The dressed atom revisited: Hamiltonian-independent treatment of the radiative cascade

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The dressed atom approach provides a tool to investigate the dynamics of an atom-laser system by fully retaining the quantum nature of the coherent mode. In its standard derivation, the internal atom-laser evolution is described within the rotating-wave approximation, which determines a doublet structure of the spectrum and the peculiar fluorescence triplet in the steady state. However, the rotating wave approximation may fail to apply to atomic systems subject to femtosecond light pulses, light-matter systems in the strong-coupling regime or sustaining permanent dipole moments. This work aims to demonstrate how the general features of the steady-state radiative cascade are affected by the interaction of the dressed atom with propagating radiation modes. Rather than focusing on a specific model, we analyze how these features depend on the parameters characterizing the dressed eigenstates in arbitrary atom-laser dynamics, given that a set of general hypotheses is satisfied. Our findings clarify the general conditions in which a description of the radiative cascade in terms of transition between dressed states is self-consistent. We provide a guideline to determine the properties of photon emission for any atom-laser interaction model, which can be particularly relevant when the model should be tailored to enhance a specific line. We apply the general results to the case in which a permanent dipole moment is a source of low-energy emission, whose frequency is of the order of the Rabi coupling.

The dressed atom approach is a tool to investigate the dynamics of an atom interacting with a laser mode in a fully quantum way, along with its coupling to the propagating radiation modes [1]. Compared to other powerful methods to investigate the atom-laser interactions such as the optical Bloch equations [2, 3], the dressed-atom approach has various advantages [4]. First, the atom-laser dynamics is typically described by time-independent evolution equations, either in a closed-system or in an open-system framework; thus, by avoiding at all the introduction of a classical time-dependent field, one can potentially extend the analysis well beyond the rotating-wave approximation (RWA), where time dependence can hardly be integrated out of the Bloch equations. Second, the quantum nature of the laser mode allows for a consistent description in both the time and frequency domains [4], as well as for the formulation of a decay dynamics that properly takes into account hybridization of the unstable atom state and non-perturbative energy shifts induced by the atom-laser coupling. The dressed-atom approach was originally introduced to describe *natural* systems interacting with light, in a condition where a quasi-resonant laser and weak coupling made RWA safely applicable. However, recent developments in the implementation of *artificial* atoms allowed for the realization of strong light-matter coupling [5–10]. Especially in view of their relevance for the implementation of quantum protocols [11], strong-coupling effects call for a specific theoretical focus on light-matter interaction beyond RWA [12].

The purpose of this Letter is to present a general framework describing the dynamics of a dressed two-level atom when one of the atomic states is unstable, yielding a radiative cascade in the asymptotic regime [4]. While the dressed-atom approach in its standard form is based on the RWA (i.e., the Jaynes-Cummings model [13]), and can be potentially corrected by perturbing the system with additional terms, we do not aim at a formulation tailored on a specific Hamiltonian model of atom-laser interactions. Instead, we show how the general features of the radiative cascade depend on the parameters that characterize the dressed atom-laser eigenstates, given a set of minimally restrictive phenomenological hypotheses on the system dynamics. The general results can then be applied to a variety of cases, including, without claiming to be exhaustive, the presence of a permanent dipole moment for one or both the atomic levels [14–16], the non-negligible contribution of counter-rotating terms [10, 12, 17–20], the presence of nonlinear coupling in the field operators [21]. We will apply the theory to the specific case systems with permanent dipole moments, that describe, e.g., polar molecules, or engineered condensed-matter systems such as asymmetric quantum dots. Their optical response enriched by the interplay of permanent and induced electric dipoles leads to application potentials for THz radiation sources [14, 15, 22, 23], squeezed light generation [24, 25] and for nonlinear optical absorption [22]. Their spontaneous decay dynamics was analyzed in Ref. [26] and, with the addition of counter-rotating terms, in Ref. [16].

The results here derived are based on the following set of hypotheses:

1. The atom can be treated as a two-level system, whose Hilbert space is spanned by the states $\{|g\rangle, |e\rangle\}$. While $|g\rangle$ is stable, in the absence of coupling with the laser the state $|e\rangle$ spontaneously decays towards $|g\rangle$ by single-photon emissions in a non-dispersive medium, with a decay rate Γ_0. Such an atom-photon interaction can be treated in the Markovian approximation.

2. The average $\langle N\rangle$ and the standard deviation $\Delta N = \sqrt{\langle (N-\langle N\rangle)^2\rangle}$ of the photon number in the laser mode satisfy

$$\langle N\rangle \gg \Delta N \gg 1. \tag{1}$$

   This assumption is crucial to approximate the action of the mode annihilation operator a on number states $|N\rangle$

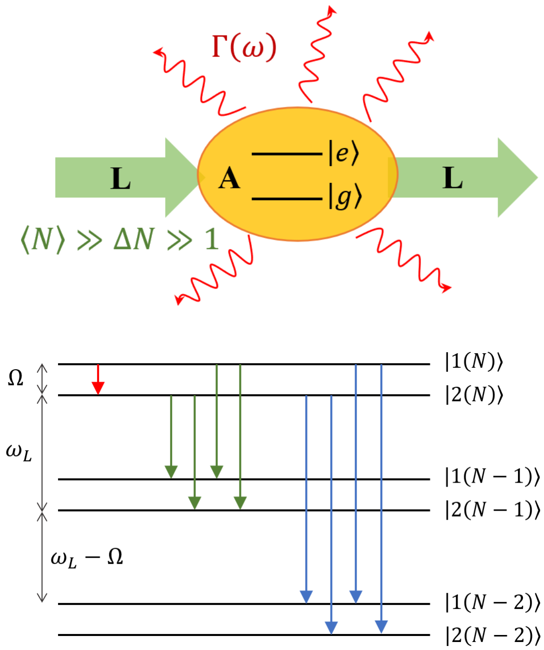

Figure 1. *Upper panel.* Schematic representation of a two-level atom coupled both to a stable laser mode, and to radiation modes characterized by a continuous spectrum, weighed at different frequencies by the form factor $\Gamma(\omega)$. *Lower panel.* Doublet structure of the spectrum of the atom-laser Hamiltonian $H_{\rm AL}$ and possible transitions between them, due to the contributions of the atomic excited and ground states in the eigenstates. The elementary frequency scales characterizing the spectrum are all much larger than the excited atom decay rate.

as $a|N\rangle \simeq \sqrt{\langle N\rangle}|N-1\rangle$ and obtain atom-laser transition matrix elements that are constant for all practical purposes. Moreover, the system is observed for a time T such that the decay-induced depletion does not significantly alter the laser number statistics, i.e. $\Gamma_0 T \ll \Delta N$.

3. The energy diagram of the atom-laser system is made of non-degenerate doublets of *dressed eigenstates* $\{|1(N)\rangle, |2(N)\rangle\}$, with eigenvalues

$$E_j(N) = \hbar\left(N\omega_L - (-1)^j \frac{\Omega}{2}\right) \quad \text{for } j \in \{1,2\}, \tag{2}$$

where $\Omega < \omega_L$, at least in the region $N \simeq \langle N\rangle$ that is relevant for dynamics.

4. The inverse time scale of the decay processes is the smallest one in the system dynamics, satisfying

$$\omega_L, \Omega, \omega_L - \Omega \gg \Gamma_0. \tag{3}$$

This property allows neglecting interference between emission lines centered on the three frequencies at the left-hand side, and *a fortiori* on higher ones, which is equivalent to applying the secular approximation in the standard treatment of the radiative cascade [4].

Give the above conditions, the whole system dynamics, including the radiation field modes, can be expressed by the Hamiltonian $H = H_{\rm AL} + H_{\rm R} + V_{\rm AR}$, where

$$H_{\rm AL} = \sum_{j=1,2}\sum_N E_j(N)|j(N)\rangle\langle j(N)|, \tag{4}$$

$$H_{\rm R} = \hbar \int {\rm d}\omega\, \omega a^\dagger(\omega) a(\omega) \tag{5}$$

respectively determine the dynamics of the atom-laser system and the free radiation field, represented by the operators $a(\omega)$ that satisfy canonical commutation relations $[a(\omega), a^\dagger(\omega')] = \delta(\omega - \omega')$. Note the general form of the atom-light Hamiltonian $H_{\rm AL}$, for which we do not assume specific form of the interaction. Finally,

$$V_{\rm AR} = \hbar \int {\rm d}\omega \sqrt{\frac{\Gamma(\omega)}{2\pi}} \left(\mathcal{S}_+ a(\omega) + \mathcal{S}_- a^\dagger(\omega)\right), \tag{6}$$

with $\mathcal{S}_+ = |e\rangle\langle g|$ and $\mathcal{S}_- = |g\rangle\langle e|$, describes the RWA coupling of the atom with the continuum field modes, with an arbitrary form factor $\Gamma(\omega)$ vanishing for $\omega \leq 0$ that determines the properties of the continuum. The expressions of the transition operators $\mathcal{S}_\pm$ in the dressed eigenstate basis are determined by the coefficients (see the Supplemental Material)

$$A_{ij}^{(q)} = \sum_p \langle i(N+p+q)|e, N\rangle\langle g, N|j(N+p)\rangle, \tag{7}$$

with $\{|e,N\rangle, |g,N\rangle\}$ being the simultaneous eigenstates of the free atom Hamiltonian and of laser photon number, can be assumed N-independent due to the hypothesis (1) on the laser state. The form of the term V_{AR}, where all the irrelevant field degrees of freedom have been integrated out, contains all the relevant information to determine, using Fermi's golden rule, the decay rate $\Gamma_0 = \Gamma(\omega_0)$ of the state $|e\rangle$ when the laser is off, with ω_0 the bare transition frequency between the two atomic states. Neglecting the frequency dependence of the form factor is a common *flat coupling* approximation [4], which we show below may lead to incorrect results beyond the RWA regime.

The frequency scale hierarchy allows the formulation of Markovian dynamics of the atom-laser system by separately considering dissipative processes associated with transitions at different frequencies. We assume that the field continuum starts in its vacuum state. Since no restriction is required on the composition of the dressed eigenstates in terms of the atomic states $|e\rangle$ and $|g\rangle$, the dissipative part of the Gorini-Kossakowski-Lindblad-Sudarshan equation reads

$$\begin{aligned}\mathcal{L}_{\rm diss}(\sigma) = \sum_{q,N,N'} \Big[\sum_{i,j=1,2} \Gamma_{ij}^{(q)} \Big(&-\frac{\delta_{N,N'}}{2}\{\sigma, |i(N)\rangle\langle i(N)|\} \\ &+ \sigma_{i(N+q),i(N'+q)} |j(N)\rangle\langle j(N')| \Big) \\ &+ K_{12}^{(q)} \sigma_{1(N+q),2(N'+q)} |1(N)\rangle\langle 2(N')| \\ &+ (K_{12}^{(q)})^* \sigma_{2(N'+q),1(N+q)} |2(N')\rangle\langle 1(N)| \Big],\end{aligned} \tag{8}$$

where $\sigma_{i(N),j(N')} = \langle i(N)|\sigma|j(N')\rangle$ are the matrix elements of the atom-laser density operator σ in the dressed basis. The decay rates appearing in Eq. (8) are determined by the coefficients (7) as $\Gamma_{jj}^{(q)} = \Gamma(q\omega_L)|A_{jj}^{(q)}|^2$ (for $j = 1, 2$), $\Gamma_{12}^{(q)} = \Gamma(q\omega_L + \Omega)|A_{12}^{(q)}|^2$, and $\Gamma_{21}^{(q)} = \Gamma(q\omega_L - \Omega)|A_{21}^{(q)}|^2$. Due to the properties of the form factor $\Gamma(\omega)$, dissipative $j \to j$ and $2 \to 1$ transitions are possible only if $q > 0$, while the $1 \to 2$ transition can occur also in the case $q = 0$, which is particularly interesting since the frequency of the emitted photons can be much smaller than the pump frequency ω_L. As we will show in the following, such a transition can be determined by the presence of permanent dipole moments. The transitions with the lowest q determined by the dissipative dynamics (8) are graphically represented in Figure 1. The strength of coherence transfer terms between dressed "1" and "2" states in Eq. (8) is instead determined by $K_{12}^{(q)} = \sqrt{\Gamma_{11}^{(q)}\Gamma_{22}^{(q)}}e^{i\varphi(q)}$, where the phase factor depends on the relative phase between $(A_{11}^{(q)})^*$ and $A_{22}^{(q)}$. The detailed derivation of all the term in Eq. (8) is reported in the Supplemental Material.

The approximate invariance of atom-laser couplings with respect to N due to assumption (1), expressed in the N-independence of the amplitudes $A_{ij}^{(q)}$ and *a fortiori* of all the coefficients appearing in Eq. (8), allows reducing the dynamics of the atom-laser system, complicated even in the standard RWA case, to a solvable set of differential equations. Specifically, one can define the reduced coherences $\sigma_{ij}^{(\ell)} = \sum_N \sigma_{i(N),j(N+\ell)}$, which, for the case $i \neq j$, follow decoupled evolution equations:

$$\dot{\sigma}_{12}^{(\ell)} = [i(\ell\omega_L - \Omega) - \Gamma_{coh}]\,\sigma_{12}^{(\ell)}, \tag{9}$$

$$\dot{\sigma}_{21}^{(\ell)} = [i(\ell\omega_L + \Omega) - \Gamma_{coh}]\,\sigma_{12}^{(\ell)}, \tag{10}$$

with

$$\Gamma_{coh} = \frac{1}{2}\sum_{q,i,j}\Gamma_{ij}^{(q)} - \mathrm{Re}\left(\sum_q K_{12}^{(q)}\right) \tag{11}$$

being the ℓ-independent decay rate of such terms. Notice that the sum of the imaginary parts of $K_{12}^{(q)}$ provides a correction of $O(\Gamma_0)$ to the imaginary part of the coefficients, which can be consistently neglected due to the condition (3). Instead, the evolution equations of the equal-index reduced coherences are coupled to each other,

$$\dot{\sigma}_{11}^{(\ell)} = [i\ell\omega_L - \Gamma_{12}]\,\sigma_{11}^{(\ell)} + \Gamma_{21}\sigma_{22}^{(\ell)} \tag{12}$$

$$\dot{\sigma}_{22}^{(\ell)} = [i\ell\omega_L - \Gamma_{21}]\,\sigma_{22}^{(\ell)} + \Gamma_{12}\sigma_{11}^{(\ell)} \tag{13}$$

with

$$\Gamma_{ij} = \sum_q \Gamma_{ij}^{(q)}. \tag{14}$$

As opposed to the case $i \neq j$, all the above equations admit a steady-state solution. In particular, the quantities $\Pi_1 = \sigma_{11}^{(0)}$ and $\Pi_2 = \sigma_{22}^{(0)}$, representing the total population of the "1" and "2" dressed states, respectively, approach the asymptotic values

$$\Pi_1^{st} = \frac{\Gamma_{21}}{\Gamma_{pop}}, \quad \Pi_2^{st} = \frac{\Gamma_{12}}{\Gamma_{pop}}, \quad \Gamma_{pop} = \Gamma_{12} + \Gamma_{21}, \tag{15}$$

determined by the balance between dissipative transitions that change the dressed state label. The quantity Γ_{pop} defined in Eq. (15) is not a mere normalization factor, but represents also the rate at which populations approach the steady state. The same-index coherences follow a similar dynamics, asymptotically evolving as $\sigma_{jj}^{(\ell)}(t) = e^{i\ell\omega_L}\Pi_j^{st}$. It is worth noticing that the reduced evolution equations appear identical to the standard dressed-atom treatment, despite the population transition rates Γ_{ij} and the coherence decay rate Γ_{coh} contain in principle contributions from an infinite number of transitions. The reason of such correspondence essentially lies in hypotheses (1)-(3) and are independent of the model details.

The asymptotic spectral density of emitted photons can be evaluated using the field operator evolution in the Heisenberg picture, the condition (3) of emission line separation, and the values of steady-state populations:

$$\begin{aligned}\mathcal{J}(\omega) = {} & \frac{\Gamma(\omega)}{\pi} \\ & \times \mathrm{Re}\lim_{T\to\infty}\int_0^T \frac{dt}{T}\int_0^\infty d\tau\langle \mathcal{S}_+(t+\tau)\mathcal{S}_-(t)\rangle e^{-i\omega\tau} \\ & = \sum_q\left(\mathcal{J}_+^{(q)}(\omega) + \mathcal{J}_-^{(q)}(\omega) + \mathcal{J}_c^{(q)}(\omega)\right) = \sum_q \mathcal{J}^{(q)}(\omega).\end{aligned} \tag{16}$$

The details of the derivation are reported in the Supplemental Material. In Eq. (16), the average is computed on an initial state in the form $\sigma(0) \otimes |\mathrm{vac}\rangle\langle\mathrm{vac}|$, where $\sigma(0)$ is the initial state of the atom-laser system and $a(\omega)|\mathrm{vac}\rangle = 0$ for all ω's. Time average is made necessary by the presence of persistent contributions to the correlator in the steady-state regime. At the same time, the limit allows to retain only the contributions that scale with T, averaging out the oscillating terms. Notice that the latter issue is not present when the radiative cascade is treated in the RWA case [4].

The evaluation of the spectral density (16), described in detail in the Supplemental Material, makes use of the quantum regression theorem, consistent with the Markovian treatment of the atom-radiation interaction [4]. The contribution of sideband photons, for any value of q, consists in the Lorentzian lines

$$\mathcal{J}_+^{(q)}(\omega) = \frac{\Gamma_{12}^{(q)}\Pi_1^{st}}{\pi}\frac{\Gamma_{coh}}{[\omega - (q\omega_L + \Omega)]^2 + \Gamma_{coh}^2}, \tag{17}$$

$$\mathcal{J}_-^{(q)}(\omega) = \frac{\Gamma_{21}^{(q)}\Pi_2^{st}}{\pi}\frac{\Gamma_{coh}}{[\omega - (q\omega_L - \Omega)]^2 + \Gamma_{coh}^2}, \tag{18}$$

all characterized by the same width Γ_{coh}, which can be integrated to provide the total line intensities

$$I_+^{(q)} = \Gamma_{12}^{(q)}\Pi_1^{st}, \qquad I_-^{(q)} = \Gamma_{21}^{(q)}\Pi_2^{st}. \tag{19}$$

Multiplet label	Number of lines	Central frequency	Line weight scaling
$q = 0$	1	Ω	$\left(\frac{\Omega_{\rm AS}}{\omega_L}\right)^2\left(\frac{\Omega}{\omega_L}\right)^\alpha$
$q = 1$	3	ω_L	1
$q = 2$	3	$2\omega_L$	$\left(\frac{\Omega_{\rm AS}}{\omega_L}\right)^2$
$q = 3$	3	$3\omega_L$	$\left(\frac{\Omega_{\rm R}}{\omega_L}\right)^2\left(\frac{\Omega}{\omega_L}\right)^2$

Table I. Properties of lowest-energy emission multiplets in steady-state radiative cascade. RWA atom-laser interaction induces doublet splitting Ω. Perturbation due to atomic permanent dipole moment and counter-rotating terms gives rise to new lines with magnitudes scaled with $\Omega_{\rm AS}$ and $\Omega_{\rm R}$, respectively.

The central lines have a more complicated structure,

$$\begin{aligned}\mathcal{J}_c^{(q)}(\omega) = &\left[\Gamma_{11}^{(q)}\left(\Pi_1^{\rm st}\right)^2 + \Gamma_{22}^{(q)}\left(\Pi_2^{\rm st}\right)^2 + 2\,{\rm Re}K_{12}^{(q)}\Pi_1^{\rm st}\Pi_2^{\rm st}\right] \\ &\times\delta(\omega - q\omega_L) \\ &+\left(\Gamma_{11}^{(q)} + \Gamma_{22}^{(q)} - 2\,{\rm Re}K_{12}^{(q)}\right)\Pi_1^{\rm st}\Pi_2^{\rm st} \\ &\times\frac{1}{\pi}\frac{\Gamma_{\rm pop}}{(\omega - q\omega_L)^2 + \Gamma_{\rm pop}^2},\end{aligned} \tag{20}$$

with the coherent peaks at $\omega = q\omega_L$ surrounded by Lorenzian lines whose width corresponds, for any q, to $\Gamma_{\rm pop}$. It is worth noticing that such a component of the spectral density depends on the coherence transfer coefficients $K_{12}^{(q)}$ only through their real parts, in accordance with the independently obtained result in Eq. (11). Integration over a central line erases the dependence on ${\rm Re}K_{12}^{(q)}$ (hence, of the relative phase between $A_{11}^{(q)}$ and $A_{22}^{(q)}$), leading to the intuitive form

$$I_c^{(q)} = \Gamma_{11}^{(q)}\Pi_1^{\rm st} + \Gamma_{22}^{(q)}\Pi_2^{\rm st}. \tag{21}$$

These results open the possibility to compare the strength of emission lines for any specific atom-laser Hamiltonian satisfying the phenomenological assumptions. Interestingly, the ratios of transition line weights (as well as the steady state populations) do not depend on the magnitude of Γ_0, which can be considered arbitrarily small to satisfy condition (3). In particular, one can compare the strength of the small-freqency singlet around $\omega = \Omega$ with that of the higher-energy triplets, obtaining

$$\frac{I_+^{(0)}}{I^{(q)}} = \frac{\Gamma_{12}^{(0)}\Gamma_{21}}{\left(\Gamma_{11}^{(q)} + \Gamma_{12}^{(q)}\right)\Gamma_{21} + \left(\Gamma_{22}^{(q)} + \Gamma_{21}^{(q)}\right)\Gamma_{12}}, \tag{22}$$

with $I^{(q)} = I_c^{(q)} + I_+^{(q)} + I_+^{(q)}$ the total triplet strength. The above derivation of the spectral characteristics of the radiative emission from the dressed system is performed with a minimal set of assumptions (1)–(3), and is Hamiltonian-independent: The Hamiltonian enters the description at the stage of evaluation of the model parameters $A_{ij}^{(q)}$ according to Eq. (7).

After deriving the general results, let us focus on the case of an atom-laser Hamiltonian

$$\begin{aligned}H_{\rm AL} = &\,\omega_0|e\rangle\langle e| \otimes \mathbb{1}_L + \mathbb{1}_A \otimes \sum_N N\omega_L|N\rangle\langle N| \\ &+\left[\frac{\Omega_{\rm R}}{2}\left(|e\rangle\langle g| + |g\rangle\langle e|\right) + \Omega_{\rm AS}|e\rangle\langle e|\right]\otimes h_L,\end{aligned} \tag{23}$$

where $h_L = \sum_N \left(|N\rangle\langle N-1| + |N-1\rangle\langle N|\right)$, and $\mathbb{1}_L$ ($\mathbb{1}_A$) is the identity operator on the laser (atom). The Rabi frequency $\Omega_{\rm R}$ and the coefficient $\Omega_{\rm AS}$, describe respectively the laser coupling with the transition dipole moments proportional to $|e\rangle\langle g|$, $|g\rangle\langle e|$ and the permanent dipole moments of the excited state $|e\rangle\langle e|$. They are both proportional to $\sqrt{\langle N\rangle}$. Let us consider the case of $\Omega_{\rm AS}, \Omega_{\rm R} \ll \omega_L$, when permanent-dipole and counter-rotating terms can be viewed as a correction to RWA. Notice that these conditions are unrelated to the request (3), since Ω is mostly determined by the RWA terms. Considering the composition of the dressed states, we obtain the lowest-order relative weight of the low-frequency singlet with respect to the main triplet as

$$\frac{I_+^{(0)}}{I^{(1)}} \simeq \frac{\Gamma(\Omega)}{\Gamma(\omega_L)}\left(\frac{\Omega_{\rm AS}}{\omega_L}\right)^2\frac{\Omega_{\rm R}^2}{4\left[\Omega_{\rm R}^2 + (\omega_0 - \omega_L)^2\right]}. \tag{24}$$

As shown in the Supplemental Material, such an expression remains effective with good approximation in a range that goes beyond the $\Omega_{\rm R}, \Omega_{\rm AS} \ll \omega_L$ regime. The low-frequency transition is naturally suppressed by a factor of $(\Omega_{\rm AS}/\omega_L)^2$, due to the dependence of $\Gamma_{12}^{(0)}$ on the permanent-dipole perturbation. Moreover, suppose one desires emission at a frequency $\Omega \ll \omega_L$. In that case, the transition is also at risk of heavy suppression by the behavior of the interaction form factor, which, in homogeneous media and in the range of validity of the dipole approximation, scales like ω^3. For the same reason, the low-energy singlet competes even with the $q = 2$ triplet, which has the same scaling with $\Omega_{\rm AS}$:

$$\frac{I_+^{(0)}}{I^{(2)}} \simeq \frac{\Gamma(\Omega)}{\Gamma(2\omega_L)}\frac{\Omega_{\rm R}^2}{4\left[\Omega_{\rm R}^2 + (\omega_0 - \omega_L)^2\right]}. \tag{25}$$

Though affected by an even amplified problem with the form factor, comparison with the $q = 3$ triplet is mitigated by the fact that the weight of the latter is proportional to the fourth power of Ω/ω_L (see Supplemental Material), yielding

$$\frac{I_+^{(0)}}{I^{(3)}} \sim \frac{\Gamma(\Omega)}{\Gamma(3\omega_L)}\left(\frac{\Omega_{\rm AS}}{\omega_L}\right)^2\left(\frac{\omega_L}{\Omega}\right)^4 \sim \left(\frac{\Omega_{\rm AS}}{\Omega}\right)^2\left(\frac{\Omega}{\omega_L}\right)^{\alpha-2} \tag{26}$$

in the case of a form factor behaving like $\Gamma(\omega) \sim \omega^\alpha$. The α parameter can be tailored, e.g., by exploiting low-dimensional geometries. The parameters characterizing the lowest-frequency emission lines in the perturbative case are reported in Table I. Enhancing the asymptotic production of low-frequency photons is therefore a difficult task in free space, but can be obtained if the form factor is properly tailored to be peaked at energies around $\hbar\Omega$. One possibility to perform such an enhancement would be to buffer the

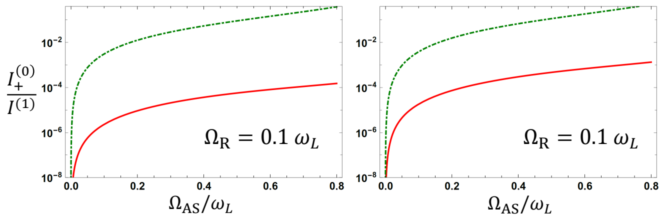

Figure 2. Relative weight of the low-frequency emission ($q = 0$) compared to the Mollow triplet ($q = 1$), computed for the atom-laser Hamiltonian (23) with $\omega_0 = \omega_L$, and for $\Gamma(\omega) \propto \omega^3$. The numerical evaluation based on Eq. (24) (solid red lines) is compared with the result obtained neglecting the dependence of decay rates on the dressed state energies (dot-dashed green lines). The emission intensity for all cases tends to 0 as $\Omega_{AS} \to 0$.

coupling between the atom and radiation propagating in free space with a lossy resonant cavity (see also discussions in Ref. [16]). Inspection of the spectral contributions provides a further quality factor of the emission lines, highlighting their prominence above the background [16]. Focusing again on the low-frequency transition, we can compare the peak value $\mathcal{J}_+^{(0)}(\Omega)$ with the total tail contributions generated by the remaining transition at the same frequency. Assuming that the $q = 1$ triplet is dominant, and neglecting the difference among its frequencies, the quality factor reads

$$\frac{\mathcal{J}_+^{(0)}(\Omega)}{\sum_{q>0}\mathcal{J}^{(q)}(\Omega)} \simeq \left(\frac{\omega_L}{\Gamma_{\rm coh}}\right)^2 \frac{\Gamma_{12}^{(0)}}{\Gamma_{12}^{(1)}} \frac{1}{4 - \Gamma_{\rm pop}/\Gamma_{\rm coh}}, \qquad (27)$$

highlighting the fact that it can be made larger by decreasing the dipole transition coupling constant, which is not fixed by the parameters entering the relative integrated peak intensities. From all the above results, the crucial role of dressed-state decay rates is evident. When transitions occur on different energy scales, neglecting their dependence on the coupling form factor which comes from knowledge of the dressed states can lead to orders-of-magnitude errors in the evaluation of photon production rates, as shown in Figure 2.

The framework introduced in this Letter extend the dressed-atom analysis of a steady-state radiative cascade to the case of an arbitrary coupling between atom and laser. The conditions for the validity of the results here presented are not related to the specific form of the interaction Hamiltonian, but to spectral features and relations between energy scales. Given these properties, we obtain a general form of the spectral density of all the emerging fluorescence multiplets, from which the emission intensity at different frequencies can be determined. At the same time, it is worth noticing that the fully quantum treatment of the atom-laser system provides information that does not emerge if the laser mode is treated as a classical field, such as the dependence of line weights on the on-shell electromagnetic form factors and on the atom-laser eigenstate mixing, which determines the coefficients in Eq. (7) and the role of interference effects in determining the coherence decay rates. Perspectives in this research line can involve the generalization to many-emitter ensembles, thus extending the semiclassical treatment in Refs. [15, 23], and the introduction of different dissipation mechanisms or interaction with more than one laser mode [22, 23, 25]. A qualitatively interesting extension of this work would be to investigate the case of a multi-level atom, where more low-energy lines can emerge, possibly with new coherence effects in the case in which more than two or more transitions inside a multiplet occur at the same frequency.

Acknowledgments.— F.V.P. acknowledges support by the PNRR MUR project PE0000023-NQSTI and by the INFN project QUANTUM. K.S. is grateful for the support by the National Science Centre, Poland (Project No. 2020/39/I/ST3/00526).

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