Subresiduated Nelson Algebras

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Abstract

In this paper we generalize the well known relation between Heyting algebras and Nelson algebras in the framework of subresiduated lattices. In order to make it possible, we introduce the variety of subresiduated Nelson algebras. The main tool for its study is the construction provided by Vakarelov. Using it, we characterize the lattice of congruences of a subresiduated Nelson algebra through some of its implicative filters. We use this characterization to describe simple and subdirectly irreducible algebras, as well as principal congruences. Moreover, we prove that the variety of subresiduated Nelson algebras has equationally definable principal congruences and also the congruence extension property. Additionally, we present an equational base for the variety generated by the totally ordered subresiduated Nelson algebras. Finally, we show that there exists an equivalence between the algebraic category of subresiduated lattices and the algebraic category of centedred subresiduated Nelson algebras.

Keywords: Subresiduated lattices, Nelson algebras, twist construction, Kleene algebras.

1 Introduction

In this paper we study the convergence of ideas arising from different varieties of algebras related to intuitionistic logics: Heyting algebras, subresiduated lattices and Nelson algebras.

Subresiduated lattices, which are a generalization of Heyting algebras, were introduced during the decade of 1970 by Epstein and Horn [7] as an algebraic counterpart of some logics with strong implication previously studied by Lewy and Hacking [8]. These logics are examples of subintuitionistic logics, i.e., logics in the language of intuitionistic logic that are defined semantically by using Kripke models, in the same way as intuitionistic logic is defined but without requiring from the models some of the properties required in the intuitionistic case. Also in relation with the study of subintuitionistic logics, Celani and Jansana [4] got these algebras as the elements of a subvariety of the variety of weak Heyting algebras (see also [3, 5]). It is known that the variety $\mathsf{S4}$, whose members are the S4-algebras, is the algebraic semantics of the modal logic **S4**. This means that ϕ is a theorem of **S4** if and only if the variety $\mathsf{S4}$ satisfies $\phi \approx 1$. The variety of subresiduated lattices corresponds to the variety of algebras defined for all the equations $\phi \approx 1$ satisfied in the variety $\mathsf{S4}$ where the only connectives that appear are conjunction $\wedge$, disjunction $\vee$, bottom $\perp$, top $\top$ and a new connective of implication $\Rightarrow$, called strict implication, defined by $\varphi \Rightarrow \psi := \Box(\varphi \to \psi)$, where $\to$ denotes the classical implication.

Nelson's constructive logic with strong negation, which was introduced in [14] (see also [16, 18, 20]), is a well-known and by now fairly well-understood non-classical logic that combines the constructive approach of positive intuitionistic logic with a classical (i.e. De Morgan) negation. The algebraic models of this logic, forming a variety whose members are called Nelson algebras, have been studied since at least the late 1950's (firstly by Rasiowa; see [16] and references therein) and are also by now a fairly well-understood class of algebras. One of the main algebraic insights on this variety came, towards the end of the 1970's, with the realisation (independently due to Fidel and Vakarelov) that every Nelson algebra can be represented as a special binary product (here called a twist structure) of a Heyting algebra.

The main goal of this manuscript is to extend the twist construction in the framework of subresiduated lattices, thus obtaining a new variety, whose members will be called subresiduated Nelson algebras. More precisely, we will show that every subresiduated Nelson algebra can be represented as a twist structure of a subresiduated lattice. Another central objective of this paper is to study the subvariety of its totally ordered members.

The paper is organized as follows. In Section 2 we recall the definition of Nelson algebra and also sketch the main constructions linking Heyting algebras with Nelson algebras. Moreover, we recall the definition of a subresiduated lattice and some of its properties. In Section 3 we introduce subresiduated Nelson algebras, proving that the class of subresiduated Nelson algebras (which is a variety) properly contains the variety of Nelson algebras. We also show that every subresiduated Nelson algebra can be represented as a twist structure of a subresiduated lattice. In Section 4 we prove that given an arbitrary subresiduated Nelson algebra, there exists an order isomorphism between the lattice of its congruences and the lattice of its open implicative filters (which are a kind of implicative filters). We use it in order to give a characterization of the principal congruences. In particular, the mentioned characterization proves that the variety of subresiduated Nelson algebras has equationally definable principal congruences (EDPC). We also give a description of the simple and subdirectly irreducible algebras, and we prove that the variety of subresiduated Nelson algebras has the congruence extention property (CEP). In Section 5 we study the class of totally ordered subresiduated Nelson algebras in order to give an equational base for the class generated by this variety. Finally, in Section 6 we characterize the subresiduated Nelson algebras that can be represented as a twist structure of a subresiduated lattice and we also prove that there exists an equivalence between the algebraic category of subresiduated lattices and the algebraic category of centered subresiduated Nelson algebras, where a centered subresiduated Nelson algebra is a subresiduated Nelson algebra endowed with a center, i.e., a fixed element with respect to the involution (this element is necessarily unique).

2 Basic results

In this section we recall the definition of Nelson algebra [1, 6, 21] and some links between Heyting algebras and Nelson algebras [20], as well as the definition of subresiduated lattice and some of its properties [7].

A Kleene algebra [1, 6, 11] is a bounded distributive lattice endowed with a unary operation $\sim$ which satisfies the following identities:

Ne1) $\sim\sim x = x$,

Ne2) $\sim (x \wedge y) = \sim x \vee \sim y$,

Ne3) $(x \wedge \sim x) \wedge (y \vee \sim y) = x \wedge \sim x$.

Definition 2.1. *An algebra* $\langle T, \wedge, \vee, \rightarrow, \sim, 0, 1\rangle$ *of type* $(2, 2, 2, 1, 0, 0)$ *is called a Nelson algebra if* $\langle T, \wedge, \vee, \sim, 0, 1\rangle$ *is a Kleene algebra and the following identities are satisfied:*

Ne4) $x \rightarrow x = 1$,

Ne5) $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z$,

Ne6) $x \wedge (x \rightarrow y) = x \wedge (\sim x \vee y)$,

Ne7) $x \rightarrow y \leq \sim x \vee y$,

Ne8) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

If $\langle T, \wedge, \vee, \rightarrow, \sim, 0, 1\rangle$ is an algebra of type $(2, 2, 2, 1, 0, 0)$ where $\langle T, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and the condition Ne6) is satisfied, then conditions Ne7) and Ne8) are also satisfied [12, 13]. We write NA for the variety of Nelson algebras.

There are two key constructions that relate Heyting algebras and Nelson algebras. Given a Heyting algebra A, we define the set

$$\mathrm{K}(A) = \{(a, b) \in A \times A : a \wedge b = 0\} \tag{1}$$

and then endow it with the following operations:

- $(a, b) \wedge (c, d) = (a \wedge c, b \vee d)$,
- $(a, b) \vee (c, d) = (a \vee c, b \wedge d)$,
- $\sim (a, b) = (b, a)$,
- $(a, b) \Rightarrow (c, d) = (a \rightarrow c, a \wedge d)$,
- $\perp = (0, 1)$,
- $\top = (1, 0)$.

Then $\langle \mathrm{K}(A), \wedge, \vee, \Rightarrow, \bot, \top \rangle \in \mathsf{NA}$ [20]. In the same manuscript, Vakarelov proves that if $T \in \mathsf{NA}$, then the relation θ defined by

$$x\theta y \text{ if and only if } x \to y = 1 \text{ and } y \to x = 1 \tag{2}$$

is an equivalence relation such that $\langle T/\theta, \wedge, \vee, \to, 0, 1 \rangle$ is a Heyting algebra with the operations defined by

- $x/\theta \wedge y/\theta := x \wedge y/\theta$,
- $x/\theta \vee y/\theta := x \vee y/\theta$,
- $x/\theta \to y/\theta := x \to y/\theta$,
- $0 := 0/\theta$,
- $1 := 1/\theta$.

It is a natural question whether these constructions can be extended to subresiduated lattices.

Definition 2.2. *A subresiduated lattice (sr-lattice for short) is a pair (A, D), where A is a bounded distributive lattice and D is a bounded sublattice of A such that for each $a, b \in A$ there exists the maximum of the set $\{d \in D : a \wedge d \leq b\}$. This element is denoted by $a \to b$.*

Let (A, D) be a subresiduated lattice. This pair can be regarded as an algebra $\langle A, \wedge, \vee, \to, 0, 1 \rangle$ of type $(2, 2, 2, 0, 0)$ where $D = \{a \in A : 1 \to a = a\} = \{1 \to a : a \in A\}$. Moreover, an algebra $\langle A, \wedge, \vee, \to, 0, 1 \rangle$ is an sr-lattice if and only if $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the following conditions are satisfied for every $a, b, c \in A$:

1) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$,

2) $c \to (a \wedge b) = (c \to a) \wedge (c \to b)$,

3) $(a \to b) \wedge (b \to c) \leq a \to c$,

4) $a \to a = 1$,

5) $a \wedge (a \to b) \leq b$,

6) $a \to b \leq c \to (a \to b)$.

We write SRL to denote the variety whose members are sr-lattices. In every sr-lattice the following cuasi-identity is satisfied: if $a \leq b \to c$ then $a \wedge b \leq c$.

The following example of sr-lattice will be used throughout the paper.

Example 2.1. *Let A be the Boolean algebra of four elements, where a and b are the atoms. This algebra can be seen as a bounded distributive lattice. Define $D = \{0, 1\}$. We have that (A, D) is an sr-lattice. With an abuse of notation we write A for this sr-lattice. Note that since $a \to 0 = 0 \neq b$ then A is not a Heyting algebra.*

In this work, we attempt to find a more general definition than the one of a Nelson algebra in order to take the first steps toward exploiting its relation to sr-lattices.

3 Subresiduated Nelson algebras

In this section we define subresiduated Nelson algebras and we show that the class of subresiduated Nelson algebras, which is a variety, properly contains the variety of Nelson algebras. We also prove that every subresiduated Nelson algebra can be represented as a twist structure of a subresiduated lattice.

Let $A \in \mathsf{SRL}$. We define $\mathrm{K}(A)$ as in (1). Then $\langle \mathrm{K}(A), \wedge, \vee, \sim, (0,1), (1,0) \rangle$ is a Kleene algebra [6, 11]. On $\mathrm{K}(A)$ we also define the binary operation $\Rightarrow$ as in Section 2. Note that this is a well defined map because if (a,b) and (c,d) are elements of $\mathrm{K}(A)$ then $(a \to c) \wedge a \wedge d \leq c \wedge d = 0$, i.e., $(a \to c) \wedge a \wedge d = 0$. Thus, the structure $\langle \mathrm{K}(A), \wedge, \vee, \Rightarrow, \sim, (0,1), (1,0) \rangle$ is an algebra of type $(2,2,2,1,0,0)$.

Remark 3.1. *Let $A \in \mathsf{SRL}$ and (a,b), (c,d) in $\mathrm{K}(A)$. Then $(a,b) \Rightarrow (c,d) = (1,0)$ if and only if $a \leq c$.*

Definition 3.1. *An algebra $\langle T, \wedge, \vee, \to, \sim, 0, 1 \rangle$ of type $(2,2,2,1,0,0)$ is said to be a subresiduated Nelson algebra if $\langle T, \wedge, \vee, \sim, 0, 1 \rangle$ is a Kleene algebra and the following conditions are satisfied for every $a, b, c \in T$:*

1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

2) $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$,

3) $((x \to y) \wedge (y \to z)) \to (x \to z) = 1$,

4) $x \to x = 1$,

5) $x \wedge (x \to y) \leq x \wedge (\sim x \vee y)$,

6) $x \to y \leq z \to (x \to y)$,

7) $\sim (x \to y) \to (x \wedge \sim y) = 1$,

8) $(x \wedge \sim y) \to \sim (x \to y) = 1$.

We write SNA to denote the variety whose members are subresiduated Nelson algebras.

Proposition 3.1. *The variety NA is a subvariety of SNA.*

Proof. Let $T \in \mathsf{NA}$ and $x, y, z \in T$. Condition 1) of Definition 3.1 is condition (1.9) of [21], 2) is Ne8), 4) is Ne4), 5) is a direct consequence of Ne6), 7) is (1.24) of [21] and 8) is (1.23) of [21].

Now we will see 3), i.e, we will show that

$$((x \to y) \wedge (y \to z)) \to (x \to z) = 1. \tag{3}$$

It follows from (1.17) of [21] that (3) holds if and only if $(y \to z) \to ((x \to y) \to (x \to z)) = 1$. But it follows from (1.11) of [21] and Ne5) that

$$(x \to y) \to (x \to z) = x \to (y \to z) = (x \wedge y) \to z.$$

Hence, (3) holds if and only if $(y \to z) \to ((x \wedge y) \to z) = 1$. Since $x \wedge y \leq y$ then it follows from (1.7) of [21] that $y \to z \leq (x \wedge y) \to z$. Thus, (1.3) of [21] shows that $(y \to z) \to ((x \wedge y) \to z) = 1$, so 3) is satisfied.

Finally we will show 6). Note that it follows from Ne5) that

$$z \to (x \to y) = (z \wedge x) \to y. \tag{4}$$

Besides, since $z \wedge x \leq x$ then it follows from (1.7) of [21] that

$$x \to y \leq (z \wedge x) \to y. \tag{5}$$

Thus, from (4) and (5) we get $x \to y \leq z \to (x \to y)$, which is 6). □

Proposition 3.2. *If* $A \in \mathsf{SRL}$ *then* $\mathrm{K}(A) \in \mathsf{SNA}$.

Proof. Let $A \in \mathsf{SRL}$. We will show that $\mathrm{K}(A)$ satisfies the conditions of Definition 3.1. In order to prove it, let $x = (a, b)$, $y = (c, d)$ and $z = (e, f)$ be elements of $\mathrm{K}(A)$.

A direct computation shows that

$$(x \vee y) \to z = ((a \vee c) \to e, (a \vee c) \wedge f),$$

$$(x \to z) \wedge (y \to z) = ((a \to e) \wedge (c \to e), (a \wedge f) \vee (c \wedge f)).$$

Since $(a \vee c) \to e = (a \to e) \wedge (c \to e)$ and $(a \vee c) \wedge f = (a \wedge f) \vee (c \wedge f)$ then $(x \vee y) \to z = (x \to z) \wedge (y \to z)$, so 1) is satisfied. The fact that $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$ can be proved following a similar reasoning, so 2) is also satisfied. In order to show 3), note that it follows from Remark 3.1 that $((x \to y) \wedge (y \to z)) \to (x \to z) = (1, 0)$ if and only if $(a \to c) \wedge (c \to e) \leq a \to e$, and the last inequality holds in sr-lattices. Hence, condition 3) holds. Taking into account that $a \to a = 1$ we get $x \to x = (1, 0)$, so 4) is verified too. In order to see 5), note that a straightforward computation shows that

$$x \wedge (x \to y) = (a \wedge (a \to c), b \vee (a \wedge d)),$$

$$x \wedge (\sim x \vee y) = (a \wedge (b \vee c), b \vee (a \wedge d)).$$

Since $a \wedge (a \to c) \leq a \wedge c = (a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$ then it follows from Remark 3.1 that $((x \wedge (x \to y)) \to (x \wedge (\sim x \vee y)) = (1, 0)$. Hence, we have proved 5). Now we will show 6). Note that

$$x \to y = (a \to c, a \wedge d)$$

$$z \to (x \to y) = (e \to (a \to c), e \wedge (a \wedge d)).$$

Since $a \to c \leq e \to (a \to c)$ and $e \wedge a \wedge d \leq a \wedge d$ then $x \to y \leq z \to (x \to y)$. Hence, 6) is satisfied. In order to prove 7) and 8), note that

$$\sim (x \to y) = (a \wedge d, a \to c),$$

$$x \wedge \sim y = (a \wedge d, b \vee c).$$

Since $a \wedge d = a \wedge d$ then it follows from Remark 3.1 that $\sim (x \to y) \to (x \wedge \sim y) = (1, 0)$ and $(x \wedge \sim y) \to \sim (x \to y) = (1, 0)$. Thus, we have proved 8).

Therefore, $\mathrm{K}(A) \in \mathsf{SNA}$. □

It is important to note that the variety NA is a proper subvariety of SNA. Indeed, let A be the subresiduated lattice given in Example 2.1. In particular, $\mathrm{K}(A) \in \mathsf{SNA}$. Take $x = (1, 0)$ and $y = (a, b)$, which are elements of $\mathrm{K}(A)$. A direct computation shows that $x \wedge (x \to y) = (0, b) \neq x \wedge (\sim x \vee y)$, so condition N8) is not satisfied. Therefore, $\mathrm{K}(A) \notin \mathsf{NA}$.

Proposition 3.3. *Let $T \in \mathsf{SNA}$. The following conditions are satisfied for every $x, y, z \in T$:*

1. $1 \to x \leq x$,
2. *if* $x \leq y$ *then* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$,
3. *if* $x \leq y$ *then* $x \to y = 1$,
4. $(x \wedge (x \to y)) \to y = 1$,
5. *if* $x \to y = 1$ *then* $x = x \wedge (\sim x \vee y)$,
6. *if* $x \to y = 1$ *and* $\sim y \to \sim x = 1$ *then* $x \leq y$,
7. *if* $x \to y = 1$ *and* $y \to z = 1$ *then* $x \to z = 1$,
8. *if* $x \to y = 1$ *then* $(x \wedge z) \to (y \wedge z) = 1$ *and* $(x \vee z) \to (y \vee z) = 1$,
9. *if* $x \to y = 1$ *then* $(y \to z) \to (x \to z) = 1$ *and* $(z \to x) \to (z \to y) = 1$.

Proof. 1. By 5) of Definition 3.1,

$$(1 \to x) = 1 \wedge (1 \to x) \leq 1 \wedge (\sim 1 \vee x) = x.$$

2. Suppose that $x \leq y$. Then $x = x \wedge y$, so it follows from 2) of Definition 3.1 that

$$z \to x = z \to (x \wedge y) = (z \to x) \wedge (z \to y) \leq z \to y.$$

Using 1) of Definition 3.1, the proof of the other inequality is analogous.

3. It follows from the previous item and 4) of Definition 3.1.

4. It follows from 5) that $x \wedge (x \to y) \leq x \wedge (\sim x \vee y)$, so

$$(x \wedge (\sim x \vee y)) \to y \leq ((x \wedge (x \to y)) \to y. \tag{6}$$

Also note that $(x \wedge \sim x) \to y = 1$. Indeed, since $0 \leq y$ then $(x \wedge \sim x) \to 0 \leq (x \wedge \sim x) \to y$. But it follows from 8) that

$$(x \wedge \sim x) \to 0 = (x \wedge \sim x) \to \sim (x \to x) = 1,$$

so

$$(x \wedge \sim x) \to y = 1. \tag{7}$$

Thus, $(x \wedge (\sim x \vee y)) \to y = ((x \wedge \sim x) \vee (x \wedge y)) \to y$. It follows from (7) and the previous item that $((x \wedge \sim x) \vee (x \wedge y)) \to y = ((x \wedge \sim x) \to y) \wedge ((x \wedge y) \to y) = 1$.

Hence, it follows from (6) that $1 \leq (x \wedge (x \to y)) \to y$, i.e., $(x \wedge (x \to y)) \to y = 1$.

5. It follows from 5) of Definition 3.1.

6. Suppose that $x \to y = 1$ and $\sim y \to \sim x = 1$. It follows from 5. of this proposition that

$$x = x \wedge (\sim x \vee y) = (x \wedge \sim x) \vee (x \wedge y),$$

$$\sim y = \sim y \wedge (y \vee \sim x) = (\sim y \wedge y) \vee (\sim y \wedge \sim x),$$

so

$$y = (y \vee \sim y) \wedge (x \vee y).$$

In particular,

$$x = (x \wedge \sim x) \vee (x \wedge y) \leq (y \vee \sim y) \vee (x \wedge y) = y \vee \sim y,$$

so

$$x = x \wedge (x \vee y) \leq (y \vee \sim y) \wedge (x \vee y) = y.$$

Therefore, $x \leq y$.

7. Suppose that $x \to y = 1$ and $y \to z = 1$. By 3),

$$1 \to (x \to z) = ((x \to y) \wedge (y \to z)) \to (x \to z) = 1.$$

Thus, it follows from 1. of this proposition that $1 = 1 \to (x \to z) \leq x \to z$. Then $x \to z = 1$.

8. Suppose that $x \to y = 1$. Thus, it follows from 2) that

$$(x \wedge z) \to (y \wedge z) = ((x \wedge z) \to y) \wedge ((x \wedge z) \to z)) = (x \wedge z) \to y.$$

From $x \wedge z \leq x$ we get $1 = x \to y \leq (x \wedge z) \to y$ and therefore $(x \wedge z) \to (y \wedge z) = 1$. The other implication is analogous using 1).

9. Suppose that $x \to y = 1$. Then by 3),

$$1 = ((x \to y) \wedge (y \to z)) \to (x \to z) = (y \to z) \to (x \to z).$$

Also by 3),

$$1 = ((z \to x) \wedge (x \to y)) \to (z \to y) = (z \to x) \to (z \to y).$$

□

Let $A \in \mathsf{SNA}$. We define the binary relation θ as in (2) of Section 2.

Lemma 3.4. *Let $T \in \mathsf{SNA}$. Then θ is an equivalence relation compatible with $\wedge$, $\vee$ and $\to$.*

Proof. The reflexivity and symmetry of θ are immediate. The transitivity of the relation follows from 7. of Proposition 3.3. Thus, θ is an equivalence relation. In order to show that θ is compatible with $\wedge$, $\vee$ and $\to$, let $x, y, z \in T$ such that $x\theta y$. It follows from 8. of Proposition 3.3 that $(x \wedge z, y \wedge z) \in \theta$ and $(x \vee z, y \vee z) \in \theta$, so θ is compatible with $\wedge$ and $\vee$. Finally, it follows from 9. of the previous proposition that $(x \to z, y \to z) \in \theta$ and $(z \to x, z \to y) \in \theta$, which implies that θ is compatible with respect to $\to$. □

Let $\langle T, \wedge, \vee, \to, \sim, 0, 1\rangle$ be a subresiduated Nelson algebra. Then it follows from Lemma 3.4 that we can define on T/θ the operations $\wedge$, $\vee$, $\to$, 0 and 1 as in Section 2. In particular, $\langle T/\theta, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice. We denote by $\preceq$ to the order relation of this lattice.

Lemma 3.5. *Let $T \in \mathsf{SNA}$ and $x, y \in A$. Then $x/\theta \preceq y/\theta$ if and only if $x \to y = 1$. Moreover, $x \to y = 1$ if and only if $x \to y/\theta = 1/\theta$.*

Proof. Suppose that $x/\theta \preceq y/\theta$, i.e., $x/\theta = x \wedge y/\theta$, so $x \to (x \wedge y) = 1$. But $x \to (x \wedge y) = (x \to y) \wedge (x \to x) = (x \to y) \wedge 1 = x \to y$, so $x \to y = 1$. Conversely, suppose that $x \to y = 1$. Thus, it follows from 8. of Proposition 3.3 that $(x \wedge x) \to (x \wedge y) = 1$, i.e., $x \to (x \wedge y) = 1$. Since we also have that $(x \wedge y) \to x = 1$, we conclude that $x/\theta = x \wedge y/\theta$, i.e., $x/\theta \preceq y/\theta$.

Finally suppose that $x \to y/\theta = 1/\theta$, so $1 \to (x \to y) = 1$. Therefore, it follows from 1. of Proposition 3.3 that $x \to y = 1$. □

Proposition 3.6. *If $T \in \mathsf{SNA}$ then $T/\theta \in \mathsf{SRL}$.*

Proof. Let $T \in \mathsf{SNA}$ and $x, y, z \in T$. We only need to show that the inequalities $x \wedge (x \to y)/\theta \preceq y/\theta$ and $(x \to y) \wedge (y \to z)/\theta \preceq x \to z/\theta$ are satisfied. It follows from Lemma 3.4 that $x \wedge (x \to y)/\theta \preceq y/\theta$ if and only if $(x \wedge (x \to y)) \to y = 1$. Since by 4. of Proposition 3.3 we have that the previous equality holds, so $x \wedge (x \to y)/\theta \preceq y/\theta$. Finally, also note that $(x \to y) \wedge (y \to z)/\theta \preceq x \to z/\theta$ if and only if $((x \to y) \wedge (y \to z)) \to (x \to z) = 1$. But the equality $((x \to y) \wedge (y \to z)) \to (x \to z) = 1$ is satisfied, so $(x \to y) \wedge (y \to z)/\theta \preceq x \to z/\theta$. □

Theorem 3.7. *Let $T \in \mathsf{SNA}$. Then the map $\rho_T : T \to \mathrm{K}(T/\theta)$ given by $\rho_T(x) = (x/\theta, \sim x/\theta)$ is a monomorphism.*

Proof. We write ρ in place of ρ_T. First we will show that ρ is a well defined map. Let $x \in T$. Note that $x/\theta \wedge \sim x/\theta = 0/\theta$ if and only if $(x \wedge \sim x) \to 0 = 1$. It follows from 8) of Definition 3.1 that $(x \wedge \sim x) \to 0 = (x \wedge \sim x) \to \sim (x \to x) = 1$. Thus, ρ is well defined. The fact that ρ is a bounded lattice homomorphism is immediate. It is also immediate that ρ preserves $\sim$. Moreover, 7) and 8) of Definition 3.1 show that h preserves the implication. Hence, ρ is a homomorphism. Finally, a direct computation based in item 6. of Proposition 3.3 proves that ρ is an injective map. Therefore, ρ is a monomorphism. □

4 Congruence relations on subresiduated Nelson algebras

We start this section by giving some elemental definitions.

Definition 4.1. *Let $T \in \mathsf{SNA}$ and $F \subseteq T$.*

1) *We say that F is a filter of T if $1 \in F$, F is an upset (i.e., for every $x, y \in T$, if $x \leq y$ and $x \in F$ then $y \in F$) and $x \wedge y \in F$, for all $x, y \in F$. If in addition $1 \to x \in F$ for every $x \in F$, we say that F is an open filter of T.*

2) *We say that F is an implicative filter of T if $1 \in F$ and for every $x, y \in F$, if $x \in F$ and $x \to y \in F$ then $y \in F$. If in addition $1 \to x \in F$ for every $x \in F$, we say that F is an open implicative filter of T.*

In this section we prove that for every $T \in \mathsf{SNA}$ there exists an order isomorphism between the lattice of congruences of A and the lattice of open implicative filters of T. We use it in order to give a characterization of the principal congruences of T. In particular, the mentioned characterization proves that the variety SNA has EDPC. We also give a description of the simple and subdirectly irreducible algebras of SNA, and we prove that the variety SNA has CEP.

We start by giving some elemental properties of sr-lattices, which will then be used to transfer them into the framework of subresiduated Nelson algebras.

Let $A \in \mathsf{SRL}$ and $a \in A$. We define $\Box a := 1 \to a$. In the same way, given $T \in \mathsf{SNA}$ and $x \in T$, we define $\Box x := 1 \to x$.

Lemma 4.1. *Let $A \in \mathsf{SRL}$ and $a, b, c \in A$. Then the following conditions are satisfied:*

1) $a \to (b \to c) \leq (a \to b) \to (a \to c)$,

2) $\Box a \to (\Box b \to c) = \Box b \to (\Box a \to c)$,

3) $\Box b \leq a \to (a \wedge b)$.

Proof. Conditions 1) and 2) follow from results of [3]. Condition 3) follows from a direct computation. ☐

Lemma 4.2. *Let $T \in \mathsf{SNA}$ and $v, w, x, y, z \in T$. Then for every $x, y, z \in T$ the following conditions are satisfied:*

1) $x \to (y \to z) \leq (x \to y) \to (x \to z)$,

2) $(x \to v) \to ((y \to w) \to z) = (y \to w) \to ((x \to v) \to z)$,

3) $\Box(x \to y) = x \to y$,

4) $\Box y \leq x \to (x \wedge y)$,

5) $\Box x \leq \sim x \to 0$.

Proof. By Theorem 3.7 we can assume that T is a subalgebra of $\mathrm{K}(A)$ for some $A \in \mathsf{SRL}$. Conditions 1) and 2) follows from a direct computation based in Lemma 4.1. Condition 3) and 4) follows from a direct computation. In order to show 4), let $x = (a, b) \in T$. We will see that $\Box x \leq \sim x \to 0$, i.e., $\Box a \leq b \to 0$. Since $\Box a \wedge b \leq a \wedge b = 0$ then $\Box a \leq b \to 0$, and hence our result is proved. $\square$

Let $T \in \mathsf{SNA}$. We write $\mathrm{Con}(T)$ to indicate the set of congruences of T. Given $\theta \in \mathrm{Con}(T)$ and $x \in T$, we write x/θ for the equivalence class of x associated to the congruence θ.

Lemma 4.3. *Let $T \in \mathsf{SNA}$, $\theta \in \mathrm{Con}(T)$ and $x, y \in T$. Then $(x, y) \in \theta$ if and only if $x \to y$, $\sim y \to \sim x$, $y \to x$, $\sim x \to \sim y \in 1/\theta$.*

Proof. Let $\theta \in \mathrm{Con}(T)$ and $x, y \in T$. It is immediate that if $(x, y) \in \theta$ then $x \to y$, $\sim y \to \sim x$, $y \to x$, $\sim x \to \sim y \in 1/\theta$. Conversely, assume that $x \to y$, $\sim y \to \sim x$, $y \to x$, $\sim x \to \sim y \in 1/\theta$. Following a similar reasoning than the one employed in item 6. of Proposition 3.3 it can be proved that $(x, (x \wedge \sim x) \vee (x \wedge y)) \in \theta$ and $(y, (y \vee \sim y) \wedge (x \vee y)) \in \theta$. Taking into account the inequality $x \wedge \sim x \leq y \vee \sim y$ and the distributivity of the underlying lattice of A we get $(x \wedge y, x) \in \theta$. Similarly it can be showed that $(y \wedge x, y) \in \theta$. Therefore, $(x, y) \in \theta$. $\square$

Lemma 4.4. *Let $T \in \mathsf{SNA}$ and $\theta, \psi \in \mathrm{Con}(T)$. Then $\theta \subseteq \psi$ if and only if $1/\theta \subseteq 1/\psi$. In particular, $\theta = \psi$ if and only if $1/\theta = 1/\psi$.*

Proof. Let $\theta, \psi \in \mathrm{Con}(T)$. It is immediate that if $\theta \subseteq \psi$ then $1/\theta \subseteq 1/\psi$. The converse follows from a direct computation based in Lemma 4.3. $\square$

Lemma 4.5. *Let $T \in \mathsf{SNA}$ and $F \subseteq T$. If F is an open implicative filter then F is an open filter.*

Proof. Let F be an open implicative filter. In order to show that F is an upset, let $x, y \in T$ be such that $x \in F$ and $x \leq y$. Then $x \to y = 1 \in F$, so $y \in F$. Hence, F is an upset. Finally, let $x, y \in F$. We will see that $x \wedge y \in F$. Since F is open then $1 \to y \in F$. From 4.2, we get that $1 \to y \leq x \to (x \wedge y)$, so $x \to (x \wedge y) \in F$. Using that $x \in F$ we obtain that $x \wedge y \in F$. Therefore, F is an open filter. $\square$

Let $T \in \mathsf{SNA}$ and F an implicative filter of T. For every $x, y \in T$ we define $s(x, y) = (x \to y) \wedge (y \to x) \wedge (\sim x \to \sim y) \wedge (\sim y \to \sim x)$. We also define the set

$$\Theta(F) = \{(x, y) \in T \times T : s(x, y) \in F\}.$$

Note that $s(x, y) \in F$ if and only if $x \to y, y \to x, \sim x \to \sim y, \sim y \to \sim x \in F$.

Lemma 4.6. *Let $T \in \mathsf{SNA}$ and F be an open implicative filter. Then $\Theta(F) \in \mathrm{Con}(T)$.*

Proof. It is immediate that $\Theta(F)$ is reflexive and symmetric. In order to show that it is transitive, let $(x,y),(y,z) \in \Theta(F)$. Since it follows from Lemma 4.5 that F is a filter and $x \to y, y \to z \in F$ then $(x \to y) \wedge (y \to z) \in F$. But $((x \to y) \wedge (y \to z)) \to (x \to z) = 1 \in F$, so $x \to z \in F$. In a similar way it can be proved that $z \to x, \sim z \to \sim x, \sim x \to \sim z \in F$. Thus, $(x,z) \in \Theta(F)$. Hence, $\Theta(F)$ is an equivalence relation.

Now we will show that $\Theta(F)$ is a congruence. Let $x,y,z \in T$ be such that $(x,y) \in \Theta(F)$. First we will show that $(x \vee z, y \vee z) \in \Theta(F)$. Note that $(x \vee z) \to (y \vee z) = (x \to (y \vee z)) \wedge (z \to (y \vee z)) = x \to (y \vee z) \geq x \to y$. Since $x \to y \in F$ and F is an upset we get $(x \vee z) \to (y \vee z) \in F$. In a similar way it can be proved that $(y \vee z) \to (x \vee z) \in F$. Besides, note that since

$$\sim (x \vee z) \to \sim (y \vee z) = (\sim x \wedge \sim z) \to (\sim y \wedge \sim z) = (\sim x \wedge \sim z) \to \sim y,$$

and $(\sim x \wedge \sim z) \to \sim y \geq \sim x \to \sim y$ then $\sim x \to \sim y \leq \sim (x \vee z) \to \sim (y \vee z)$. But $\sim x \to \sim y \in F$, so $\sim (x \vee z) \to \sim (y \vee z) \in F$. Analogously it can be showed that $\sim (y \vee z) \to \sim (x \vee z) \in F$. Thus, $(x \vee z, y \vee z) \in \Theta(F)$. A similar argument proves that $(x \wedge z, y \wedge z) \in \Theta(F)$. It is also immediate that $(\sim x, \sim y) \in \Theta(F)$. We have proved that $\Theta(F)$ preserves the operations $\vee$, $\wedge$ and $\sim$.

Following this, we will see that $\Theta(F)$ preserves $\to$. In order to show it, let $x,y,z \in T$ such that $(x,y) \in \Theta(F)$. First we will prove that $(z \to x, z \to y) \in \Theta(F)$. Since $x \to y \leq z \to (x \to y)$ and $x \to y \in F$ then $z \to (x \to y) \in F$. It follows from Lemma 4.2 that $z \to (x \to y) \leq (z \to x) \to (z \to y)$, so $(z \to x) \to (z \to y) \in F$. In a similar way we can show that $(z \to y) \to (z \to x) \in F$. Now we will prove that $\sim (x \to z) \to \sim (y \to z) \in F$. First note that since $\sim y \to \sim x \leq (\sim y \wedge z) \to (\sim x \wedge z)$ and $\sim y \to \sim x \in F$ then $(\sim y \wedge z) \to (\sim x \wedge z) \in F$. We also have that $(z \wedge \sim x) \to \sim (z \to x) = 1$. Taking into account that

$$(((\sim y \wedge z) \to (\sim x \wedge z)) \wedge ((\sim x \wedge z) \to \sim (z \to x))) \to ((\sim y \wedge z) \to \sim (z \to x)) = 1$$

we get

$$((\sim y \wedge z) \to (\sim x \wedge z)) \to ((\sim y \wedge z) \to \sim (z \to x)) = 1.$$

But $(\sim y \wedge z) \to (\sim x \wedge z) \in F$ and $1 \in F$, so

$$(\sim y \wedge z) \to \sim (z \to x) \in F. \tag{8}$$

Besides, since

$$((\sim (z \to y) \to (\sim y \wedge z)) \wedge ((\sim y \wedge z) \to \sim (z \to x))) \to (\sim (z \to y) \to \sim (z \to x)) = 1$$

and $\sim (z \to y) \to (\sim y \wedge z) = 1$, so

$$((\sim y \wedge z) \to \sim (z \to x)) \to (\sim (z \to y) \to \sim (z \to x)) = 1 \in F.$$

Then it follows from (8) that $\sim (z \to y) \to \sim (z \to x) \in F$. In a similar way we can see that $\sim (z \to x) \to \sim (z \to y) \in F$. Hence, $(z \to x, z \to y) \in \Theta(F)$.

Finally we will show that $(x \to z, y \to z) \in \Theta(F)$. It follows from Lemma 4.2 that

$$(y \to z) \to ((x \to z) \to (y \to z)) = (x \to z) \to ((y \to z) \to (y \to z)) = 1 \in F.$$

Thus, since $y \to z \in F$ then $(x \to z) \to (y \to z) \in F$. Analogously, $(y \to z) \to (x \to z) \in F$. Now we will see that $\sim (x \to z) \to\sim (y \to z) \in F$. Note that

$$((\sim (x \to z) \to (\sim z \wedge x)) \wedge ((\sim z \wedge x) \to\sim (y \to z))) \to (\sim (x \to z) \to\sim (y \to z)) = 1.$$

But $\sim (x \to z) \to (\sim z \wedge x) = 1$, so

$$((\sim z \wedge x) \to\sim (y \to z)) \to (\sim (x \to z) \to\sim (y \to z)) = 1.$$

Since $1 \in F$ and F is a filter, in order to show that $\sim (x \to z) \to\sim (y \to z) \in F$ it is enough to see that $(\sim z \wedge x) \to\sim (y \to z) \in F$. Since $x \to y \leq (x\wedge \sim z) \to (y\wedge \sim z)$ and $x \to y \in F$ then

$$(x\wedge \sim z) \to (y\wedge \sim z) \in F. \tag{9}$$

Besides, since $(((\sim z \wedge x) \to (y\wedge \sim z)) \wedge ((y\wedge \sim z) \to\sim (y \to z))) \to ((\sim z \wedge x) \to\sim (y \to z)) = 1$ and $(y\wedge \sim z) \to\sim (y \to z) = 1$ then

$$((\sim z \wedge x) \to (y\wedge \sim z))) \to ((\sim z \wedge x) \to\sim (y \to z)) = 1.$$

Since $1 \in F$ then it follows from (9) that $(\sim z \wedge x) \to\sim (y \to z) \in F$, which was our aim. Then $\sim (x \to z) \to\sim (y \to z) \in F$. Analogously, $\sim (y \to z) \to\sim (x \to z) \in F$. Hence, $\Theta(F)$ preserves $\to$. □

For $T \in \mathsf{SNA}$ we write $\mathrm{IF}^o(T)$ to denote the set of open implicative filters of T.

Theorem 4.7. *Let $T \in \mathsf{SNA}$. The assignments $\theta \mapsto 1/\theta$ and $F \mapsto \Theta(F)$ establish an order isomorphism between $\mathrm{Con}(T)$ and $\mathrm{IF}^o(T)$.*

Proof. Let $H : \mathrm{Con}(T) \to \mathrm{IF}^o(T)$ be the function given by $H(\theta) = 1/\theta$. In order to show that H is a well defined map, let $\theta \in \mathrm{Con}(T)$. In particular, $1 \in 1/\theta$. Let $x, x \to y \in 1/\theta$, so $x \wedge (x \to y) \in 1/\theta$. Since $(x \wedge (x \to y)) \to y = 1$ then $1 \to y \in 1/\theta$. But $1 \to y \leq y$, and $(y \wedge (1 \to y), y) \in \theta$, so $(1 \to y, y) \in \theta$. Thus, $y \in 1/\theta$. Hence, $1/\theta$ is an implicative filter. The fact that $1/\theta$ is open is immediate, so $1/\theta \in \mathrm{IF}^o(A)$. Hence, H is a well defined map. The injectivity of H follows from Lemma 4.3. In order to show that H is suryective, let $F \in \mathrm{IF}^o(T)$. Then it follows from Lemma 4.6 that $\Theta(F) \in \mathrm{Con}(A)$. We will show that $H(\Theta(F)) = F$, i.e., $1/\Theta(F) = F$. In order to prove it, let $x \in 1/\Theta(F)$, i.e., $(x, 1) \in \Theta(F)$. In particular, $1 \to x \in F$. But $1 \to x \leq x$, so $x \in F$. Conversely, let $x \in F$. In particular, $x \to 1 = 1 \in F$. Besides, since F is open then $1 \to x \in F$. We also have that $\sim 1 \to\sim x = 1 \in F$. Finally, it follows from Lemma 4.2 that $1 \to x \leq\sim x \to 0 =\sim x \to\sim 1$, so $\sim x \to\sim 1 \in F$. Then $x \in 1/\Theta(F)$. Thus, $1/\Theta(F) = F$. We have proved that H is a suryective map. Hence, H is a bijective function. Therefore, it follows from Lemma 4.3 that H is an order isomorphism. □

Let $T \in \mathsf{SNA}$ and $X \subseteq T$. We write $\langle X \rangle$ in order to indicate the open implicative filter generated by X, i.e., the least open implicative filter (with respect to the inclusion) which contains the set X. In other words, $\langle X \rangle$ is the intersection of all the open implicative filters that contain X.

Lemma 4.8. *Let $T \in \mathsf{SNA}$ and X a non empty subset of T. Then*

$$\langle X \rangle = \{x \in T : \Box(x_1 \wedge \cdots \wedge x_n) \to x = 1 \textit{ for some } x_1, \ldots, x_n \in X\}.$$

Proof. Let $S = \{x \in T : \Box(x_1 \wedge \cdots \wedge x_n) \to x = 1$ for some $x_1, \ldots, x_n \in X\}$. We will show that S is an open implicative filter. It is immediate that $1 \in S$. Let now $x, y \in T$ be such that $x, x \to y \in S$. Then there exist $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ such that $\Box(x_1 \wedge \cdots \wedge x_n) \to x = 1$ and $\Box(y_1 \wedge \cdots \wedge y_m) \to (x \to y) = 1$. Let $z = x_1 \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots y_m$. Since $\Box(z) \leq \Box(x_1 \wedge \cdots \wedge x_n), \Box(y_1 \wedge \cdots y_m)$ then $\Box(z) \to x = 1$ and $\Box(z) \to (x \to y) = 1$. Thus, it follows from Lemma 4.2 that

$$1 = \Box(z) \to (x \to y) \leq (\Box(z) \to x) \to (\Box(z) \to y) = 1 \to (\Box(z) \to y)$$

so $\Box(z) \to y = 1 \to (\Box(z) \to y) = 1$. Hence, $y \in S$. We have proved that S is an implicative filter. In order to see that S is open, let $x \in S$ so $\Box(x_1 \wedge \cdots \wedge x_n) \to x = 1$ for some $x_1, \ldots, x_n \in X$. Then

$$1 = \Box(1) = \Box(\Box(x_1 \wedge \cdots \wedge x_n) \to x) \leq \Box(x_1 \wedge \cdots \wedge x_n) \to \Box(x),$$

so $\Box(x_1 \wedge \cdots \wedge x_n) \to \Box(x) = 1$. Hence, S is an open implicative filter. Finally, let F be an open implicative filter such that $X \subseteq F$. A direct computation based in the fact that every open implicative filter is an open filter shows that $S \subseteq F$. Therefore, $\langle X \rangle = S$, which was our aim. ∎

A straightforward computation based in Lemma 4.8 proves the following result.

Corollary 4.9. *Let $T \in \mathsf{SNA}$, $F \in \mathrm{IF}^o(T)$ and $x \in T$. Then*

$$\langle F \cup \{x\} \rangle = \{y \in T : (f \wedge \Box(x)) \to y = 1 \textit{ for some } f \in F\}.$$

Let $T \in \mathsf{SNA}$ and $x \in X$. We write $\langle x \rangle$ instead of $\langle \{x\} \rangle$.

Corollary 4.10. *Let $T \in \mathsf{SNA}$ and $x \in T$. Then $\langle x \rangle = \{y \in T : \Box(x) \to y = 1\}$.*

Let $T \in \mathsf{SNA}$ and $x, y \in T$. For every $\theta \in \mathrm{Con}(T)$, it follows from Theorem 4.7 that $(x, y) \in \theta$ if and only if $(s(x, y), 1) \in \theta$. Thus, SNA is a term variety where $(s(x, y), 1)$ is a pair associated to SNA.

Let $T \in \mathsf{SNA}$ and $x, y \in T$. We write $\theta(x, y)$ for the congruence generated by the pair (x, y), i.e., the least congruence which contains the pair (x, y) [17].

Proposition 4.11. *Let $T \in \mathsf{SNA}$ and $x, y, z, w \in T$. Then $(z, w) \in \theta(x, y)$ if and only if $s(x, y) \to s(z, w) = 1$.*

Proof. It follows from Theorem 3.7 and [17, Theorem 2.4] that $(z,w) \in \theta(x,y)$ if and only if $s(z,w) \in \langle s(x,y)\rangle$. That is, $(z,w) \in \theta(x,y)$ if and only if $\Box(s(x,y)) \to s(z,w) = 1$. But $\Box(s(x,y)) = s(x,y)$, so $(z,w) \in \theta(x,y)$ if and only if $s(x,y) \to s(z,w) = 1$. □

Corollary 4.12. *The variety* SNA *has* EDPC.

Other application of Theorem 4.7 are the following two propositions, where we give a description of the simple and subdirectly irreducible algebras of the variety SNA respectively.

Proposition 4.13. *Let $T \in$ SNA. The following conditions are equivalent:*

1) *T is simple.*

2) *For every $x \in T$, if $x \neq 1$ then $\Box(x) \to 0 = 1$.*

Proof. Suppose that T is simple, so $\mathrm{IF}^o(T) = \{\{1\}, T\}$. Let $x \in T$ such that $x \neq 1$. Then $\langle x\rangle \neq \{1\}$, so $\langle x\rangle = T$. Since $0 \in T$ then $\Box(x) \to 0 = 1$. Conversely, let $F \in \mathrm{IF}^o(T)$ be such that $F \neq \{1\}$, so there exists $x \in T$ such that $x \neq 1$. It follows from hypothesis that $\Box(x) \to 0 = 1 \in F$. Since $x \in F$ then $\Box(x) \in F$, so $0 \in F$, i.e., $F = T$. Therefore, T is simple. □

Proposition 4.14. *Let $T \in$ SNA and suppose that T is not trivial. The following conditions are equivalent:*

1) *T is subdirectly irreducible.*

2) *There exists $x \in T - \{1\}$ such that for every $y \in T - \{1\}$, $\Box(y) \to x = 1$.*

Proof. Let T be a non trivial subresiduated Nelson algebra. Suppose that T is subdirectly irreducible. Thus, there exists $F \in \mathrm{IF}^o(T)$ such that $F \neq \{1\}$ and for every open implicative filter $G \neq \{1\}$, $F \subseteq G$. Since $F \neq \{1\}$ there exists $x \in F$ such that $x \neq 1$. Let $y \in T$ be such that $y \neq 1$, i.e., $\langle y\rangle \neq \{1\}$. Thus, $x \in F \subseteq \langle y\rangle$, i.e., $\Box(y) \to x = 1$. Conversely, suppose that 2) is satisfied. Let $x \neq 1$ be an element which satisfies 2). Let $F = \langle x\rangle$, so $F \neq \{1\}$. Let $G \in \mathrm{IF}^o(T)$ be such that $G \neq \{1\}$. Note that $F \subseteq G$ if and only if $x \in G$. In order to see that $x \in G$, note that since $G \neq \{1\}$ then there exists $y \in G$ such that $y \neq 1$. Thus, it follows from hypothesis that $\Box(y) \to x = 1 \in G$. But $y \in G$, so $\Box(y) \in G$. Hence, $x \in G$. Therefore, T is subdirectly irreducible. □

Finally, we use Theorem 4.7 in order to show that SNA has the congruence extension property.

Proposition 4.15. *The variety* SNA *has the congruence extension property.*

Proof. Let $T, U \in$ SNA be such that U is a subalgebra of T and $\theta \in \mathrm{Con}(U)$. We will show that there exists $\hat{\theta} \in \mathrm{Con}(T)$ such that $\theta = \hat{\theta} \cap U^2$. We define $\hat{\theta}$ as the congruence of T generated by θ. Thus, it follows from Lemma 4.4 and Theorem 4.7 that $1/\hat{\theta}$ is the open implicative filter of T generated by the set

$1/\theta$. In order to see that $\theta = \hat{\theta} \cap U^2$, let $(x, y) \in \hat{\theta} \cap U^2$, so $x, y \in U$ and $s(x, y) \in 1/\hat{\theta}$. In particular, $s(x, y) \in U$ and it follows from Lemma 4.8 that there exist $x_1, \ldots, x_n \in 1/\theta$ such that $\Box(x_1 \wedge \cdots \wedge x_n) \to s(x, y) = 1$. Since $\Box(x_1 \wedge \cdots \wedge x_n)$ and 1 are elements of $1/\theta$ then $s(x, y) \in 1/\theta$, i.e., $(x, y) \in \theta$. Therefore, $\theta = \hat{\theta} \cap U^2$. □

5 The variety generated by the class of totally ordered subresiduated Nelson algebras

The aim of this final section is to give an equational base for the variety generated by the totally ordered subresiduated Nelson algebras.

Let $T \in \mathsf{SNA}$. A proper implicative filter P of T is called prime if for every $x, y \in T$ we have that if $x \vee y \in P$ then $x \in P$ or $y \in P$. We write $\mathrm{X}^o(T)$ to denote the set of open prime implicative filters of T.

Lemma 5.1. *Let $T \in \mathsf{SNA}$ such that*

$$T \vDash ((x \to y) \wedge (\sim y \to \sim x)) \vee ((y \to x) \wedge (\sim x \to \sim y)) = 1.$$

Let $P \in \mathrm{X}^o(T)$. Then T/P is a chain.

Proof. Let $P \in \mathrm{X}^o(T)$ and $x, y \in T$. Taking into account that

$$((x \to y) \wedge (\sim y \to \sim x)) \vee ((y \to x) \wedge (\sim x \to \sim y)) = 1 \in P$$

we get $(x \to y) \wedge (\sim y \to \sim x) \in P$ or $(y \to x) \wedge (\sim x \to \sim y) \in P$. Thus, $(x, x \wedge y) \in \Theta(P)$ or $(y, y \wedge x) \in \Theta(P)$, i.e., $x/P \leq y/P$ or $y/P \leq x/P$. Hence, T/P is a chain. □

Lemma 5.2. *Let $T \in \mathsf{SNA}$ such that $T \vDash \Box(x \vee y) = \Box(x) \vee \Box(y)$. Let $F \in \mathrm{IF}^o(A)$ and I an ideal such that $F \cap I = \emptyset$. Then there exists a filter $P \in \mathrm{X}^o(T)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let $\Sigma = \{G \in \mathrm{IF}^o(T) : F \subseteq G \text{ and } G \cap I = \emptyset\}$. Since $F \in \Sigma$ then $\Sigma \neq \emptyset$. A direct computation shows that Σ is under the hypothesis of Zorn's lemma, so there exists a maximal element P in Σ. In particular, $F \subseteq P$ and $P \cap I = \emptyset$. Moreover, it is immediate that P is a proper open implicative filter. Suppose that P is not prime, so there exist $x, y \in T$ such that $x \vee y \in P$ and $x, y \notin P$. Let $P_x = \langle P \cup \{x\} \rangle$ and $P_y = \langle P \cup \{y\} \rangle$. The maximality of P in Σ implies that $P_x \cap I \neq \emptyset$ and $P_y \cap I \neq \emptyset$, so there exist $z, w \in I$ and $p_1, p_2 \in P$ such that $(p_1 \wedge \Box(x)) \to z = 1$ and $(p_2 \wedge \Box(y)) \to w = 1$. Let $p = p_1 \wedge p_2 \in P$. Thus, $(p \wedge \Box(x)) \to (z \vee w) = 1$ and $(p \wedge \Box(y)) \to (z \vee w) = 1$. Thus,

$$((p \wedge \Box(x)) \to (z \vee w)) \wedge ((p \wedge \Box(y)) \to (z \vee w)) = 1.$$

But

$$((p \wedge \Box(x)) \to (z \vee w)) \wedge ((p \wedge \Box(y)) \to (z \vee w)) = ((p \wedge \Box(x)) \vee (p \wedge \Box(y))) \to (y \vee z)$$

and

$$((p \wedge \Box(x)) \vee (p \wedge \Box(y))) \to (y \vee z) = (p \wedge \Box(x \vee y)) \to (y \vee z).$$

Then

$$(p \wedge \Box(x \vee y)) \to (y \vee z) = 1 \in P$$

and $p \wedge \Box(x \vee y) \in P$, so $y \vee z \in P$. Besides, since $y, z \in I$ then $y \vee z \in I$, so $P \cap I \neq \emptyset$, which is a contradiction. Therefore, $P \in \mathrm{X}^o(T)$. $\square$

Let $T \in \mathsf{SNA}$ be such that $T \vDash \Box(x \vee y) = \Box(x) \vee \Box(y)$. Suppose that T is not trivial. Then $\mathrm{X}^o(T) \neq \emptyset$. Indeed, let $x \in T$ such that $x \neq 1$. Then $(x] \cap \{1\} = \emptyset$, where $(x] := \{y \in T : y \leq x\}$. Then there exists a filter $P \in \mathrm{X}^o(T)$. Therefore, $\mathrm{X}^o(T) \neq \emptyset$.

Corollary 5.3. *Let $T \in \mathsf{SNA}$ be such that $T \vDash \Box(x \vee y) = \Box(x) \vee \Box(y)$. Suppose that T is not trivial. Then the intersection of all open prime implicative filters of T is equal to $\{1\}$.*

Let $T \in \mathsf{SNA}$ and $x, y \in T$. We define

$$t(x, y) = ((x \to y) \wedge (\sim y \to \sim x)) \vee ((y \to x) \wedge (\sim x \to \sim y)).$$

We also define

$$\mathsf{SNA}^c = \mathsf{SNA} + \{\Box(x \vee y) = \Box(x) \vee \Box(y)\} + \{t(x, y) = 1\}.$$

Let C be the class of totally ordered members of SNA.

Theorem 5.4. $\mathrm{V}(\mathsf{C}) = \mathsf{SNA}^c$.

Proof. A straightforward computation shows that $\mathsf{C} \subseteq \mathsf{SNA}^c$, so $\mathrm{V}(\mathsf{C}) \subseteq \mathsf{SNA}^c$. In order to show the converse inclusion, let $T \in \mathsf{SNA}^c$. If T is trivial then $T \in \mathrm{V}(\mathsf{C})$. Now suppose that T is not trivial. Let $\alpha : T \to \prod_{P \in \mathrm{X}^o(T)} T/P$ be the homomorphism defined by $\alpha(x) = (x/P)_{P \in \mathrm{X}^o(T)}$. We will show that the homomorphism α is injective. Let $x, y \in T$ such that $\alpha(x) = \alpha(y)$. Then $x/P = y/P$ for every $P \in \mathrm{X}^o(T)$. Thus, $s(x, y) \in P$ for every $P \in \mathrm{X}^o(T)$. It follows from Corollary 5.3 that $s(x, y) = 1$. Hence, $x = y$. We have proved that α is a monomorphism. Besides, it follows from Lemma 5.1 that T/P is a chain for every $P \in \mathrm{X}^o(T)$. Therefore, $T \in \mathrm{V}(\mathsf{C})$. Then, $\mathsf{SNA}^c \subseteq \mathrm{V}(\mathsf{C})$. Therefore, $\mathrm{V}(\mathsf{C}) = \mathsf{SNA}^c$. $\square$

Note that SNA^c is a proper subvariety of SNA. Indeed, let A be the subresiduated lattice given in Example 2.1. We have that $\mathrm{K}(A) \in \mathsf{SNA}$. Consider $x = (a, 0)$ and $y = (b, 0)$, which are elements of $\mathrm{K}(A)$. Then $\Box x \vee \Box y = (0, 0)$ and $\Box(x \vee y) = (1, 0)$, so $\mathrm{K}(A) \notin \mathsf{SNA}^c$.

6 Centered subresiduated Nelson algebras

A Kleene algebra (Nelson algebra) is called centered if there exists an element which is a fixed point with respect to the involution, i.e., an element c such that $\sim \mathsf{c} = \mathsf{c}$. This element is necessarily unique. If $T = \mathrm{K}(A)$ where A is a bounded distributive lattice, the center is $\mathsf{c} = (0, 0)$. Let T be a centered Kleene algebra. We define the following condition:

(CK) For every $x, y \in T$ if $x, y \geq \mathsf{c}$ and $x \wedge y \leq \mathsf{c}$ then there exists $z \in T$ such that $z \vee \mathsf{c} = x$ and $\sim z \vee \mathsf{c} = y$.

The condition (CK) is not necessarily satisfied in every centered Kleene algebra, see for instance Figure 1 of [2]. However, every centered Nelson algebra satisfies the condition (CK) (see [6, Theorem 3.5] and [2, Proposition 6.1]).

The following two properties are well known:

- Let T be a Kleene algebra. Then T is isomorphic to $\mathrm{K}(A)$ for some bounded distributive lattice A if and only if T is centered and satisfies the condition (CK) (see [6, Theorem 2.3] and [2, Proposition 6.1]).
- Let T be a Nelson algebra. Then T is isomorphic to $\mathrm{K}(A)$ for some Heyting algebra A if and only if T is centered (see [6, Theorem 3.7]).

An algebra $\langle T, \wedge, \vee, \rightarrow, \sim, 0, 1, \mathsf{c}\rangle$ is a centered subresiduated Nelson algebra if $\langle T, \wedge, \vee, \rightarrow, \sim, 0, 1\rangle$ is as a subresiduated Nelson algebra and c is a center. We write SNA^c for the variety whose members are centered subresiduated Nelson algebras.

In this section we prove that given $T \in \mathsf{SNA}$, T is isomorphic to $\mathrm{K}(A)$ for some subresiduated lattice A if and only if T is centered and satisfies the condition (CK) (we also show that the condition (CK) is not necessarily satisfied in every centered subresiduated Nelson algebra). Finally we show that there exists a categorical equivalence between SRL and the full subcategory of SNA^c whose objects satisfy the condition (CK) [1].

We start with some preliminary results.

Lemma 6.1. *Let $T \in \mathsf{SNA}^c$ and $x, y \in T$. Then the following conditions are satisfied:*

1) $\mathsf{c} \rightarrow x = 1$.

2) $((x \wedge y) \vee \mathsf{c}) \rightarrow 0 = (x \wedge y) \rightarrow 0$.

3) $(x \wedge y) \rightarrow 0 = 1$ *if and only if* $x \wedge y \leq \mathsf{c}$.

Proof. In order to show 1), first we will see that $\mathsf{c} \rightarrow 0 = 1$. Indeed, $\mathsf{c} \rightarrow 0 = (\mathsf{c} \wedge \sim \mathsf{c}) \rightarrow \sim (\mathsf{c} \rightarrow \mathsf{c}) = 1$, so $c \rightarrow 0 = 1$. Taking into account that $\mathsf{c} \rightarrow 0 = 1$ and $0 \rightarrow x = 1$, it follows from Proposition 3.3 that $\mathsf{c} \rightarrow x = 1$. The condition 2) is a direct consequence of 1).

[1]If V is a variety of algebras, with an abuse of notation we write it in this way to refer to the algebraic category associated to this variety.

In order to prove 3), suppose that that $(x \wedge y) \to 0 = 1$, so

$$x \wedge y = (x \wedge y) \wedge ((x \wedge y) \to 0) \leq (x \wedge y) \wedge \sim (x \wedge y) \leq \mathsf{c}.$$

Hence, $x \wedge y \leq \mathsf{c}$. The converse follows from 1). ☐

Let $T \in \mathsf{SNA}^c$. We define the following condition:

(C) For every $x, y \in T$, if $(x \wedge y) \to 0 = 1$ then there exists $z \in T$ such that $z \vee \mathsf{c} = x \vee \mathsf{c}$ and $\sim z \vee \mathsf{c} = y \vee \mathsf{c}$.

Lemma 6.2. *Let $T \in \mathsf{SNA}^c$. Then ρ is surjective if and only if T satisfies the condition* (C).

Proof. Suppose that ρ is surjective. Let $x, y \in T$ such that $(x \wedge y) \to 0 = 1$, i.e., $x/\theta \wedge y/\theta = 0/\theta$. Since ρ is surjective, there exists $z \in T$ such that $z/\theta = x/\theta$ and $\sim z/\theta = y/\theta$, i.e., $z \to x = 1$, $x \to z = 1$, $\sim z \to y = 1$, $y \to \sim z = 1$. We will show that $x \vee \mathsf{c} \leq z \vee \mathsf{c}$. It follows from Proposition 3.3 that $(x \vee \mathsf{c}) \to (z \vee \mathsf{c}) = 1$. Thus,

$$x \vee \mathsf{c} = (x \vee \mathsf{c}) \wedge ((x \vee \mathsf{c}) \to (z \vee \mathsf{c})) \leq (x \vee \mathsf{c}) \wedge (\sim (x \vee \mathsf{c}) \vee (z \vee c)) = (x \vee \mathsf{c}) \wedge (z \vee \mathsf{c}),$$

so $x \vee \mathsf{c} \leq z \vee \mathsf{c}$. The same argument shows that $z \vee \mathsf{c} \leq x \vee \mathsf{c}$, so $x \vee \mathsf{c} = z \vee \mathsf{c}$. Similarly, we get $\sim z \vee \mathsf{c} = y \vee \mathsf{c}$. Hence, condition (C) is satisfied.

Conversely, suppose that (C) is satisfied and let $x, y \in T$ be such that $x/\theta \wedge y/\theta = 0/\theta$, i.e., $(x \wedge y) \to 0 = 1$. It follows from hypothesis that there exists $z \in T$ such that $z \vee \mathsf{c} = x \vee \mathsf{c}$ and $\sim z \vee \mathsf{c} = y \vee \mathsf{c}$. Then $(x \vee \mathsf{c}) \to x = (z \vee \mathsf{c}) \to x$, i.e., $(z \vee \mathsf{c}) \to x = 1$. Besides, $(z \vee \mathsf{c}) \to x = (z \to x) \wedge (\mathsf{c} \to x)$, and it follows from Lemma 6.1 that $\mathsf{c} \to x = 1$, so $z \to x = 1$. The same argument shows that $x \to z = 1$, so $x/\theta = z/\theta$. For the same reason, we get $\sim z/\theta = y/\theta$. Therefore, ρ is surjective. ☐

Lemma 6.3. *Let $T \in \mathsf{SNA}^c$. Conditions* (C) *and* (CK) *are equivalent.*

Proof. Suppose that T satisfies (CK) and let $x, y \in T$ such that $(x \wedge y) \to 0 = 1$. It follows from Lemma 6.1 that $x \wedge y \leq \mathsf{c}$. Let $\hat{x} = x \vee \mathsf{c}$ and $\hat{y} = y \vee \mathsf{c}$. Then $\hat{x}, \hat{y} \geq \mathsf{c}$ and $\hat{x} \wedge \hat{y} \leq \mathsf{c}$, so it follows from hypothesis that there exists $z \in T$ such that $z \vee \mathsf{c} = \hat{x}$ and $\sim z \vee \mathsf{c} = \hat{y}$, i.e., $z \vee \mathsf{c} = x \vee \mathsf{c}$ and $\sim z \vee \mathsf{c} = y \vee \mathsf{c}$.

Conversely, suppose that (C) is satisfied. Let $x, y \in T$ be such that $x, y \geq \mathsf{c}$ and $x \wedge y \leq \mathsf{c}$ It follows from Lemma 6.1 that $(x \wedge y) \to 0 = 1$. Thus, it follows from hypothesis that there exists $z \in T$ such that $z \vee \mathsf{c} = x \vee \mathsf{c}$ and $\sim z \vee \mathsf{c} = y \vee \mathsf{c}$, i.e., $z \vee \mathsf{c} = x$ and $\sim z \vee \mathsf{c} = y$. ☐

Theorem 6.4. *Let $T \in \mathsf{SNA}$. Then T is isomorphic to $\mathrm{K}(A)$ for some subresiduated lattice A if and only if T has center and it satisfies the condition* (CK).

Proof. Suppose that $T \cong \mathrm{K}(A)$ for some $A \in \mathsf{SRL}$. Let $x, y \in \mathrm{K}(A)$ such that $x, y \geq \mathsf{c}$ and $x \wedge y \leq \mathsf{c}$. Thus, there exists $(a, b) \in \mathrm{K}(A)$ such that $x = (a, 0)$ and $y = (b, 0)$. The element $z = (a, b) \in \mathrm{K}(A)$ satisfies that $z \vee \mathsf{c} = x$ and $\sim z \vee \mathsf{c} = y$. Hence, $\mathrm{K}(A)$ satisfies (CK). Since this condition is preserved by isomorphisms then T satisfies (CK). Furthermore, the fact that $\mathrm{K}(A)$ has got a center, which is $(0, 0)$, implies that T has a center. The converse follows from Lemma 6.2, Lemma 6.3 and Theorem 3.7. ☐

Let A be the subresiduated lattice given in Example 2.1. Then $\mathrm{K}(A) \in \mathsf{SNA}^c$. Let T be a subset of $\mathrm{K}(A)$ given by the following Hasse diagram, which is the Hasse diagram of Figure 1 of [2]:

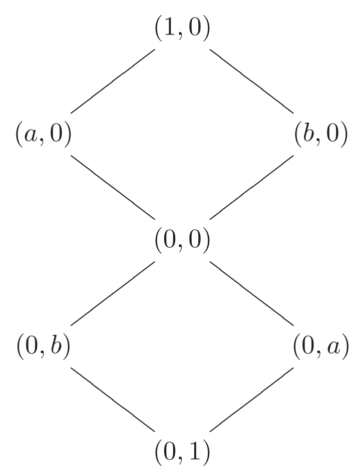

Figure 1: A centered subresiduated Nelson algebra that does not meet Condition (CK).

Since T is a subalgebra of $\mathrm{K}(A)$, we get $T \in \mathsf{SNA}^c$. Note that we have $(a, 0), (b, 0) \geq (0, 0)$ and $(a, 0) \wedge (b, 0) = (0, 0)$. However there is not $z \in T$ such that $z \vee (0, 0) = (a, 0)$ and $\sim z \vee (0, 0) = (b, 0)$. Therefore, T does not satisfy (CK). It is also interesting to note that $U = \{(0, 1), (1, 0)\} \in \mathsf{SNA}$ but U does not have a center.

If $A \in \mathsf{SRL}$ then $\mathrm{K}(A) \in \mathsf{SNA}$. Besides, if $f : A \to B$ is a morphism in SRL then it follows from a direct computation that the map $\mathrm{K}(f) : \mathrm{K}(A) \to \mathrm{K}(B)$ given by $\mathrm{K}(f)(a, b) := (f(a), f(b))$ is a morphism in SNA. Moreover, K can be extended to a functor from SRL to SNA. Conversely, if $T \in \mathsf{SNA}$ then $\mathsf{C}(T) =: T/\theta \in \mathsf{SRL}$. If $f : T \to U$ is a morphism in SNA then $\mathsf{C}(f) : T/\theta \to U/\theta$ given by $\mathsf{C}(f)(x/\theta) = f(x)/\theta$ is a morphism in SRL. Moreover, C can be extended to a functor from SNA to SRL.

Lemma 6.5. *Let $A \in \mathsf{SRL}$. Then the map $\alpha_A : A \to \mathsf{C}(\mathrm{K}(A))$ given by $\alpha_A(a) = (a, \neg a)/\theta$ is an isomorphism.*

Proof. We write α in place of α_A. First we will show that α is a well defined map. Let $a \in A$. Then $a \wedge \neg a = 0$. Thus, $(a, \neg a)/\theta \in \mathrm{K}(A)/\theta$. Let $a \in A$. Then $(a, \neg a)/\theta = (a, 0)/\theta$. It is immediate that α is a homomorphism. The injectivity of α is also immediate. In order to show that α is suryective, let $y \in \mathsf{C}(\mathrm{K}(A))$, so $y = (a, b)/\theta$ for some $a, b \in A$ such that $a \wedge b = 0$. Moreover, $y = (a, \neg a)/\theta$, so $y = \alpha(a)$. Thus, α is suryective. Therefore, α is an isomorphism. □

A direct computation shows that if $f : A \to B$ is a morphism in SRL and $a \in A$ then $\mathsf{C}(K(f))(\varphi_A(a)) = \varphi_B(f(a))$, and that if $f : T \to U$ is a morphism in SNA and $x \in T$ then $\mathrm{K}(\mathsf{C}(f))(\rho_T(x)) = \rho_U(f(x))$.

Remark 6.1. *Note that if T, U are Kleene algebras, T has center and f is a morphism in SNA from T to U, then U has center and $f(\mathsf{c}) = \mathsf{c}$. Indeed, $f(\mathsf{c}) = f(\sim \mathsf{c}) = \sim f(\mathsf{c})$.*

Therefore, the following result follows from the previous results of this section, and Lemmas 6.2 and 6.3.

Theorem 6.6. *There exists a categorical equivalence between SRL and SNA^c.*

7 Conclusions and two open problems

In this paper we extended, in the framework of sr-lattices, the well known twist construction given for Heyting algebras. In order to make it possible, we introduced and studied the variety SNA by showing that every subresiduated Nelson algebra can be represented as a twist structure of an sr-lattice. We also characterized the congruences of subresiduated Nelson algebras and we applied this result in order to obtain some additional properties for these algebras. In particular, we described simple and subdirectly irreducible algebras, we proved that SNA has EDPC and CEP, we presented an equational base for the variety of SNA generated by the class of its totally ordered members and finally we proved that there exists a categorical equivalence between SRL and SNA^c.

We finish this paper by considering two (open) problems concerning the matter of this paper.

Problem 1: Generalize the term equivalence between Nelson algebras and Nelson lattices

We assume the reader is familiar with commutative residuated lattices [9]. An involutive residuated lattice is a bounded, integral and commutative residuated lattice $(T, \wedge, \vee, *, \to, 0, 1)$ such that for every $x \in T$ it holds that $\neg\neg x = x$, where $\neg x := x \to 0$ and 0 is the first element of T [1]. In an involutive residuated lattice it holds that $x * y = \neg(x \to \neg y)$ and $x \to y = \neg(x * \neg y)$. A Nelson lattice [1] is an involutive residuated lattice $(T, \wedge, \vee, *, \to, 0, 1)$ which satisfies the additional inequality $(x^2 \to y) \wedge ((\neg y)^2 \to \neg x) \leq x \to y$, where $x^2 := x * x$. See also [20].

Remark 7.1. *Let $(T, \wedge, \vee, \Rightarrow, \sim, 0, 1)$ be a Nelson algebra. We define on T the binary operations $*$ and $\rightarrow$ by $x * y :=\sim (x \Rightarrow \sim y) \vee \sim(y \Rightarrow \sim x)$ and $x \rightarrow y := (x \Rightarrow y) \wedge (\sim y \Rightarrow \sim x)$. Then [1, Theorem 3.1] says that $(T, \wedge, \vee, \rightarrow, *, 0, 1)$ is a Nelson lattice. Moreover, $\sim x = \neg x = x \rightarrow 0$.*

Let $(T, \wedge, \vee, *, \rightarrow, 0, 1)$ be a Nelson lattice. We define on T a binary operation $\Rightarrow$ and a unary operation $\sim$ by $x \Rightarrow y := x^2 \rightarrow y$ and $\sim x := \neg x$, where $x^2 = x * x$. Then Theorem 3.6 of [1] says that the $(T, \wedge, \vee, \Rightarrow, \sim, 0, 1)$ is a Nelson algebra. In [1, Theorem 3.11] it was also proved that the category of Nelson algebras and the category of Nelson lattices are isomorphic. Taking into account the construction of this isomorphism in that paper we have that the variety of Nelson algebras and the variety of Nelson lattices are term equivalent and the term equivalence is given by the operations we have defined before.

Remark 7.2. *Let A be a Heyting algebra. Then $(\mathrm{K}(A), \wedge, \vee, \Rightarrow, \sim, \mathsf{c}, 0, 1)$ is a centered Nelson algebra. Thus, it follows from Remark 7.1 that $\hat{\mathrm{K}}(A) := (\mathrm{K}(A), \wedge, \vee, *, \rightarrow, \mathsf{c}, 0, 1)$ is a centered Nelson lattice, where for (a, b) and (d, e) in $\mathrm{K}(A)$ the operations $*$ and $\rightarrow$ are given by*

$$(a, b) * (c, d) = (a \wedge c, (a \rightarrow d) \wedge (c \rightarrow b)),$$
$$(a, b) \rightarrow (c, d) = ((a \rightarrow c) \wedge (d \rightarrow b), a \wedge d).$$

The following question naturally arises:

- Is it there a variety of algebras, in the language of Nelson lattices, which is term equivalent to the variety of subresiduated Nelson lattices?

We do not have an answer for this question.

In [2, Corollary 4.18] it was proved that there exists an equivalence between SRL and an algebraic category whose objects are in the language of centered Nelson lattices. We write KSRL for this algebraic category. In particular, if $A \in \mathsf{SRL}$ then $\hat{\mathrm{K}}(A) \in \mathsf{KSRL}$, where the binary operation $\rightarrow$ is defined as in Remark 7.2. Moreover, for every $T \in \mathsf{KSRL}$ there exists $A \in \mathsf{SRL}$ such that T and $\hat{\mathrm{K}}(A)$ are isomorphic algebras.

The following result follows from Theorem 6.6 and [2, Corollary 4.18].

Proposition 7.1. *The categories KSRL and SNA^c are equivalent.*

The following question also naturally arises:

- Is there a variety of algebras, in the language of centered Nelson lattices, which is term equivalent to the variety of centered subresiduated Nelson lattices?

We do not have an answer for this question. However, we know that the usual construction (the one used to show the term equivalence between Nelson algebras and Nelson lattices) does not work by considering centered subresiduated Nelson algebras and the objects of the algebraic category KSRL. In order to show it, assume that the construction works, which is equivalent to say that this works for the centered subresiduated Nelson algebra $(\mathrm{K}(A), \wedge, \vee, \Rightarrow, \sim, \mathsf{c}, 0, 1)$ and the algebra $\hat{\mathrm{K}}(A)$ where A is an arbitrary sr-lattice. Then, for every $A \in \mathsf{SRL}$ and $a, b, c, d \in A$ such that $a \wedge b = c \wedge d = 0$ the equality $(a, b) \Rightarrow (c, d) = (a, b)^2 \to (c, d)$ is satisfied, i.e., the inequality $a \to c \leq d \to (a \to b)$ is satisfied. Consider the Boolean algebra of four elements, where a and b are the atoms, and $D = \{0, a, 1\}$. We have that (A, D), or directly A, is an sr-lattice. Define $c = a$ and $d = b$. We have that $a \to c = 1$ and $d \to (a \to b) = d \to 0 = a$, so $a \to c \nleq d \to (a \to b)$, which is a contradiction.

Problem 2: Generalize the equivalence between NA and a category of enriched Heyting algebras

We know that every Nelson algebra can be represented as a special twist structure of a Heyting algebra. This correspondence was formulated as a categorical equivalence (by Sendlewski in the early 1990's, see [19] and also in [21]) between Nelson algebras and a category of enriched Heyting algebras, which made it possible to transfer a number of fundamental results from the more widely studied theory of intuitionistic algebras to the realm of Nelson algebras.

The objects of the category of enriched Heyting algebras above mentioned are pairs (A, R), where A is a Heyting algebra and R is a Boolean congruence of A (i.e., R is a congruence such that A/R is a Boolean algebra). Congruences of any Heyting algebra can be represented by filters, and filters corresponding to Boolean congruences are precisely those containing all dense elements, i.e., elements a such that $\neg a = 0$ [15]. This allows to replace the notion of a Boolean congruence by the notion of a filter containing dense elements, which will be called Boolean filter. Thus, we may consider pairs (A, F) where A is a Heyting algebra and F is a Boolean filter. The categorical equivalence for NA can be presented as follows. We define Hey^* as the category whose objects are pairs (A, F), where A is a Heyting algebra and F is a Boolean filter, and whose morphisms $f : (A, F) \to (B, G)$ are homomorphisms such that $f(F) \subseteq G$. If $(A, F) \in \mathsf{Hey}^*$ then

$$\mathrm{K}(A, F) := \{(a, b) \in A \times A : a \wedge b = 0 \text{ and } a \vee b \in F\}$$

is a Nelson algebra with the operations mentioned in Section 3. If $f : (A, F) \to (B, G)$ is a morphism in Hey^* then $\mathrm{K}(f) : \mathrm{K}(A, F) \to \mathrm{K}(B, G)$ defined by $\mathrm{K}(f)(a, b) := (f(a), f(b))$ is a morphism in NA. Moreover, K can be extended to a functor from Hey^* to NA. Conversely, if $T \in \mathsf{NA}$ then $\mathsf{C}(T) := (T/\theta, T^+/\theta) \in \mathsf{Hey}^*$, where $T^+ := \{x \in T : x \geq \sim x\}$. If $f : T \to U$ is a morphism in NA then $\mathsf{C}(f) : \mathsf{C}(T) \to \mathsf{C}(U)$ given by $\mathsf{C}(f)(x/\theta) := f(x)/\theta$ is a morphism in Hey^*. Moreover, C can be extended to a functor from NA to Hey^*. Furthermore, if $(A, F) \in \mathsf{Hey}^*$ then $\alpha : (A, F) \to \mathsf{C}((\mathrm{K}(A, F))$, given as in Lemma 6.5, is an isomorphism in Hey^*, and if $T \in \mathsf{NA}$ then $\rho : T \to \mathrm{K}(\mathsf{C}(T))$, given as in Theorem 3.7, is an isomorphism in NA. The functors K and C establish a categorical equivalence between Hey^* and NA [20, 21]. Therefore, it is natural to think in the following problem:

- Is it possible to generalize the above mentioned categorical equivalence in the framework of SNA?

Let $A \in \mathsf{SRL}$. One might think that a possible solution is to consider filters of an algebra $A \in \mathsf{SRL}$ that contain the set of dense elements $\mathsf{De}(A) := \{a \in A : \neg a = 0\}$ (as in the case of Heyting algebras). It is known that in Heyting algebras the set of dense elements is a filter. However, in sr-lattices it is not necessarily satisfied. Indeed, let A be the sr-lattice given in Example 2.1. Then $\mathsf{De}(A) = \{a, b, 1\}$, which is not a filter. In sr-lattices the congruences are given by open filters, i.e., filters F such that $1 \to a \in F$ whenever $a \in F$ [7, 4]. In order to try to generalize the equivalence between Hey^* and NA, it seems natural to work with pairs (A, F), where A is a sr-lattice and F is an open filter such that $\mathsf{De}(A) \subseteq F$ (these filters will be called subresiduated filters).

Lemma 7.2. *Let $A \in \mathsf{SRL}$. Then $\mathsf{De}(A) = \{a \vee \neg a : a \in A\}$.*

Proof. Let $a \in \mathsf{De}(A)$. Then $\neg a = 0$, so $a = a \vee \neg a$. Hence $a \in \{b \vee \neg b : b \in A\}$. Conversely, let $a \in A$. We shall see that $\neg(a \vee \neg a) = 0$. Indeed, $\neg(a \vee \neg a) = \neg a \wedge \neg\neg a = 0$. Hence, $a \vee \neg a \in \mathsf{De}(A)$. □

We write SRL^* for the set whose elements are pairs (A, F), with $A \in \mathsf{SRL}$ and F a subresiduated filter.

Let $(A, F) \in \mathsf{SRL}^*$. We define

$$\mathrm{K}(A, F) := \{(a, b) \in A \times A : a \wedge b = 0 \text{ and } a \vee b \in F\}.$$

Note that $\mathrm{K}(A, A) = \mathrm{K}(A)$. Let $(a, b), (c, d) \in \mathrm{K}(A, F)$. Then, we have that $(a, b) \wedge (c, d), (a, b) \vee (c, d)$ and $\sim (a, b) \in \mathrm{K}(A, F)$ (for details see for instance [21]).

Lemma 7.3. *Let $(A, F) \in \mathsf{SRL}^*$ and $(a, b), (c, d) \in \mathrm{K}(A, F)$. Then $(a, b) \Rightarrow (c, d) \in \mathrm{K}(A, F)$.*

Proof. Let $(a, b), (c, d) \in \mathrm{K}(A, F)$. We will prove that $(a \to c, a \wedge d) \in \mathrm{K}(A, F)$. First, note that since $(a \to c, a \wedge d) \in \mathrm{K}(A)$, $(a \to c) \wedge a \wedge d = 0$. We will see that $(a \to c) \vee (a \wedge d) \in F$. By distributive law, $(a \to c) \vee (a \wedge d) = ((a \to c) \vee a) \wedge ((a \to c) \vee d)$. It is immediate to see that $\neg a \leq a \to c$ and in consequence $\neg a \vee a \leq (a \to c) \vee a$. Since $\neg a \vee a \in \mathsf{De}(A) \subseteq F$, we get $(a \to c) \vee a \in F$. We only need to check that $(a \to c) \vee d \in F$. To do so, we know that $(1 \to (c \vee d)) \wedge ((c \vee d) \to c) \leq 1 \to c$. Since, $(c \vee d) \to c = (c \to c) \wedge (d \to c) = d \to c$, and $d \to c = (d \to c) \wedge (d \to d) = d \to (c \wedge d) = d \to 0$, we get $(1 \to (c \vee d)) \wedge \neg d \leq 1 \to c$. Then, $((1 \to (c \vee d)) \wedge \neg d) \vee d \leq (1 \to c) \vee d$, i.e.,

$$((1 \to (c \vee d)) \vee d) \wedge (\neg d \vee d) \leq (1 \to c) \vee d.$$

Since $c \vee d \in F$ and F is an open filter, it follows that $1 \to (c \vee d) \in F$ and thus, $(1 \to (c \vee d)) \vee d \in F$. From $\neg d \vee d \in F$, we get $(1 \to c) \vee d \in F$. It is easy to see that $(1 \to c) \vee d \leq (a \to c) \vee d$, therefore we get $(a \to c) \vee d \in F$, which was our aim. □

Thus, for $(A, F) \in \mathsf{SRL}^*$ we have that $\langle \mathrm{K}(A, F), \wedge, \vee, \Rightarrow, \sim, (0,1), (1,0) \rangle$ is a subresiduated Nelson algebra because this is a subalgebra of the subresiduated Nelson algebra $\langle \mathrm{K}(A), \wedge, \vee, \Rightarrow, \sim, (0,1), (1,0) \rangle$. However, this construction can not be extended to a categorical equivalence, unless following the construction which shows the categorical equivalence for the category of enriched Heyting algebras we have mentioned in this section. Indeed, let us consider T as the subresiduated Nelson algebra whose Hasse diagram is given in Figure 1. Then T/θ is isomorphic to the sr-lattice A given in the Example 2.1. Note that $\mathsf{De}(A) = \{a, b, 1\}$. So, the only open filter F that contains the set $\mathsf{De}(T/\theta)$ is $F = T/\theta$ and we know that T is not isomorphic to $\mathrm{K}(T/\theta, T/\theta) = \mathrm{K}(T/\theta)$.

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References

[1] Busaniche M. and Cignoli R., *Constructive Logic with Strong Negation as a Substructural Logic*. Journal of Logic and Computation 20 (4), 761–793 (2010).

[2] Castiglioni J.L., Celani S. and San Martín H.J., *Kleene algebras with implication*. Algebra Universalis, vol. 77, no. 4, 375–393 (2017).

[3] Castiglioni J.L., Fernández V., Mallea H. and San Martín H.J., *On subreducts of subresiduated lattices and some related logics*. Journal of Logic and Computation (2023).

[4] Celani S. and Jansana R., *Bounded distributive lattices with Strict implication*. Math. Log. Quart. 51, No. 3, 219–246 (2005).

[5] Celani S., Nagy A. and San Martín H.J., *Dualities for subresiduated lattices*. Algebra Universalis 82, 59 (2021).

[6] Cignoli R., *The class of Kleene algebras satisfying an interpolation property and Nelson algebras*. Algebra Universalis 23, 262–292 (1986).

[7] Epstein G. and Horn A., *Logics wich are characterized by subresiduated lattices*. Z. Math. Logik Grundlagen Math. 22, 199–210 (1976).

[8] Hacking I., *What is strict implication?* J. Symb. Logic 28, 51–71 (1963).

[9] Hart J., Raftery L. and Tsinakis C., *The structure of commutative residuated lattices*, Internat. J. Algebra Comput. 12, 509–524 (2002).

[10] Jansana R. and San Martín H.J., *On Kalman's functor for bounded hemi-implicative semilattices and hemi-implicative lattices.* Logic Journal of the IGPL, vol. 26, no. 1, 47–82 (2018).

[11] Kalman J.A, *Lattices with involution.* Trans. Amer. Math. Soc. 87, 485–491 (1958).

[12] Monteiro A. and Monteiro L., *Axiomes indépendants pour les algèbres de Nelson, de Lukasiewicz trivalentes, de De Morgan et de Kleene.* Informe Técnico Interno 47. Instituto de Matemática. Universidad Nacional del Sur, Bahía Blanca (1995).

[13] Monteiro A. and Monteiro L., *Unpublished papers I, Notas de Lógica Matemática 40.* Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca (1996).

[14] Nelson D.,*Constructible falsity.* Journal of Symbolic Logic 14, 16–26 (1949).

[15] Rasiowa H. and Sikorski R., *The Mathematics of Metamathematics.* PWN Warszawa (1970).

[16] Rasiowa H., *An algebraic approach to non-classical logics*, volume 78 of Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam (1974).

[17] San Martín H.J., *On relative principal congruences in term quasivarieties.* Studia Logica 110, 1465–1491 (2022).

[18] Sendlewski A., *Some investigations of varieties of N-lattices.* Studia Logica 43, 257–280 (1984).

[19] Sendlewski A., *Nelson algebras through Heyting ones. I.* Studia Logica 49, 1, 105–126 (1990).

[20] Vakarelov D., *Notes on N-lattices and constructive logic with strong negation.* Studia Logica 36, 109–125 (1977).

[21] Viglizzo I., *Álgebras de Nelson.* Tesis de Magíster, Universidad Nacional del Sur, Bahía Blanca, Buenos Aires, Argentina (1999).

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