcal{L}_{\mathsf{BCL}}$). *A model for $\mathcal{L}_{\mathsf{BCL}}$ (based on relation R) is an ordered pair $\langle v, R\rangle$, where:*

- $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ *is a valuation of variables*
- $R \subseteq \mathsf{For}_{BCL} \times \mathsf{For}_{BCL}$ *is a binary relation between formulas.*

We assume the following truth-conditions:

Definition 1.2 (Truth-conditions in model for $\mathcal{L}_{\mathsf{BCL}}$). *Let $A \in \mathsf{For}_{BCL}$. A is true in $\mathfrak{M} = \langle v, R\rangle$ (in short: $\mathfrak{M} \vDash A$; $\mathfrak{M} \nvDash A$, if it is not the case) iff for all $B, C \in \mathsf{For}_{BCL}$:*

(Var) $v(A) = 1$, *if* $A \in \mathsf{Var}$

($\neg$) $\mathfrak{M} \nvDash B$, *if* $A = \neg B$

($\wedge$) $\mathfrak{M} \vDash B$ *and* $\mathfrak{M} \vDash C$, *if* $A = B \wedge C$

($\vee$) $\mathfrak{M} \vDash B$ *or* $\mathfrak{M} \vDash C$, *if* $A = B \vee C$

($\rightarrow$) $[\mathfrak{M} \nvDash B$ *or* $\mathfrak{M} \vDash C]$ *and* $R(B, C)$, *if* $A = B \rightarrow C$.

As we can see from the previous definition, the implication ($\rightarrow$) has a non–classical interpretation in **BCL**. One part of the truth condition is a value–assignment of its components, and the second part is a relation between those components.

Let us denote the class of all relations R $\subseteq \mathsf{For}_{\mathsf{BCL}} \times \mathsf{For}_{\mathsf{BCL}}$ as $\mathbf{R}$, and to denote subsets of $\mathbf{R}$ we will use $\mathbf{Q}, \mathbf{Q}_1, \mathbf{Q}_2, ...$ respectively. Then we will denote the class of all models for $\mathsf{For}_{\mathsf{BCL}}$ by **MR**, and by **MQ** the class of all models based on some subset $\mathbf{Q} \subseteq \mathbf{R}$. The same notation will be used for $\mathsf{For}_{\mathsf{MBCL}}$, if it is not misleading. So, when we talk about **MBCL**, $\mathbf{R}$ will denote $\mathsf{For}_{\mathsf{MBCL}} \times \mathsf{For}_{\mathsf{MBCL}}$ etc. Now can introduce the notion of validity.

Definition 1.3 (Validity for $\mathsf{For}_{\mathsf{BCL}}$). *Let $A \in \mathsf{For}_{BCL}$. A is valid in a relation $R \in \mathbf{R}$ (in short: $R \vDash A$) iff for all valuations v, $\langle v, R\rangle \vDash A$. Similarly, A is valid in a class of relations $\mathbf{Q} \subseteq \mathbf{R}$ (in short: $\vDash_{\mathbf{Q}} A$) iff for all relations $R \in \mathbf{Q}$, $R \vDash A$.*

Due to the fact that **BCL** is a connexive logic, it is crucial for Aristotle's and Boethius' laws to be satisfied by the models. Henceforth, the relating relation has to be limited by some specific conditions. We will use the abbreviation $\sim \mathrm{R}(A, B)$ when $\mathrm{R}(A, B)$ does not hold. Let $A, B \in \mathsf{For}_{\mathsf{BCL}}$, and $\mathrm{R} \in \mathbf{R}$. In [5] the following properties were defined:

(a1) $\sim \mathrm{R}(A, \neg A)$

(a2) $\sim \mathrm{R}(\neg A, A)$

(b0) $\mathrm{R}(A, B) \Rightarrow \sim \mathrm{R}(A, \neg B)$

(b1) $\mathrm{R}((A \to B), \neg(A \to \neg B))$

(b2) $\mathrm{R}((A \to \neg B), \neg(A \to B))$

(cun) $\mathrm{R}(A, B) \Rightarrow \mathrm{R}(\neg A, \neg B)$.

The conditions (a1) and (a2) correspond to Aristotle's theses, whereas the combination of conditions (b0), (b1) and (b2) correspond to Boethius' theses. Let us denote the class of all relations that satisfy those conditions as **BC** and the class of all relations satisfying stated conditions with the addition of (cun) as $\mathbf{BC}^c$.

The mentioned correspondence between theorems (A1), (A2), (B1), (B2) and (a1), (a2), (b0), (b1), (b2) was proved in [5]. Two theorems state that if the class of relations satisfy conditions (a1), (a2), (b0), (b1), (b2) it is the case that Aristotle's and Boethius' theses are valid in such structure. But the backward implication is not true, unless all the relations in the class satisfy condition (cun).

Theorem 1.4 (Theorem 5.1 from [5]). *Let $R \in \mathbf{R}$, then:*

(a) *R satisfies* (a1) $\Rightarrow R \vDash \neg(A \to \neg A)$

(b) *R satisfies* (a2) $\Rightarrow R \vDash \neg(\neg A \to A)$

(c) *R satisfies* (b0) *and* (b1) $\Rightarrow R \vDash (A \to B) \to \neg(A \to \neg B)$

(d) *R satisfies* (b0) *and* (b1) $\Rightarrow R \vDash (A \to \neg B) \to \neg(A \to B)$.

Theorem 1.4 can be summed up by the following fact. The proof is obvious due to the definition of the class **BC**.

Fact 1.5. *If $R \in \mathbf{BC}$ then* (A1), (A2), (B1), (B2) *are valid in R.*

Theorem 1.6 (Theorem 6.1 from [5]). *Let $R \in \mathbf{R}$ and R satisfies* (cun) *then:*

(a) *R satisfies* (a1) $\Leftrightarrow R \models \neg(A \to \neg A)$

(b) *R satisfies* (a2) $\Leftrightarrow R \models \neg(\neg A \to A)$

(c) *R satisfies* (b0) *and* (b1) $\Leftrightarrow R \models (A \to B) \to \neg(A \to \neg B)$

(d) *R satisfies* (b0) *and* (b1) $\Leftrightarrow R \models (A \to \neg B) \to \neg(A \to B)$.

Theorem 1.6 can be summed up by the following fact. The proof is obvious due to the definition of the class $\mathbf{BC}^c$.

FACT 1.7. *$R \in \mathbf{BC}^c$ iff* (A1), (A2), (B1), (B2) *are valid in R.*

The classes of models **MBC** and $\mathbf{MBC}^c$ define the least **BCL** and the least **BCL** closed under negation respectively (in short: $\mathbf{BCL}^\neg$). This became clear when axiom systems were introduced for these logics. The systems were published in [7]. For the logic characterized by **MBC** it was assumed:

(CPL) A, where A is a substituition instance of a classical propositional tautology

(A1) $\neg(A \to \neg A)$

(A2) $\neg(\neg A \to A)$

(B1) $(A \to B) \to \neg(A \to \neg B)$

(B2) $(A \to \neg B) \to \neg(A \to B)$

(Imp) $(A \to B) \supset (A \supset B)$

And the disjunctive syllogism:

$$(\mathsf{DS}_\supset) \qquad \frac{\begin{array}{c} A \\ A \supset B \end{array}}{B}$$

The described axiom system will be denoted by Ax_{BCL}. Whereas the system with addition of below axiom schemata for logic defined by $\mathbf{MBC}^c$ will be denoted as Ax^c_{BCL}:

(C1) $(A \to B) \supset ((\neg A \to \neg B) \vee (\neg A \wedge B))$

(C2) $(A \to B) \supset (\neg\neg A \to \neg\neg B)$

Since Ax_{BCL} contains additionally all classical tautologies, both systems are closed under the Modus Ponens for relating implication (by (Imp) and ($\mathsf{DS}_{\supset}$)):

$$(\mathsf{MP}_{\rightarrow}) \qquad \frac{\begin{array}{c} A \\ A \rightarrow B \end{array}}{B}$$

Semantics for **MBCL** is more complicated, due to the fact that the relating relation has to be associated to each possible world. Hence, relational structure has to be combined with a modal structure in the definition of a model for $\mathcal{L}_{\mathsf{MBCL}}$. Such structures are of the form $\langle \mathsf{W}, \mathsf{Q}, \{\mathrm{R}_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}} \rangle$, which will be called *combined frames*, or just *frames* for simplicity. The combined frame is composed of a modal frame $\langle \mathsf{W}, \mathsf{Q} \rangle$, and of a relating frame $\{\mathrm{R}_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}} \subseteq \{\mathrm{R}_{\mathsf{w}} \colon \mathrm{R} \in \mathbf{R} = \mathsf{For}_{\mathsf{MBCL}} \times \mathsf{For}_{\mathsf{MBCL}}, \mathsf{w} \in \mathsf{W}\}$.

Definition 1.8 (Model for $\mathcal{L}_{\mathsf{MBCL}}$). *A model for $\mathcal{L}_{MBCL}$ is an ordered quadruple $\langle \mathsf{W}, \mathsf{Q}, \{R_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}}, v \rangle$ such that:*

- $\mathsf{W} \neq \emptyset$
- $\mathsf{Q} \subseteq \mathsf{W} \times \mathsf{W}$
- $\{R_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}}$ *is a family of relations,* $R_{\mathsf{w}} \subseteq \mathsf{For}_{MBCL} \times \mathsf{For}_{MBCL}$ *for any* $\mathsf{w} \in \mathsf{W}$
- $v \colon \mathsf{Var} \longrightarrow \mathsf{W} \times \{1, 0\}$ *is a valuation of variables.*

Definition 1.9 (Truth-conditions in model for $\mathcal{L}_{\mathsf{MBCL}}$). *Let $A \in \mathsf{For}_{MBCL}$. Let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, \{R_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}}, v \rangle$ and $\mathsf{w} \in \mathsf{W}$. A is true in $\mathfrak{M}, \mathsf{w}$ (in short: $\mathfrak{M}, \mathsf{w} \vDash A$; $\mathfrak{M}, \mathsf{w} \not\vDash A$, if it is not the case) iff for all $B, C \in \mathsf{For}_{MBCL}$:*

(Var) $v(\mathsf{w}, A) = 1$, *if* $A \in \mathsf{Var}$

($\neg$) $\mathfrak{M}, \mathsf{w} \not\vDash B$, *if* $A = \neg B$

($\wedge$) $\mathfrak{M}, \mathsf{w} \vDash B$ *and* $\mathfrak{M}, \mathsf{w} \vDash C$, *if* $A = B \wedge C$

($\vee$) $\mathfrak{M}, \mathsf{w} \vDash B$ *or* $\mathfrak{M}, \mathsf{w} \vDash C$, *if* $A = B \vee C$

($\Box$) $\forall_{\mathsf{u} \in \mathsf{W}}(\mathsf{Q}(\mathsf{w}, \mathsf{u}) \Rightarrow \mathfrak{M}, \mathsf{u} \vDash B)$, *if* $A = \Box B$

($\Diamond$) $\exists_{\mathsf{u} \in W}(\mathsf{Q}(\mathsf{w}, \mathsf{u})$ *and* $\mathfrak{M}, \mathsf{u} \vDash B)$, *if* $A = \Diamond B$

($\rightarrow$) ($\mathfrak{M}, \mathsf{w} \not\vDash B$ *or* $\mathfrak{M}, \mathsf{w} \vDash C$) *and* $R_{\mathsf{w}}(B, C)$, *if* $A = B \rightarrow C$.

Definition 1.10 (Validity for $\mathsf{For}_{\mathsf{MBCL}}$). *Let $A \in \mathsf{For}_{MBCL}$. A is valid in a frame $\langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}\rangle$ (in short: $\langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}\rangle \models A$) iff for all valuations v and all worlds $\mathsf{w} \in \mathsf{W}$, $\langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}, v\rangle, \mathsf{w} \models A$. Similarly, A is valid in a class of combined frames $\boldsymbol{F}$ (in short: $\models_{\boldsymbol{F}} A$), iff for every frame $\langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}\rangle \in \boldsymbol{F}$, $\langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}\rangle \models A$.*

In analogy to **BCL**, in **MBCL** we can provide the list of conditions that have to be met by relating frames for Aristotle's and Boethius' laws to be valid. These conditions are the already mentioned (a1), (a2), (b0), (b1), (b2), noticing that now the relation symbol R is indexed by some world w, so R_w[1].

Let us denote the class of combined structures that satisfy conditions (a1), (a2), (b0), (b1), (b2) as $\mathbf{F}_B$ and the class of combined structures satisfying the previous conditions with the addition of the modalized (cun) as $\mathbf{F}_{B^c}$. Let moreover $\mathbf{MF}_B$ be the class of all models built over frames from $\mathbf{F}_B$, and similarly for $\mathbf{MF}_{B^c}$. The theorems regarding the relations between $\mathbf{F}_B$, $\mathbf{F}_{B^c}$ and the laws of connexive logic are analogous to the theorems for **BCL** (see [6]).

Theorem 1.11 (Theorem 3.1 from [6]). *Let $\{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}$ be a relating frame, $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}, v\rangle$ and $\mathsf{w} \in \mathsf{W}$ then:*

(a) *R_w satisfies (a1) $\Rightarrow \mathfrak{M}, \mathsf{w} \models \neg(A \to \neg A)$*

(b) *R_w satisfies (a2) $\Rightarrow \mathfrak{M}, \mathsf{w} \models \neg(\neg A \to A)$*

(c) *R_w satisfies (b0) and (b1) $\Rightarrow \mathfrak{M}, \mathsf{w} \models (A \to B) \to \neg(A \to \neg B)$*

(d) *R_w satisfies (b0) and (b1) $\Rightarrow \mathfrak{M}, \mathsf{w} \models (A \to \neg B) \to \neg(A \to B)$.*

Theorem 1.11 can be summed up by the following fact. The proof is obvious due to the definition of the class $\mathbf{F}_B$.

Fact 1.12. *If $\langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}\rangle \in \mathbf{F}_B$ then (A1), (A2), (B1), (B2) are valid in $\langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}\rangle$.*

Theorem 1.13 (Theorem 4.1 from [6]). *Let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, \{R_\mathsf{w}\}_{\mathsf{w}\in\mathsf{W}}, v\rangle$ and $\mathsf{w} \in \mathsf{W}$. Let R_w satisfy (cun). Then:*

(a) *R_w satisfies (a1) $\Leftrightarrow \mathfrak{M}, \mathsf{w} \models \neg(A \to \neg A)$*

(b) *R_w satisfies (a2) $\Leftrightarrow \mathfrak{M}, \mathsf{w} \models \neg(\neg A \to A)$*

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[1] Strictly speaking, all these conditions are formulated in $\mathsf{For}_{\mathsf{BCL}}$ and they should be translated into the richer language $\mathsf{For}_{\mathsf{MBCL}}$. Since this abuse of notation is harmless, we will keep using the same names for the new conditions in the modal setting.

(c) R_{w} *satisfies* (b0) *and* (b1) $\Leftrightarrow \mathfrak{M}, \mathsf{w} \models (A \to B) \to \neg(A \to \neg B)$

(d) R_{w} *satisfies* (b0) *and* (b1) $\Leftrightarrow \mathfrak{M}, \mathsf{w} \models (A \to \neg B) \to \neg(A \to B)$.

The proof of another fact is obvious due to the definition of the class $\mathbf{F}_{B^c}$.

FACT 1.14. $\langle \mathsf{W}, \mathsf{Q}, \{R_{\mathsf{w}}\}_{\mathsf{w}\in\mathsf{W}}\rangle \in \mathbf{F}_{B^c}$ *iff* (A1), (A2), (B1), (B2) *are valid in* $\langle \mathsf{W}, \mathsf{Q}, \{R_{\mathsf{w}}\}_{\mathsf{w}\in\mathsf{W}}\rangle$.

Facts 1.12 and 1.14 sum up the theorems (1.11, 1.13) about the classes of $\mathbf{F}_B$ and $\mathbf{F}_{B^c}$ in the same way as corresponding facts for **BCL**. The classes $\mathbf{F}_B$ and $\mathbf{F}_{B^c}$ characterize the least **MBCL** and the least **MBCL** closed under negation (in short: $\mathbf{MBCL}^{\neg}$) respectively. The axiom systems for **MBCL** and $\mathbf{MBCL}^{\neg}$ were also provided in the work of Klonowski (see [7]) are the same listed for **BCL**, plus:

(Dual) $\Diamond A \equiv \neg\Box\neg A$

($\mathsf{K}^{\supset}$) $\Box(A \supset B) \supset (\Box A \supset \Box A)$

and the necessitation rule:

$$(\mathsf{Nec}) \qquad \frac{A}{\Box A}$$

The described axioms we denote as Ax_{MBCL}.

It is worth noting that none of these modal Boolean connexive systems systems have principles characteristic of the logic (K) or stronger normal modal logics expressed by relating implication:

(D1) $\Diamond A \to \neg\Box\neg A$

(D2) $\neg\Box\neg A \to \Diamond A$

(K) $\Box(A \to B) \to (\Box A \to \Box B)$

(T) $\Box A \to A$

(D) $\Box A \to \Diamond A$

(B) $A \to \Diamond\Box A$

(4) $\Box A \to \Box\Box A$

(5) $\Diamond A \to \Box\Diamond A$.

Only their weaker versions, where we replace $\rightarrow$ with $\supset$, hold. In [6] two ways were presented in which we can validate these modal laws in stronger versions. The first one assumes some restrictions on modal and relating frames that we present below. The second is a demodalization move that we discuss in the section 2 and explore extensively in the section 4.

The listed modal laws could be true in the model, if following properties of combined frames are fulfilled. (Their designations match the designations of the corresponding modal principles.) Let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, v, \{\mathrm{R}_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}} \rangle$ and $\mathsf{w} \in \mathsf{W}$, then:

(d1) $\mathrm{R}_{\mathsf{w}}(\Diamond A, \neg\Box\neg A)$

(d2) $\mathrm{R}_{\mathsf{w}}(\neg\Box\neg A, \Diamond A)$

(k1) $\mathrm{R}_{\mathsf{w}}(\Box(A \rightarrow B), (\Box A \rightarrow \Box B))$

(k2) $\forall_{\mathsf{u} \in \mathsf{W}}((\mathsf{Q}(\mathsf{w}, \mathsf{u}) \Rightarrow \mathrm{R}_{\mathsf{u}}(A, B)) \Rightarrow \mathrm{R}_{\mathsf{w}}(\Box A, \Box B))$

(t) $\mathrm{R}_{\mathsf{w}}(\Box A, A)$ and the modal frame (i.e. $\langle \mathsf{W}, \mathsf{Q} \rangle$) is reflexive

(d) $\mathrm{R}_{\mathsf{w}}(\Box A, \Diamond A)$ and the modal frame (i.e. $\langle \mathsf{W}, \mathsf{Q} \rangle$) is serial

(b) $\mathrm{R}_{\mathsf{w}}(A, \Box\Diamond A)$ and the modal frame (i.e. $\langle \mathsf{W}, \mathsf{Q} \rangle$) is symmetrical

(iv) $\mathrm{R}_{\mathsf{w}}(\Box A, \Box\Box A)$ and the modal frame (i.e. $\langle \mathsf{W}, \mathsf{Q} \rangle$) is transitive

(v) $\mathrm{R}_{\mathsf{w}}(\Diamond A, \Box\Diamond A)$ and the modal frame (i.e. $\langle \mathsf{W}, \mathsf{Q} \rangle$) is Euclidean.

In [6] it was shown that by imposing the individual properties: (d1), (d2), (k1), (k2), (t), (d), (b), (iv), (v) on frames from $\mathbf{F}_B$ or $\mathbf{F}_{B^c}$, we get the classes of frames that validates the modal principles: (D1), (D2), (K), (T), (D), (B), (4), (5). It should be noted that the axiom (K) corresponds to two semantic conditions: (k1), (k2). The other axioms are semantically characterized by only one condition.

Let $X_1, ..., X_n \in \{$(D1),(D2), (K), (T), (D), (B), (4), (5)$\}$. We assume that $\mathbf{F}_{B,X_1,...,X_n}$ and $\mathbf{F}_{B^c,X_1,...,X_n}$ mean that the $\mathbf{F}_{B,X_1,...,X_n}$ and $\mathbf{F}_{B^c,X_1,...,X_n}$ are classes of all frames from $\mathbf{F}_B$ and respectively $\mathbf{F}_{B^c}$ that satisfy the properties corresponding to $X_1, ..., X_n$. So, for example if $\mathbf{F}_{B,X_1,...,X_4} = \mathbf{F}_{B,\text{(D1)},\text{(D2)},\text{(K)},\text{(4)}}$ then $\mathbf{F}_{B,X_1,...,X_4}$ is the class of all combined frames in $\mathbf{F}_B$ that satisfy the conditions (d1), (d2), (k1), (k2), (iv). Similarly, we will denote the systems of axioms reinforced by individual modal rules: $Ax_{\mathsf{MBCL},X_1,...,X_n}$, where $X_1, \ldots, X_n$ are designation of the modal principles.

In the work of [7], the author presented the completeness theorems of these axiom systems in respect to **BCL**, $\mathbf{BCL}^{\neg}$, and **MBCL** with the reinforcements .

The notion of syntactic consequence for a logic **L** is constructed in the standard way and is denoted by $\vdash_{Ax_{\mathbf{L}}}$. For readability, we will omit the subscript everywhere it will be clear which logic we mean.

THEOREM 1.15 (Theorem 5.1 from [7]). *Let* $A \in \mathsf{For}_{\mathit{BCL}}$. *Then:*

(a) $\vdash_{Ax_{\mathit{BCL}}} A \Leftrightarrow \models_{\mathbf{BC}} A$

(b) $\vdash_{Ax^c_{\mathit{BCL}}} A \Leftrightarrow \models_{\mathbf{BC}^c} A$.

THEOREM 1.16 (Theorem 5.2 from [7]). *Let* $A \in \mathsf{For}_{\mathit{MBCL}}$. *Then:*

$$\vdash_{Ax_{\mathit{MBCL},X_1,\dots,X_n}} A \Leftrightarrow \models_{\mathbf{F}_{B,X_1,\dots,X_n}} A$$

for all $X_1, \dots, X_n \in \{$(D1),(D2), (K), (T), (D), (B), (4), (5)$\}$.

2. Motivation: syncategorematic negation and modalities

One of the first logical systems defined by relating semantics were presented by Richard Epstein in [1]. By means of the considered relating implication, Epstein captured a content-based conditional, where the content relationship was understood as overlapping of the subject matter of sentences.[2] The idea aroused the interest of other logicians (like D. Walton, D. Lewis and R. Goldblatt), who noticed that Epstein's implication could be used to express the content relationship understood as overlapping of sentence content (see [2, p. 113]; [1, pp. 156–158]; [3, pp. 68–70]).

The logics he considered were named relatedness logics: the logic **S** and the logic **R**, which is a sublogic of **S**.[3] The monograph [3] contains the most significant results on the logics defined by Epstein, for instance, the sound and complete axiomatization (cf. [1]).[4]

The main postulate of Epstein's content-related sentences is that the subject matter of sentences is independent from the logical constants occurring in that sentence. This is explicitly given in the following passage:

> The logical connectives are syncategorematic: they are neutral with respect to subject matter ([3, p. 66]).

---

[2]It is also worth noting that in the paper of Gärdenfors [4] there was an attempt to give postulates for relevance of two sentences in terms of a binary relation R which resembles pretty much the approach proposed by Epstein in [1].

[3]The logic **R** should not be confused with Anderson and Belnap's relevant logic **R**.

[4]More about the history of relating logic and Epstein's contribution can be found in [8].

Considering a language with $\neg$, $\wedge$, $\rightarrow$ in the signature, the postulate is implemented by five assumptions ([3, p. 66]):

R1 $\mathrm{R}(A, \neg B) \Leftrightarrow \mathrm{R}(A, B)$

R2 $\mathrm{R}(A, B \wedge C) \Leftrightarrow \mathrm{R}(A, B \rightarrow C)$

R3 $\mathrm{R}(A, B) \Leftrightarrow \mathrm{R}(B, A)$

R4 $\mathrm{R}(A, A)$

R5 $\mathrm{R}(A, B \wedge C) \Leftrightarrow \mathrm{R}(A, B)$ or $\mathrm{R}(A, C)$.

Since Boolean Connexive Logics are defined by relating semantics and additionally connexive logic is motivated by the idea of some special relationship between the sentences in the conditional, it seems natural to consider at least some of the R1–R5 conditions in the context of (a1), (a2), (b0), (b1), (b2).

It is clear, however, that the combination of all these conditions leads to a contradiction, as stated by the following fact:

FACT 2.1. *Let* $\mathbf{Q} \subseteq \mathbf{R}$. *If* $\mathbf{Q}$ *contains exactly those relations that satisfiy one of the following sets of conditions:*

- $\{$(a1), $R1$, $R4\}$
- $\{$(a2), $R1$, $R3$, $R4\}$
- $\{$(b0), (b1), $R1\}$
- $\{$(b0), (b2), $R1\}$,

$\mathbf{Q}$ *is the empty set.*

The fact 2.1 gives the exemplary sets of conditions that are inconsistent. Thus, the entire set of Epstein's content–related assumptions, together with the semantic assumptions of Boolean Connexive Logics, is inconsistent. And since it defines the empty set of relations $\mathbf{Q}$, we consequently have an empty set of models $\mathbf{MQ}$, but $\emptyset$ determines the trivial logic (by 1.3 or by 1.10 in the modal context) – the set of all formulas.

However, instead of adding all these assumptions, or even some of them, we could consider incremental moves that bring the concept of connexivity closer to the relation between sentences as a kind of subject matter relation. Such a small move was made when in the semantics for Boolean Connexive Logic a closure condition for negation was imposed on the relating relation:

(cun) $\mathrm{R}(A, B) \Rightarrow \mathrm{R}(\neg A, \neg B)$.

Indeed, the (cun) condition states that adding the negation to two sentences does not invalidate the fact that the sentences are connected. Of course, this condition implies that we can add the negation n–times in this way by applying the condition (cun) n–times.

$$\mathrm{R}(A, B) \Rightarrow \mathrm{R}(\underbrace{\neg \ldots \neg}_{n} A, \underbrace{\neg \ldots \neg}_{n} B)$$

But why would we add the negation the same number of times on both sides? Also, different numbers of additional negations could preserve the fact that the two sentences are connected.

Moreover, the assumption that removing negations on both sides preserves the relationship between sentences seems to be another step towards Epstein's postulate. Thus, the opposite implication to (cun) deserves a separate study.

$$\mathrm{R}(\neg A, \neg B) \Rightarrow \mathrm{R}(A, B).$$

The implication can be generalized as described earlier, so that we can remove a different number of negations on both sides of the relation.

Putting the ideas together we can present the generalised closure under negation:

(gcun) $\mathrm{R}(\underbrace{\neg \ldots \neg}_{k} A, \underbrace{\neg \ldots \neg}_{l} B) \Rightarrow \mathrm{R}(\underbrace{\neg \ldots \neg}_{m} A, \underbrace{\neg \ldots \neg}_{n} B)$

Condition (gcun) states that the negation is syncategorematic or partially syncategorematic, if we fix some numbers k, l, m, $n \in \mathbb{N}$. In the section 3 we discuss the condition and propose an adequate axiomatization of Boolean Connexive Logics determined by (gcun) and its variants.

The language considered by Epstein did not contain modalities, but the postulate of content independence from logical constants can also be extended to modalities. We consider **MBCL** which is defined on the set of formulas $\mathsf{For}_{\mathsf{MBCL}}$, containing $\Diamond$ and $\Box$.

In the paper [6], it was proposed to consider such a possibility. Thus, having two sentences p and q that are connected: $\mathrm{R}(p, q)$, we can assume that the addition of any modality does not invalidate this state, so for example, also $\mathrm{R}(\Diamond p, \Box q)$, etc. It is also possible to assume the opposite, i.e. that sentences with modalities that are connected do not cease to be related when the modality is removed. Thus, if $\mathrm{R}(\Diamond p, \Box q)$, then $\mathrm{R}(p, q)$. Finally, both assumptions can be made simultaneously. In order to present these three properties in general terms, the cited article provides a function that removes modalities from formulas. **MBCL** is significantly strengthened by imposing these properties on the classes of frames $\mathbf{F}_B$ and $\mathbf{F}_{B^c}$.

Undoubtedly, such an approach fits in with Epstein's postulate that the logical connectives are neutral with respect to subject matter, namely they are syncategorematic. In article [6], we suggest the same about modalities:

> Any modality can be treated – due to the Latin etymology of the word "modality" – as the way a modalized proposition holds. Term modality comes from the Latin world *modus* which means a way; a way that something happens.
>
> An option of non–treating modalities literally is to assume that modalities $\Box$, $\Diamond$ add nothing to the content of propositions modalized by them. For some people it may sound controversial, but *modus* means a way, not a content.

In the section 4 we formulate, discuss and finally axiomatize these Modal Boolean Connexive Logic systems, which are created by making assumptions about the syncategorematicity of $\Diamond$ and $\Box$.

3. Closure under multiple negations

We have found that the property of closure under negation can be generalized to cover all the possible cases. Thus allowing for any number of negation symbols to appear in the antecedent and the consequent of the implication. Symbolically, a property of being closed under negation can be generalized for any number of negations in the following way. Let $A, B \in \mathsf{For}_{\mathsf{BCL}}$, $k, l, m, n \in \mathbb{N}$, $\mathrm{R} \in \mathbf{R}$, then:

$$\mathrm{R}(\underbrace{\neg \ldots \neg}_{k} A, \underbrace{\neg \ldots \neg}_{l} B) \Rightarrow \mathrm{R}(\underbrace{\neg \ldots \neg}_{m} A, \underbrace{\neg \ldots \neg}_{n} B)$$

Let us shorten down the notation by adding a superscript to negation symbol to express the number of occurrences. Thus, $\neg^k A$ would denote the same expression as $\underbrace{\neg \ldots \neg}_{k} A$. This way, the presented property of R shall have the following form:

(gcun) $\mathrm{R}(\neg^k A, \neg^l B) \Rightarrow \mathrm{R}(\neg^m A, \neg^n B)$.

Note that (gcun) can be understood in several ways, e.g. we can fix some four numbers k, l, m, $n \in \mathbb{N}$ and only for them the given property is defined. In this case, with fixed numbers k, l, m, $n \in \mathbb{N}$, we will denote this property as (cun^{klmn}).

Another way to understand this is to consider k, l, m, n as variables and add different quantifier configurations as a prefix. Since we strive for generality, we will assume that the property is preceded by a universal quantification, so that the property is to hold for all natural numbers k, l, m, n. We will denote this variant of the property by the abbreviation (gcun).

The problem, however, is that in the general sense of the condition (gcun), the only relation class that satisfies this condition and the consequential conditions together is the empty class. These conditions are inconsistent. We have the fact:

FACT 3.1. *Let* $\mathbf{Q} \subseteq \mathbf{R}$. *If* $\mathbf{Q}$ *contains exactly those relations that satisfy* (gcun)*,* (b0)*, and* (b1) *or* (b2)*,* $\mathbf{Q}$ *is the empty set.*

PROOF. Let us make the assumptions. For the proof that imposing (b1) on $\mathbf{Q}$ makes the set empty, in the schema (gcun) we take $k = 0$, $l = 1$, $m = 0$, $n = 2$, for A: $(A \to B)$, and for B: $(A \to \neg B)$. So, we have:

$$\mathrm{R}((A \to B), \neg(A \to \neg B)) \Rightarrow \mathrm{R}((A \to B), \neg\neg(A \to \neg B)).$$

But the antecedent of the implication is (b1), so we conclude that $\mathrm{R}((A \to B), \neg\neg(A \to \neg B))$. Now we make use of (b0). Instead of A we take $(A \to B)$, and for B: $(A \to \neg B)$. So, we have:

$$\mathrm{R}((A \to B), \neg(A \to \neg B)) \Rightarrow \sim \mathrm{R}((A \to B), \neg\neg(A \to \neg B)).$$

Again we detach (b1), and we get $\sim \mathrm{R}((A \to B), \neg\neg(A \to \neg B))$. which is a contradiction.

For the case of (b2), we assume also $k = 0$, $l = 1$, $m = 0$, $n = 2$ in (gcun), but substitute (gcun) and (b0) such that we can detach two times (b2). This again leads to contradiction. ■

Thus (gcun) with (a1), (a2), (b0), (b1), and (b2) result in the trivial logic (by 1.3 or by 1.10 in the modal context). The only reasonable option is to use the property (cun^{klmn}) for some quadruples that do not lead to contradiction. Among a countable set of such quadruples, at least some do not lead to a contradiction. An example is the condition (cun), which is based on the quadruple: 0, 0, 1, 1. On the other hand, in the proof of fact

3.1, the quadruple 0, 1, 0, 2 resulted in a contradiction. In the following, we will introduce the axiomatization method for (cun^{klmn}).

For further investigations, let us assume some four natural numbers: k, l, m, n. Before we move on to the next axioms, let's consider what would happen if $k = l = m = n$. Another fact tells about this:

FACT 3.2. *Let* $\mathbf{Q} \subseteq \mathbf{R}$. *Let* $k = l = m = n$. *If* $\mathbf{Q}$ *contains exactly those relations that satisfy* (cun^{klmn}), $\mathbf{Q} = \mathbf{R}$.

PROOF. If $k = l = m = n$, then (cun^{klmn}) takes the form:

$$\mathrm{R}(\neg^k A, \neg^k B) \Rightarrow \mathrm{R}(\neg^k A, \neg^k B),$$

which is an instance of classical tautology, so any binary relation satisfies it, particularly any relation in $\mathbf{R}$.

■

So, the case when $k = l = m = n$ does not define any new class of connexive relations or frames. But what, if the four numbers are not equal?

So far we are able to provide complete syntactic calculi for classes of models closed under (cun^{klmn}) for quadruples of values k, l, m, n satisfying specific conditions. We generalize the strategy developed by Klonowski [7]. As already seen, among the axioms of $\mathbf{BCL}^{\neg}$ the ones which allow to express the property of closure under negation (cun) inside the syntax are the pair:

($\mathsf{C1}$) $(A \to B) \supset ((\neg A \to \neg B) \vee (\neg A \wedge B))$

($\mathsf{C2}$) $(A \to B) \supset (\neg\neg A \to \neg\neg B)$

Obviously ($\mathsf{C1}$) is an instance of (cun^{klmn}) when we take the quadruple of values: $0, 0, 1, 1$. For arbitrary quadruples, it is plausible to expect the schema to generalize to:

($\mathsf{GC1}$) $(\neg^k A \to \neg^l B) \supset ((\neg^m A \to \neg^n B) \vee (\neg^m A \wedge \neg^{n+1} B))$

The situation is less straightforward for a generalization of ($\mathsf{C2}$). This axiom expresses the result of a double application of the property (cun^{klmn}) Moreover we want said application to preserve the truth-value of the transformed formulas, since they are connected by the connexive arrow, which in order to be true requires also the truth of its respective material implication. Therefore the desired axiom must express an iteration of (cun^{klmn}) which preserves the truth-value of each transformed formula.

Let us fix $k, l, m, n \in \mathbb{N}$, assuming that $k \leq m, l \leq n$. We start from a graphical consideration. Property (cun^{klmn}) states that when $\neg^k A$ and $\neg^l B$ are related so are $\neg^m A$ and $\neg^n B$. Let us focus on A. The axiom we are looking for should transform $\neg^k A$ into $\neg^m A$. Now $\neg^m A$ is just a graphical variant of $\neg^k \neg^{m-k} A$, therefore this formula can be transformed again into $\neg^m \neg^{m-k} A$, which is the same as $\neg^{2m-k} A$. The same reasoning applies to $\neg^l B$, which after two tranformations becomes $\neg^{2n-l} B$. We want the iterated tranformation to preseve truth-values and since negation behaves classically in our framework, the only way for $\neg^k A$ and $\neg^{2m-k} A$ to have the same truth-values is for $2m - k$ to be even, thefore k has to be even. By the same reasoning l has to be even as well.

We conclude that accepting the constraints that $k \leq m, l \leq n$, with k, l even, the following axiom is a good candidate for a generalization of (C2):

(GC2) $(\neg^k A \to \neg^l B) \supset (\neg^{2m-k} A \to \neg^{2n-l} B)$

Before we move to the completeness proof, we need to fix some notation. Let us fix $k, l, m, n \in \mathbb{N}$, with $k \leq m, l \leq n$, and k, l even. Let $\mathbf{BC}^{gc}$ denotes the class of all relations that satisfy the $\mathbf{BCL}^{\neg}$ conditions and (cun^{klmn}). The class of all models built over those relations will be denoted by $\mathbf{MBC}^{gc}$.

The axiom system $Ax^{\mathsf{gc}}_{\mathsf{BCL}}$ is the Hilbert-style calculus obtained from (CPL), (A1), (A2), (B1), (B2), (C1), (C2), (Imp), ($\mathsf{DS}_{\supset}$), by the addition of schemata (GC1), (GC2) for the same quadruple of values fixed above.

THEOREM 3.3 (Soundness Theorem for $Ax^{\mathsf{gc}}_{\mathsf{BCL}}$). *For any* $\Gamma \subseteq \mathsf{For}_{BCL}, A \in \mathsf{For}_{BCL}$*,* $\Gamma \vdash_{Ax^{gc}_{BCL}} A \Rightarrow \Gamma \vDash_{\mathbf{BC}^{gc}} A$*.*

PROOF. Let us consider all axiom schemata and the rule contained in $Ax^{\mathsf{gc}}_{\mathsf{BCL}}$.

Classical Propositional Logic Axioms. According to the definition 1.2, we got that $\mathbf{CPL} \subseteq \mathbf{BC}$. Thus, any axiom of $\mathbf{CPL}$ is valid in $\mathbf{BC}$.

Disjunctive Syllogism Rule. Let us assume that $\vDash B$ and $\vDash B \supset C$ for some $B, C \in \mathsf{For}_{\mathsf{BCL}}$. Then according to definition 1.2, we have that $v(B) = 1$ and ($v(B) = 0$ or $v(C) = 1$). Hence, $v(C) = 1$, from which it follows that $\vDash C$.

(A1). Let $\mathfrak{M} = \langle v, \mathrm{R} \rangle \in \mathbf{MBC}^{gc}$. Then $\sim \mathrm{R}(A, \neg A)$ by (a1). Hence $\mathfrak{M} \nvDash A \to \neg A$, that is $\mathfrak{M} \vDash \neg(A \to \neg A)$.

(A2). Similar to (A1), using property (a2).

(B1). Let us assume that $\mathfrak{M} \nvDash (A \to B) \to \neg(A \to \neg B)$ for some $\mathfrak{M} \in \mathbf{MBC}^{gc}$. According to definition 1.2, (1) $\mathfrak{M} \nvDash (A \to B) \supset \neg(A \to \neg B)$ or (2) $\sim \mathrm{R}((A \to B), \neg(A \to \neg B))$. (2) cannot hold since by lemma 3.10

(b0) holds. If (1), then $\mathfrak{M} \models (A \to B)$ and $\mathfrak{M} \not\models \neg(A \to \neg B)$, which implies both $\mathrm{R}(A, B)$ and $\mathrm{R}(A, \neg B)$, contradicting (b1).

(B2). Similar to (B1), using property (b2).

(Imp). Let us assume that $\models A \to B$. Then, due to definition 1.2, $\models \neg A \vee B$ and $\mathrm{R}(A, B)$. According to the abbreviation for material implication, we obtain $\models A \supset B$.

(GC1). Let us assume that $\mathfrak{M} \not\models (\neg^k A \to \neg^l B) \supset ((\neg^m A \to \neg^n B) \vee (\neg^m A \wedge \neg^{n+1} B))$, therefore $\mathfrak{M} \models (\neg^k A \to \neg^l B)$ and $\mathfrak{M} \not\models (\neg^m A \to \neg^n B)$ and $\mathfrak{M} \not\models (\neg^m A \wedge \neg^{n+1} B)$. From $\mathfrak{M} \not\models (\neg^m A \to \neg^n B)$ it follows that either $\mathfrak{M} \not\models (\neg^m A \supset \neg^n B)$ or $\sim \mathrm{R}(\neg^m A, \neg^n B)$. The latter cannot hold, since $\mathfrak{M} \models (\neg^k A \to \neg^l B)$ implies $\mathrm{R}(\neg^m A, \neg^n B)$. The former holds iff $\mathfrak{M} \models \neg^m A$ and $\mathfrak{M} \not\models \neg^n B$, hence $\mathfrak{M} \models \neg^{n+1} B$, obtaining a contradiction.

(C1). Similar to (GC1).

(GC2). Let us assume that $\mathfrak{M} \models \neg^k A \to \neg^l B$, therefore $\mathfrak{M} \models (\neg^k A \supset \neg^l B)$ and $\mathrm{R}(\neg^k A, \neg^l B)$. Since $2m - k, 2n - l$, also $\mathfrak{M} \models (\neg^k A \supset \neg^l B)$. From $\mathrm{R}(\neg^k A, \neg^l B)$ a double application of (cun^{klmn}) yelds $\mathrm{R}(\neg^{2m-k} A, \neg^{2n-l} B)$, therefore $\mathfrak{M} \models (\neg^{2m-k} A \to \neg^{2n-l} B)$.

(C2). Similar to (GC2). ■

We are going to employ the standard notions of consistent and maximal consistent sets of formulas. Let us denote $\mathsf{For}_{\mathsf{Gen}}$ as the formulas of some language $\mathcal{L}_{\mathsf{Gen}}$ and Ax_{Gen} be some axiom system expressed in the language $\mathcal{L}_{\mathsf{Gen}}$.

Definition 3.4 (Consistent Set of Formulas). *Let $\Gamma \subseteq \mathsf{For}_{BCL}$. Then, Γ is a Ax_{Gen}–consistent set of formulas iff for some $A \in \mathsf{For}_{BCL}$, $\Gamma \nvdash_{Ax_{Gen}} A$. Otherwise, it is called a Ax_{Gen}–inconsistent set of formulas.*

Definition 3.5 (Maximal Consistent Set of Formulas). *Let $\Gamma \subseteq \mathsf{For}_{Gen}$. We call Γ a maximal Ax_{Gen}–consistent iff both conditions are satisfied:*

(a) *Γ is Ax_{Gen}–consistent*

(b) *for any $\Delta \subseteq \mathsf{For}_{Gen}$, $\Gamma \subset \Delta \Rightarrow \Delta$ is Ax_{Gen}–inconsistent.*

We will denote the class of all maximal Ax_{Gen}–consistent sets of formulas as $\mathbf{Max}_{\mathsf{Gen}}$. Analogous notation will be used for **BCL** and **MBCL**.

We can now proceed and provide a proof for completeness using the modified notion of canonical model for **BCL** defined by Klonowski in (see [7], p. 529). We are going to build two canonical models, such that they make true the same formulas, but only the second of them is guaranteed to belong to the desired class of models.

Definition 3.6 (First Canonical Model for $Ax^{\mathsf{gc}}_{\mathsf{BCL}}$). *Let* $\Gamma \in \mathbf{Max}_{Ax^{gc}_{BCL}}$*, the first canonical model generated by* Γ *(in short* Γ*-model) is a pair* $\langle v, R\rangle$ *such that, for any* $A, B \in \mathsf{For}_{BCL}$*:*

(a) $v(p) = 1 \Leftrightarrow p \in \Gamma$*, for* $p \in \mathsf{Var}$

(b) $R(A, B) \Leftrightarrow A \to B \in \Gamma$

Lemma 3.7. *Let* $\Gamma \in \mathbf{Max}_{Ax^{gc}_{BCL}}$ *and* $\mathfrak{M}$ *be a* Γ*–model. Then, for all* $A \in \mathsf{For}_{BCL}$*,* $\mathfrak{M} \models A \Leftrightarrow A \in \Gamma$.

Proof. Assume all the hypotheses. We will use induction over the complexity of formulas.

Base case. Let $A \in \mathsf{Var}$. Then $\mathfrak{M} \models A \Leftrightarrow v(A) = 1$. By definition 3.6, we obtain that it is equivalent to $A \in \Gamma$.

Inductive hypothesis. Let $n \in \mathbb{N}$. Let us suppose that the theorem is true for all the formulas whose complexity is lesser or equal than n.

Inductive step. Let us assume that the formula A has complexity equal to $n + 1$. Thus, the possible cases are $A = \neg B$, $A = B \wedge C$, $A = B \vee C$ and $A = B \to C$ for some $B, C \in \mathsf{For}_{\mathsf{BCL}}$. Due to the basic properties of maximal consistent sets and definition 3.6 we can easily prove the theorem for complex formulas constructed using classical connectives.

Let $A = B \to C$. '$\Rightarrow$'. Suppose that $\mathfrak{M} \models B \to C$. Then, by the definition 1.2, we obtain ($\mathfrak{M} \not\models B$ or $\mathfrak{M} \models C$) and $\mathrm{R}(B, C)$. By the definition 3.6 and from the fact that $\mathrm{R}(B, C)$, we obtain $B \to C \in \Gamma$.

'$\Leftarrow$'. Suppose that $B \to C \in \Gamma$. Thus, by (Imp) and the usual properties of a maximal consistent set, we obtain $B \notin \Gamma$ or $C \in \Gamma$. Henceforth, by the inductive hypothesis, $\mathfrak{M} \not\models B$ or $\mathfrak{M} \models C$. Moreover by the definition 3.6, $B \to C \in \Gamma$ implies $\mathrm{R}(B, C)$. By definition 1.2, we conclude $\mathfrak{M} \models B \to C$. ■

It is essential to the completeness proof for the canonical model to belong to the class $\mathbf{MBC}^{gc}$. It is for this reason that we introduce a second canonical model.

Definition 3.8 (Second Canonical Model for $Ax^{\mathsf{gc}}_{\mathsf{BCL}}$). *Let* $\Gamma \in \mathbf{Max}_{Ax^{gc}_{BCL}}$ *and* $\mathfrak{M} = \langle v_{\mathfrak{M}}, R\rangle$ *be a* Γ*-model. Let* $R^{\neg} \subseteq \mathsf{For}_{BCL} \times \mathsf{For}_{BCL}$ *be the least relation* ρ *such that:*

(a) $\mathfrak{M} \models A \to B \Rightarrow \rho(A, B)$

(b) $\mathfrak{M} \models \neg^k A \to \neg^l B$ *and* $\mathfrak{M} \models \neg^m A \wedge \neg^{n+1} B \Rightarrow \rho(\neg^m A, \neg^n B)$

The second canonical model generated by Γ *(in short* $\Gamma^{\neg}$*-model) is the pair* $\mathfrak{M}^{\neg} = \langle v_{\mathfrak{M}}, R^{\neg}\rangle$.

Lemma 3.9. *For* $\Gamma \in \mathbf{Max}_{Ax^{gc}_{BCL}}$*, let* $\mathfrak{M} = \langle v, R\rangle$ *be a* Γ*-model and* $\mathfrak{M}^{\neg} = \langle v, R^{\neg}\rangle$ *be its* $\Gamma^{\neg}$*-model. Then for all* $A \in \mathsf{For}_{BCL}, \mathfrak{M} \models A \Leftrightarrow \mathfrak{M}^{\neg} \models A$.

Proof. We proceed by induction over the complexity of formulas. Since the two models share the same valuation v, the only non-trivial case is the one concerning the connexive arrow.

Let $A = B \to C$. '$\Rightarrow$'. Suppose that $\mathfrak{M} \models B \to C$. Then, by definition 1.2, we obtain ($\mathfrak{M} \not\models B$ or $\mathfrak{M} \models C$) and $\mathrm{R}(B, C)$. By induction hypothesis, $\mathfrak{M}^{\neg} \not\models B$ or $\mathfrak{M}^{\neg} \models C$. By definition 3.8, $\mathfrak{M} \models B \to C$ implies $\mathrm{R}^{\neg}(B, C)$. We obtain $\mathfrak{M}^{\neg} \models B \to C$.

'$\Leftarrow$'. Suppose that $\mathfrak{M}^{\neg} \models B \to C$. Then, by definition 1.2, we obtain ($\mathfrak{M}^{\neg} \not\models B$ or $\mathfrak{M}^{\neg} \models C$) and $\mathrm{R}^{\neg}(B, C)$. Now, by definition 3.8 $\mathrm{R}^{\neg}(B, C)$ can hold only in two cases: either (a) $\mathfrak{M} \models B \to C$ or (b) $B = \neg^m B_1, C = \neg^n C_1, \mathfrak{M} \models \neg^k B_1 \to \neg^l C_1$ and $\mathfrak{M} \models \neg^m B_1 \wedge \neg^{n+1} C_1$. If (a) holds, we are done. If (b) is the case, since $\mathfrak{M}^{\neg} \not\models B$ or $\mathfrak{M}^{\neg} \models C$, it is the same as $\mathfrak{M}^{\neg} \not\models \neg^k B_1$ or $\mathfrak{M}^{\neg} \models \neg^l C_1$, and by induction hypothesis $\mathfrak{M} \not\models \neg^k B_1$ or $\mathfrak{M} \models \neg^l C_1$, which contradicts $\mathfrak{M} \models \neg^m B_1 \wedge \neg^{n+1} C_1$. Therefore (a) must hold. ∎

Lemma 3.10. *For* $\Gamma \in \mathbf{Max}_{Ax^{gc}_{BCL}}$*, let* $\mathfrak{M}^{\neg} = \langle v, R^{\neg}\rangle$ *be a* $\Gamma^{\neg}$*-model. Then* $\mathfrak{M}^{\neg} \in \mathbf{MBC}^{gc}$.

Proof. Let $\mathfrak{M} = \langle v, \mathrm{R}\rangle$ be the Γ-model from which $\mathfrak{M}^{\neg}$ is built. Assume $\mathrm{R}^{\neg}(\neg^k A, \neg^l B)$. By definition 3.8, this holds only in two cases: either (a) $\mathfrak{M} \models \neg^k A \to \neg^l B$ or (b) $\neg^k A = \neg^m A_1, \neg^l B = \neg^n B_1, \mathfrak{M} \models \neg^k A_1 \to \neg^l B_1$ and $\mathfrak{M} \models \neg^m A_1 \wedge \neg^{n+1} B_1$.

If (a) holds, by properties of maximal consistent sets, $\mathfrak{M} \models \neg^k A \to \neg^l B$ implies $\mathfrak{M} \models (\neg^m A \to \neg^n B) \vee (\neg^m A \wedge \neg^{n+1} B)$ by (GC1). Therefore by the same properties, $\mathfrak{M} \models \neg^m A \to \neg^n B$ or $\mathfrak{M} \models \neg^m A \wedge \neg^{n+1} B$. In case $\mathfrak{M} \models \neg^m A \to \neg^n B$, by definition 3.8, this implies $\mathrm{R}^{\neg}(\neg^m A, \neg^n B)$. If $\mathfrak{M} \models \neg^m A \wedge \neg^{n+1} B$, by the same definition and $\mathfrak{M} \models \neg^k A \to \neg^l B$ it follows $\mathrm{R}^{\neg}(\neg^m A, \neg^n B)$.

If (b) is the case, $\mathfrak{M} \models \neg^k A_1 \to \neg^l B_1$ and (GC2) imply $\mathfrak{M} \models \neg^{2m-k} A_1 \to \neg^{2n-l} B_1$. Notice that $\neg^{2m-k} A_1 = \neg^{m-k}\neg^m A_1 = \neg^{m-k}\neg^k A = \neg^m A$ and $\neg^{2n-l} B_1 = \neg^{n-l}\neg^n B_1 = \neg^{n-l}\neg^l B = \neg^n B$. Hence $\mathfrak{M} \models \neg^m A \to \neg^n B$, which by definition 3.8 allow us to conclude $\mathrm{R}^{\neg}(\neg^m A, \neg^n B)$.

For all the other properties, the proof follows [7], proposition 4.11. ∎

The second part of the last theorem, which concerns all the connexive properties, is working in particular thanks to the closure (cun). This motivates the decision to include (C1), (C2) in the axiomatization. There is more left to explore here. At least, two directions can be considered. We can study classes of models in which the connexive laws stills hold but such that (cun) is no longer guaranteed. In that sense the focus is on the interconnection between the properties characterizing connexivity and the iteration of negations, to see how far this can go while preserving both connexive laws and consistency. A diverging path is the one leading to the more general framework of relating semantics. In that sense, the main interest would shift over the way of characterizing classes of models closed under multiple negations, leaving aside connexivity and studying all the possible choices of values for (cun^{klmn}). Both suggetions will be left for future works.

Returning to the main proof, we have shown that $\mathfrak{M}$ and $\mathfrak{M}^{\neg}$ behaves semantically in the same way. Moreover $\mathfrak{M}^{\neg} \in \mathbf{MBC}^{gc}$. Therefore we can work inside $\mathfrak{M}$ and be sure that there exists another model in the desired class which satisfies precisely the same formulas.

Theorem 3.11 (Completeness Theorem for $Ax^{\mathsf{gc}}_{\mathsf{BCL}}$). *For any* $\Gamma \subseteq \mathsf{For}_{BCL}$, $A \in \mathsf{For}_{BCL}$, $\Gamma \vDash_{\mathbf{BC}^{gc}} A \Rightarrow \Gamma \vdash_{Ax^{gc}_{BCL}} A$.

Proof. By contraposition, let us assume that $\Gamma \nvdash_{Ax^{\mathsf{gc}}_{\mathsf{BCL}}} A$. Then $\Gamma \cup \{\neg A\}$ is consistent. According to the Lindebaum's lemma, there exists a maximal consistent extension of this set Δ. According to lemma 3.7, there exists a first canonical model $\mathfrak{M}$ for which $\mathfrak{M} \vDash P$ iff $P \in \Delta$, Thus, $\mathfrak{M} \nvDash A$ and, for all $B \in \Gamma, \mathfrak{M} \vDash B$. By 3.9, also the second canonical model $\mathfrak{M}^{\neg}$ based on $\mathfrak{M}$ is such that $\mathfrak{M}^{\neg} \nvDash A$ and, for all $B \in \Gamma, \mathfrak{M}^{\neg} \vDash B$. Finally $\mathfrak{M}^{\neg} \in \mathbf{MBC}^{gc}$ by 3.10. Therefore $\Gamma \nvDash_{\mathbf{BC}^{gc}} A$. ■

Analogous results can be achieved for **MBCL** closed under multiple negations. Due to the fact that the semantics is more complicated, there is a need of introducing a new notion of a canonical model. As well as in the previous case, Klonowski defined the notion of a canonical model for **MBCL** in (see [7], p. 532), which we will modify.

Symbolically, the property of being closed under negation for **MBCL** looks very similar to the corresponding property for **BCL**. The change is within the subscript of R, which is related to some possible world $\mathsf{w} \in \mathsf{W}$. Let $A, B \in \mathsf{For}_{\mathsf{MBCL}}$, $k, l, m, n \in \mathbb{N}$, $\mathsf{w} \in \mathsf{W}$, and $\mathrm{R}_{\mathsf{w}} \in \mathbf{R}$, then, with a slight abuse of notation:

(gcun) $\mathrm{R}_{\mathsf{w}}(\neg^k A, \neg^l B) \Rightarrow \mathrm{R}_{\mathsf{w}}(\neg^m A, \neg^n B)$

Let us denote the class of all the combined frames whose relations satisfy the $\mathbf{MBCL}^{\neg}$ conditions and (gcun) as $\mathbf{F}_{B^{gc}}$. The class of all models built over those frames will be denoted by $\mathbf{MF}_{B^{gc}}$. The property can be expressed in the syntax by the schemata already introduced, (GC1), (GC2). The axiom system $Ax^{\mathsf{gc}}_{\mathsf{MBCL}}$ is the Hilbert-style calculus obtained from $Ax^{\mathsf{gc}}_{\mathsf{BCL}}$ adding the modal schemata (Dual), ($\mathsf{K}^{\supset}$), and the rule (Nec).

Theorem 3.12 (Soundness Theorem for $Ax^{\mathsf{gc}}_{\mathsf{MBCL}}$). *For any* $\Gamma \subseteq \mathsf{For}_{MBCL}$, $A \in \mathsf{For}_{MBCL}$, $\Gamma \vdash_{Ax^{gc}_{MBCL}} A \Rightarrow \Gamma \vDash_{\mathbf{F}_{B^{gc}}} A$.

Proof. All the cases considered in theorem 3.3 still hold. Let us consider the remaining ones.

Neccessitation Rule. Let us assume that $\vDash A$. Then, $\mathfrak{M}, \mathsf{w} \vDash A$ for any $\mathfrak{M} \in \mathbf{MF}_{B^{gc}}$ and $\mathsf{w} \in \mathsf{W}$. In particular, for all $\mathsf{w}_1 \in \mathsf{W}$ such that $\mathsf{Q}(\mathsf{w}, \mathsf{w}_1), \mathfrak{M}, \mathsf{w}_1 \vDash A$. Therefore $\vDash \Box A$.

(Dual). Let $\mathfrak{M}, \mathsf{w} \vDash \Diamond A$. Then, according to definition 1.9, this holds iff there exists $\mathsf{w}_1 \in \mathsf{W}$ such that $\mathsf{Q}(\mathsf{w}, \mathsf{w}_1)$ and $\mathfrak{M}, \mathsf{w}_1 \vDash A$. Which is equivalent to the fact that it is not the case that for every $\mathsf{w}_2 \in \mathsf{W}$ such that $\mathsf{Q}(\mathsf{w}, \mathsf{w}_2)$ we have $\mathfrak{M}, \mathsf{w}_1 \nvDash A$. By the same definition 1.9, this amounts to $\mathfrak{M}, \mathsf{w} \vDash \neg\Box\neg A$.

(K). Let $\mathfrak{M}, \mathsf{w} \vDash \Box(A \supset B)$ and $\mathfrak{M}, \mathsf{w} \vDash \Box A$. Then for all $\mathsf{w}_1 \in \mathsf{W}$ such that $\mathsf{Q}(\mathsf{w}, \mathsf{w}_1), \mathfrak{M}, \mathsf{w}_1 \vDash A \supset B$ and $\mathfrak{M}, \mathsf{w}_1 \vDash A$. Hence $\mathfrak{M}, \mathsf{w}_1 \vDash B$, therefore $\mathfrak{M}, \mathsf{w} \vDash \Box B$. ■

For completeness, we adapt the same strategy employed for the non-modal case, building two canonical models.

Definition 3.13 (First Canonical Model for $Ax^{\mathsf{gc}}_{\mathsf{MBCL}}$). *The first canonical model for* Ax_{MBCL} *is a quadruple* $\langle \mathsf{W}, \mathsf{Q}, \{R_{\mathsf{w}}\}_{\mathsf{w}\in\mathsf{W}}, v\rangle$ *such that, for any* $A, B \in \mathsf{For}_{MBCL}$:

(a) $\mathsf{W} = \mathbf{Max}_{Ax^{gc}_{MBCL}}$

(b) $\mathsf{Q}(\mathsf{w}, \mathsf{w}_1) \Leftrightarrow$ *for all* $A \in \mathsf{For}_{MBCL}$, $(\Box A \in \mathsf{w} \Rightarrow A \in \mathsf{w}_1)$

(c) *for all* $\mathsf{w} \in \mathsf{W}, p \in \mathsf{Var}$:

$$v(\mathsf{w}, p) = \begin{cases} 1 & \textit{if } p \in \mathsf{w} \\ 0 & \textit{if } p \notin \mathsf{w} \end{cases}$$

(d) $R_{\mathsf{w}}(A, B) \Leftrightarrow A \to B \in \mathsf{w}$

Definition 3.14 (Second Canonical Model for $Ax^{\mathsf{gc}}_{\mathsf{MBCL}}$). *Let* $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, \{R_{\mathsf{w}}\}_{\mathsf{w}\in\mathsf{W}}, v\rangle$ *be the first canonical model for* Ax_{MBCL}. *For* $\mathsf{w} \in \mathsf{W}$, *let* $R^{\neg}_{\mathsf{w}} \subseteq \mathsf{For}_{MBCL} \times \mathsf{For}_{MBCL}$ *be the least relation* ρ *such that:*

(a) $\mathfrak{M}, \mathsf{w} \vDash A \to B \Rightarrow \rho(A, B)$

(b) $\mathfrak{M}, \mathsf{w} \vDash \neg^k A \to \neg^l B$ *and* $\mathfrak{M}, \mathsf{w} \vDash \neg^m A \wedge \neg^{n+1} B \Rightarrow \rho(\neg^m A, \neg^n B)$

The second canonical model for Ax^{gc}_{MBCL} *is the tuple* $\mathfrak{M}^{\neg} = \langle \mathsf{W}, \mathsf{Q}, \{R^{\neg}_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}}, v \rangle$.

The construction follows precisely the steps for the non-modal case, adapting it to the new modal setting. Now the relation R is extended to a family in which each relation is indexed to a possible world. From the first canonical model, we build a second one by substituting each relation R_{w} with its counterpart $\mathrm{R}^{\neg}_{\mathsf{w}}$ closed under (gcun).

As expected, all the facts previously proved hold in their modal version as well. Since the proofs are, with the appropriate notation, identical to the ones already given, we simply list the results.

Lemma 3.15. *Let* $\mathfrak{M}$ *be the first canonical model for* Ax^{gc}_{MBCL}. *Then, for all* $\mathsf{w} \in \mathsf{W}_{\mathfrak{M}}, A \in \mathsf{For}_{MBCL}$, $\mathfrak{M}, \mathsf{w} \vDash A \Leftrightarrow A \in \mathsf{w}$.

Lemma 3.16. *Let* $\mathfrak{M}$ *be the first canonical model for* Ax^{gc}_{MBCL}, *and* $\mathfrak{M}^{\neg}$ *be the second one. Then for all* $\mathsf{w} \in \mathsf{W}, \mathfrak{M}, \mathsf{w} \vDash A \Leftrightarrow \mathfrak{M}^{\neg}, \mathsf{w} \vDash A$.

Lemma 3.17. *Let* $\mathfrak{M}^{\neg}$ *be the second canonical model for* Ax^{gc}_{MBCL}. *Then* $\mathfrak{M}^{\neg} \in \mathbf{MF}_{B^{gc}}$.

Theorem 3.18 (Completeness Theorem for Ax^{gc}_{MBCL}). *For any* $\Gamma \subseteq \mathsf{For}_{MBCL}$, $A \in \mathsf{For}_{MBCL}$, $\Gamma \vDash_{\mathbf{F}_{B^{gc}}} A \Rightarrow \Gamma \vdash_{Ax^{gc}_{MBCL}} A$.

4. Closure under demodalization

Besides the fact that the relating relation in **MBCL** could be closed under multiple negations, there is also a possibility of closing it under a demodalization function. This function simplifies a given expression by removing all modal symbols. By closing the relation R under the mentioned function, we obtain a connection between expressions containing modal symbols and expressions resulting from removal of those symbols.

We can distinguish three different types of connections between those two pairs of formulas: (1) if two formulas containing modal symbols are in relation R, then the formulas resulting from them by removing modal symbols are also in relation R, (2) the implication the other way round, (3) the equivalence.[5]

[5]Since it will not be relevant to the current discussion, we leave aside the mixed cases in which we start from a pair containing both a modal formula and a demodalized one

In this section, we will provide semantic conditions matching described types of connections and present axiom schemata for each of them. Having this, we will use the notion of a canonical model for **MBCL** used in the previous section, to prove completeness and soundness for those three axiom systems.

Let us start with the definition of a demodalization function. The function was introduced by Malinowski and Jarmużek in (see [5], p. 227).

DEFINITION 4.1. *A function* $d : \mathsf{For}_{MBCL} \longrightarrow \mathsf{For}_{MBCL}$ *will be called demodalization if it satisfies following conditions:*

(a) $d(A) = A$ *if* $A \in \mathsf{Var}$

(b) $d(\neg A) = \neg d(A)$

(c) $d(A \star B) = d(A) \star d(B)$*, where* $\star \in \{\wedge, \vee, \rightarrow, \equiv\}$

(d) $d(*A) = A$*, where* $* \in \{\Diamond, \Box\}$.

Using defined notions, we can introduce conditions for closure under demodalization. Let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, v, \{\mathrm{R}_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}} \rangle$, and $\mathsf{w} \in \mathsf{W}$, then:

(DemR) $\mathrm{R}_{\mathsf{w}}(A, B) \Rightarrow \mathrm{R}_{\mathsf{w}}(d(A), d(B))$

(DemL) $\mathrm{R}_{\mathsf{w}}(d(A), d(B)) \Rightarrow \mathrm{R}_{\mathsf{w}}(A, B)$

(DemE) $\mathrm{R}_{\mathsf{w}}(A, B) \equiv \mathrm{R}_{\mathsf{w}}(d(A), d(B))$

To express those conditions, we propose three axiom schemata:

(CUDR) $(A \rightarrow B) \supset (d(A) \rightarrow d(B)) \vee (d(A) \wedge \neg d(B))$

(CUDL) $(d(A) \rightarrow d(B)) \supset (A \rightarrow B) \vee (A \wedge \neg B)$

(CUDE) $(A \rightarrow B) \equiv (d(A) \rightarrow d(B)) \vee (d(A) \wedge \neg d(B))$

By adding any of those axiom schemata to Ax_{MBCL} we can obtain three different axiom systems, which we will denote respectively: $Ax^{\mathsf{DemR}}_{\mathsf{MBCL}}$, $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$ and $Ax^{\mathsf{DemE}}_{\mathsf{MBCL}}$.

The main advantage of incorporating those axioms is simplification of a class of models for **MBCL** containing some specific modal axiom schemata. Let us consider the well-known modal laws:

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and obtain a pair of a demodalized and a modal ones. Using the notation that is going to explained in the following, these are the cases:

(DemM1) $\mathrm{R}_{\mathsf{w}}(A, d(B)) \Rightarrow \mathrm{R}_{\mathsf{w}}(d(A), B)$

(DemM2) $\mathrm{R}_{\mathsf{w}}(d(A), B) \Rightarrow \mathrm{R}_{\mathsf{w}}(A, d(B))$

(D1) $\Diamond A \to \neg\Box\neg A$

(D2) $\neg\Box\neg A \to \Diamond A$

(K) $\Box(A \to B) \to (\Box A \to \Box B)$

(T) $\Box A \to A$

(D) $\Box A \to \Diamond A$

(B) $A \to \Diamond\Box A$

(4) $\Box A \to \Box\Box A$

(5) $\Diamond A \to \Box\Diamond A$

Listed laws are true in a model if following properties of combined frames are fulfilled. Let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, v, \{\mathrm{R}_{\mathsf{w}}\}_{\mathsf{w}\in\mathsf{W}}\rangle$ and $\mathsf{w}, \mathsf{w}_1 \in \mathsf{W}$, then:

(d1) $\mathrm{R}_{\mathsf{w}}(\Diamond A, \neg\Box\neg A)$

(d2) $\mathrm{R}_{\mathsf{w}}(\neg\Box\neg A, \Diamond A)$

(k1) $\mathrm{R}_{\mathsf{w}}(\Box(A \to B), (\Box A \to \Box B))$

(k2) $((\mathsf{Q}(\mathsf{w}, \mathsf{w}_1) \Rightarrow \mathrm{R}_{\mathsf{w}_1}(A, B)) \Rightarrow \mathrm{R}_{\mathsf{w}}(\Box A, \Box B))$

(t) $\mathrm{R}_{\mathsf{w}}(\Box A, A)$ and the modal frame is reflexive

(d) $\mathrm{R}_{\mathsf{w}}(\Box A, \Diamond A)$ and the modal frame is serial

(b) $\mathrm{R}_{\mathsf{w}}(A, \Box\Diamond A)$ and the modal frame is symmetrical

(iv) $\mathrm{R}_{\mathsf{w}}(\Box A, \Box\Box A)$ and the modal frame is transitive

(v) $\mathrm{R}_{\mathsf{w}}(\Diamond A, \Box\Diamond A)$ and the modal frame is Euclidean

If **MBCL** axiom system contains (CUDR) schema, mentioned conditions could be simplified in the following way:

$(\mathsf{d1})_d$ $\mathrm{R}_{\mathsf{w}}(d(\Diamond A), d(\neg\Box\neg A))$

$(\mathsf{d2})_d$ $\mathrm{R}_{\mathsf{w}}(d(\neg\Box\neg A), d(\Diamond A))$

$(\mathsf{k1})_d$ $\mathrm{R}_{\mathsf{w}}(d(\Box(A \to B)), (d(\Box A) \to d(\Box B)))$

$(\mathsf{k2})_d$ $((\mathsf{Q}(\mathsf{w}, \mathsf{w}_1) \Rightarrow \mathrm{R}_{\mathsf{w}_1}(A, B)) \Rightarrow \mathrm{R}_{\mathsf{w}}(d(\Box A), d(\Box B)))$

$(\mathsf{t})_d$ $\mathrm{R}_{\mathsf{w}}(d(\Box A), A)$ and the modal frame is reflexive

$(\mathsf{d})_d$ $\mathrm{R}_{\mathsf{w}}(d(\Box A), d(\Diamond A))$ and the modal frame is serial

$(\mathsf{b})_d$ $\mathrm{R}_{\mathsf{w}}(A, d(\Box\Diamond A))$ and the modal frame is symmetrical

$(\mathsf{iv})_d$ $\mathrm{R}_{\mathsf{w}}(d(\Box A), d(\Box\Box A))$ and the modal frame is transitive

$(\mathsf{v})_d$ $\mathrm{R}_{\mathsf{w}}(d(\Diamond A), d(\Box\Diamond A))$ and the modal frame is Euclidean

After application of the demodalization function, listed conditions are of the form:

$(\mathsf{d1})_d$ $\mathrm{R}_{\mathsf{w}}(A, \neg\neg A)$

$(\mathsf{d2})_d$ $\mathrm{R}_{\mathsf{w}}((\neg\neg A, A)$

$(\mathsf{k1})_d$ $\mathrm{R}_{\mathsf{w}}((A \to B)), (A \to B))$

$(\mathsf{k2})_d$ $((\mathsf{Q}(\mathsf{w}, \mathsf{w}_1) \Rightarrow \mathrm{R}_{\mathsf{w}_1}(A, B)) \Rightarrow \mathrm{R}_{\mathsf{w}}(A, B))$

$(\mathsf{t})_d$ $\mathrm{R}_{\mathsf{w}}(A, A)$ and the modal frame is reflexive

$(\mathsf{d})_d$ $\mathrm{R}_{\mathsf{w}}(A, A)$ and the modal frame is serial

$(\mathsf{b})_d$ $\mathrm{R}_{\mathsf{w}}(A, A)$ and the modal frame is symmetrical

$(\mathsf{iv})_d$ $\mathrm{R}_{\mathsf{w}}(A, A)$ and the modal frame is transitive

$(\mathsf{v})_d$ $\mathrm{R}_{\mathsf{w}}(A, A)$ and the modal frame is Euclidean

Hence, conditions (d1), (d2), (t), (d), (b), (iv), (v) are reduced to reflexivity of the relating relations and standard conditions for modal frames. In particular, if (c) is any of the mentioned conditions, notice that under (DemL) we have that for (c) to hold it is enough for $(\mathsf{c})_d$ to hold. Therefore $(\mathsf{c})_d$ is a sufficient condition for the relevant frame property to obtain, a condition which is likely easier to check than $(\mathsf{c})_d$. This is why we are going to focus our attention over classes of models closed under (DemL).

For this reason we introduce the axiom system $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$, which is the Hilbert-style calculus obtained from (CPL), (A1), (A2), (B1), (B2), (C1), (C2), (Imp), $(\mathsf{DS}_{\supset})$, by the addition of schema (CUDL). Let us denote the class of all the combined frames whose relations satisfy the $\mathbf{MBCL}^{\neg}$ conditions and (DemL) as $\mathbf{F}^{\mathsf{DemL}}_{B^c}$.

THEOREM 4.2 (Soundness Theorem for $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$). *For any* $\Gamma \subseteq \mathsf{For}_{MBCL}, A \in \mathsf{For}_{MBCL}$, $\Gamma \vdash_{Ax^{DemL}_{MBCL}} A \Rightarrow \Gamma \models_{\mathbf{F}} {}^{DemL}_{B^c} A$.

Proof. By theorem 3.12, soundness holds for all the shared cases.

(CUDL). Let $\mathfrak{M}, \mathsf{w} \models d(A) \to d(B)$ and $\mathfrak{M}, \mathsf{w} \not\models A \to B$. The latter holds iff either (a) $\mathfrak{M}, \mathsf{w} \not\models A \supset B$ or (b) $\sim \mathrm{R}_{\mathsf{w}}(A, B)$. (b) cannot be the case, since $\mathfrak{M} \in \mathbf{MF}^{\mathsf{DemL}}_{B^c}$, therefore since $\mathrm{R}_{\mathsf{w}}(d(A), D(B))$, by (DemL) also $\mathrm{R}_{\mathsf{w}}(A, B)$. Hence (a) holds. ■

For completeness, we perform the same strategy employed in the previous section.

Definition 4.3 (Canonical Model for $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$). *The first canonical model for Ax^{DemL}_{MBCL} differs from the one for Ax^{gc}_{MBCL} only for the set of possible worlds:*

(a) $\mathsf{W} = \mathbf{Max}_{Ax^{DemR}_{MBCL}}$

Definition 4.4 (Second Canonical Model for $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$). *Let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, \{R_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}}, v\rangle$ be the first canonical model for Ax_{MBCL}. For $\mathsf{w} \in \mathsf{W}$, let $R^d_{\mathsf{w}} \subseteq \mathsf{For}_{MBCL} \times \mathsf{For}_{MBCL}$ be the least relation ρ such that:*

(a) $\mathfrak{M}, \mathsf{w} \models A \to B \Rightarrow \rho(A, B)$

(b) $\mathfrak{M}, \mathsf{w} \models d(A) \to d(B)$ *and* $\mathfrak{M}, \mathsf{w} \models A \wedge \neg B \Rightarrow \rho(A, B)$

The second canonical model for Ax^{DemL}_{MBCL} is the tuple $\mathfrak{M}^d = \langle \mathsf{W}, \mathsf{Q}, \{R^d_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}}, v\rangle$.

In the the definition of the second canonical model, now the defining characteristic of the derived relation R^d is no longer closure under negation but the role played by the demodalizing function. Point (b) of the definition in fact guarantees the relation of two modalized formulas even when the material implication between them fails but the connexive implication between their demodalized versions holds, that is to say when the demodalized versions are related. This is just a rewording of (DemL).

Since the first canonical model for $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$ is almost identical to the one for $Ax^{\mathsf{gc}}_{\mathsf{MBCL}}$, the canonical valuation lemma immediately follows from lemma 3.15.

Lemma 4.5. *Let $\mathfrak{M}$ be the first canonical model for Ax^{DemL}_{MBCL}. Then, for all $\mathsf{w} \in \mathsf{W}_{\mathfrak{M}}, A \in \mathsf{For}_{MBCL}$, $\mathfrak{M}, \mathsf{w} \models A \Leftrightarrow A \in \mathsf{w}$.*

Again, the two canonical models are semantically indistinguishable. Moreover, the second one belongs to the desired class of models.

Lemma 4.6. *Let $\mathfrak{M}$ be the first canonical model for $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$, and $\mathfrak{M}^d$ be the second one. Then for all* $\mathsf{w} \in \mathsf{W}, \mathfrak{M}, \mathsf{w} \vDash A \Leftrightarrow \mathfrak{M}^d, \mathsf{w} \vDash A$.

Proof. We proceed by induction over the complexity of formulas. The only non-trivial case is the one concerning the connexive arrow, since the two models share the same valuation v and accessibility relation Q.

Let $A = B \to C$. '$\Rightarrow$'. Suppose that $\mathfrak{M}, \mathsf{w} \vDash B \to C$. Then ($\mathfrak{M}, \mathsf{w} \nvDash B$ or $\mathfrak{M}, \mathsf{w} \vDash C$) and $\mathrm{R}_{\mathsf{w}}(B, C)$. By induction hypothesis, $\mathfrak{M}^d, \mathsf{w} \nvDash B$ or $\mathfrak{M}^d, \mathsf{w} \vDash C$. By definition 4.4, $\mathfrak{M}, \mathsf{w} \vDash B \to C$ implies $\mathrm{R}^d(B, C)$. We obtain $\mathfrak{M}^d, \mathsf{w} \vDash B \to C$.

'$\Leftarrow$'. Suppose that $\mathfrak{M}^d, \mathsf{w} \vDash B \to C$. Then ($\mathfrak{M}^d, \mathsf{w} \nvDash B$ or $\mathfrak{M}^d, \mathsf{w} \vDash C$) and $\mathrm{R}^d(B, C)$. By definition 4.4, $\mathrm{R}^d(B, C)$ can hold only if either (a) $\mathfrak{M}, \mathsf{w} \vDash B \to C$ or (b) $\mathfrak{M}, \mathsf{w} \vDash d(B) \to d(C)$ and $\mathfrak{M}, \mathsf{w} \vDash B \wedge \neg C$. If (a) holds, we are done. If (b) is the case, since $\mathfrak{M}^d, \mathsf{w} \nvDash B$ or $\mathfrak{M}^d, \mathsf{w} \vDash C$, by induction hypothesis $\mathfrak{M}, \mathsf{w} \nvDash B$ or $\mathfrak{M}, \mathsf{w} \vDash C$, which contradicts $\mathfrak{M}, \mathsf{w} \vDash B \wedge \neg C$. Therefore (a) must hold. ■

Lemma 4.7. *Let $\mathfrak{M}^d$ be the second canonical model for $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$. Then $\mathfrak{M}^d \in$* $\mathbf{MF}^{\mathsf{DemL}}_{B^c}$.

Proof. Let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{Q}, \{\mathrm{R}_{\mathsf{w}}\}_{\mathsf{w} \in \mathsf{W}}, v \rangle$. be the first canonical model for $Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}$, and $\mathfrak{M}^d$ the second one. Assume $\mathrm{R}^d(d(A), d(B))$. By definition 4.4, this holds only in two cases: either (a) $\mathfrak{M}, \mathsf{w} \vDash d(A) \to d(B)$ or (b) $d(A) = A_1, d(B) = B_1, \mathfrak{M}, \mathsf{w} \vDash d(A_1) \to d(B_1)$ and $\mathfrak{M} \vDash A_1 \wedge \neg B_1$.

If (a) holds, by properties of maximal consistent sets, $\mathfrak{M}, \mathsf{w} \vDash d(A) \to d(B)$ implies $\mathfrak{M}, \mathsf{w} \vDash (A \to B) \vee (A \wedge \neg B)$ by (CUDL). By the same properties, $\mathfrak{M}, \mathsf{w} \vDash A \to B$ or $\mathfrak{M}, \mathsf{w} \vDash A \wedge \neg B$. In the former case $\mathrm{R}_{\mathsf{w}}(A, B)$ follows, in the latter as well by definition 4.4.

If we consider (b), notice that the demodalization d is obviously an idempotent function, i.e. $d(P) = d(d(P))$. Therefore $d(A) = d(d(A)) = d(A_1)$, and similarly $d(B) = d(B_1)$. Hence $\mathfrak{M}, \mathsf{w} \vDash d(A_1) \to d(B_1)$ is the same as $\mathfrak{M}, \mathsf{w} \vDash d(A) \to d(B)$, which is subsumed under case (a).

For all the other properties, the proof is an adaptation to the modal setting of [7], proposition 4.11. ■

We can finally prove completeness, since the previous two facts prove that working with the first canonical model is enough to be guaranteed that everything that holds there holds in a model of the class $\mathbf{MF}^{\mathsf{DemL}}_{B^c}$.

Theorem 4.8. *For any* $\Gamma \subseteq \mathsf{For}_{\mathsf{MBCL}}$, $A \in \mathsf{For}_{\mathsf{MBCL}}$, $\Gamma \vDash_{\mathbf{F}^{\mathsf{DemL}}_{B^c}} A \Rightarrow \Gamma \vdash_{Ax^{\mathsf{DemL}}_{\mathsf{MBCL}}} A$.

5. Further work: a syncategorematic approch to negation and modalities

The results presented in this work are meant as a perliminary study in the direction of the more general picture of relating logics closed under arbitrary negations and demodalization. Within the case study of connexivity, we have proved that it is possible to strengthen the logics **BCL** and **MBCL** in order to obtain closures under special cases of iterated negations and under certain applications of the demodalization function. The decision to work in the setting of connexive logic was motivated by the fact that it was precisely during its study in [5] that it emerged the peculiar connection between the relating relation and the behaviour of iterated negation.

As we have claimed in the current paper, there is much work to be done. The two kinds of closures studied here have a deeper, non-technical motivation, and that is syncategorematicity. Closure under multiple negations expresses, up to a certain iteration of the negation connective, that this connective does not alter part of the intensional component of a sentence. Here we presented only particular cases of such closure, but it is an ongoing project the attempt to formulate a general theory of closure under multiple negations, either within Boolean Connexive Logics and, to the highest degree of generality, for relating semantics.

Similarly, the possibility to preserve part of the intensional content of a sentence through the process of demodalization (or modalization), is a way to clarify how the relation between formulas expands from (or to) their modal counterparts. This is an application of the syncategorematic approach to modality. While we considered left demodalization (DemL), there are still other directions of demodalization to be considered. After that, partial demodalization is the next step: what if the demodalization function operates only up to or from a certain complexity? As shown, the effect of demodalization has important effects on frame conditions, which are a key element in the study of modal logics. Different forms of demodalization may allow to find a more precise connection with said conditions. The attempt of our paper was to lay an initial framework for exploring these questions.

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