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Joanna Bruzda
Nicolaus Copernicus University in Toruń

The Haar Wavelet Transfer Function Model and Its Applications[†]

A b s t r a c t. In the paper the Haar wavelet transfer function models are suggested as a way to parsimoniously parametrise the impulse responses and construct models with parameters providing an insight into the frequency content of the relationships under scrutiny. Besides, the models enable to verify hypotheses concerning changes of the regression parameters across dyadic scales (octave frequency bands). In the paper some theoretical properties of the models are investigated and an empirical illustration is provided. In the empirical study returns on WIG are modelled with the help of returns on S&P 500. Interestingly, besides the insight into the frequency content of the relationship, the empirical wavelet transfer function models also provided good forecasts.

K e y w o r d s: wavelet transfer function model, Haar wavelet, maximal overlap discrete wavelet transform.

Introduction

There are two approaches to examine economic relationships with wavelets. In the first case, the processes under scrutiny are decomposed according to dyadic scales and the economic relationship is investigated for the separate octave frequency bands relying on DWT- or MODWT-based[1] wavelet and scaling coefficient or, alternatively, DWT- or MODWT-based details and approximations (smooths). The second approach is more prediction-oriented and consists in replacing some or all of explanatory variables with their wavelet packet coefficients. The method was introduced in Nason and Sapatinas (2002) and applied to such problems as wind speed prediction (Hunt, Nason, 2001; Nason, Sapatinas, 2002), data segmentation (Nason et al., 2001), modelling market shares of

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[1] The abbreviations DWT and MODWT refer to the discrete wavelet transform and the maximal overlap (non-decimated) discrete wavelet transform accordingly. Further we use also the acronym MODWPT, which stands for the maximal overlap discrete wavelet packet transform.

products with their relative prices (Hunt, 2002) and constructing marketing mix models (Michis, 2006). To overcome the problem of multicollinearity of wavelet packet coefficients form different decomposition levels, the coefficient vectors that show the maximum correlation with the dependent variable are usually used (Nason, Sapatinas, 2002) or the packet coefficients are replaced with their principal components (Hunt, Nason, 2001; Hunt, 2002; Michis, 2006).

The approach suggested here resembles that of Nason and Sapatinas (2002), although in the construction of our wavelet transfer function model we put more emphasis on the interpretation of parameters and make use of the notion of the wavelet best basis. In what follows some theoretical properties of the suggested model are investigated and an empirical illustration is provided. In Section 1 we introduce our Haar wavelet transfer function model and examine spectral characteristics of the underlying bivariate process, while in Section 2 the concept is confronted with some empirical data. In the empirical study returns on WIG are modelled with the help of returns on S&P 500 and the wavelet as well as conventional transfer function models are used further for forecasting purposes. The last section offers brief conclusions.

1. The Haar Wavelet Transfer Function Model

Let us consider modelling a response variable Y_t in terms of the present and previous values of an explanatory variable X_t. We assume for the moment that the processes have the mean values of zero. We start with a construction utilising the Haar wavelet and scaling coefficients and comment further on possible generalisations including the wavelet packet transfer function modelling. Using the Haar scaling and wavelet filters we have[2]:

$$X_t = \tilde{V}_{Kt}^X + \tilde{W}_{1t}^X + \tilde{W}_{2t}^X + \ldots + \tilde{W}_{Kt}^X,$$

where $K \le [\log_2 N]$ and the MODWT scaling and wavelet coefficients are obtained via the following recursive formulas:

$$\begin{aligned}
&\tilde{V}_{1t}^X = 0.5(X_t + X_{t-1}); \qquad \tilde{W}_{1t}^X = 0.5(X_t - X_{t-1});\\
&\tilde{V}_{2t}^X = 0.5(\tilde{V}_{1t}^X + \tilde{V}_{1,t-2}^X); \qquad \tilde{W}_{2t}^X = 0.5(\tilde{V}_{1t}^X - \tilde{V}_{1,t-2}^X);\\
&\ldots\\
&\tilde{V}_{Kt}^X = 0.5(\tilde{V}_{K-1,t}^X + \tilde{V}_{K-1,t-2^{K-1}}^X); \qquad \tilde{W}_{Kt}^X = 0.5(\tilde{V}_{K-1,t}^X - \tilde{V}_{K-1,t-2^{K-1}}^X).
\end{aligned}$$

[2] The level j Haar scaling and wavelet filters are obtained via the formulas: $\tilde{g}_{jl} = \frac{1}{2^j} \cdot \phi\left(\frac{l}{2^j}\right)$, $\tilde{h}_{jl} = \frac{1}{2^j} \cdot \psi\left(\frac{l}{2^j}\right)$, $l = 0, \ldots, 2^j - 1$, where $\phi(\cdot)$ and $\psi(\cdot)$ are the Haar scaling and wavelet functions defined as $\phi(x) = \mathbf{1}_{<0,1)}(x)$ and $\psi(x) = \mathbf{1}_{<0,1/2)}(x) - \mathbf{1}_{<1/2,1)}(x)$, respectively.

Our proposal consists in using the following model to describing Y_t in terms of X_t :

$$Y_t = \beta_0 \tilde{V}^X_{K,t-\tau_0} + \beta_1 \tilde{W}^X_{1,t-\tau_1} + \beta_2 \tilde{W}^X_{2,t-\tau_2} + \ldots \beta_K \tilde{W}^X_{K,t-\tau_K} + \eta_t , \qquad (1)$$

where $\tau_0, \tau_1, \ldots, \tau_K$ are nonnegative integers and it is assumed that η_t is strictly exogenous for the regressors in (1). The model enables to possibly parsimoniously parametrise an autoregressive distributed lag (ADL) model, when the regression parameter and (or) the time delay is scale-dependent. Furthermore, the model provides an insight into the frequency character of the relationship between X_t and Y_t, being at the same time a simple forecasting instrument[3].

Let the bivariate process (X_t, Y_t) defined via equation (1) be covariance stationary with an absolute summable autocovariance matrix. Then, its cross-covariance function has the form:

$$\gamma_{XY}(\tau) = EX_t Y_{t+\tau} = \beta_0 E\left\{ 1/2^K X_t \sum_{j=0}^{2^K-1} X_{t-\tau_0+\tau-j} \right\} +$$

$$+\beta_1 E\left\{ 1/2\, X_t \left(X_{t-\tau_1+\tau} - X_{t-\tau_1+\tau-1} \right) \right\} + \beta_2 E\left\{ 1/4 X_t \left(\sum_{j=0}^{1} X_{t-\tau_2+\tau-j} - \sum_{j=2}^{3} X_{t-\tau_2+\tau-j} \right) \right\}$$

$$+\ldots + \beta_K E\left\{ 1/2^K X_t \left(\sum_{j=0}^{2^{K-1}-1} X_{t-\tau_K+\tau-j} - \sum_{j=2^{K-1}}^{2^K-1} X_{t-\tau_K+\tau-j} \right) \right\} =$$

$$= \beta_0 / 2^K \sum_{j=0}^{2^K-1} \gamma_X(\tau - \tau_0 - j) + \beta_1 / 2 \left\{ \gamma_X(\tau - \tau_1) - \gamma_X(\tau - \tau_1 - 1) \right\} +$$

$$+\beta_2 / 4 \left\{ \sum_{j=0}^{1} \gamma_X(\tau - \tau_2 - j) - \sum_{j=2}^{3} \gamma_X(\tau - \tau_2 - j) \right\} + \ldots$$

$$+ \beta_K / 2^K \left\{ \sum_{j=0}^{2^{K-1}-1} \gamma_X(\tau - \tau_K - j) - \sum_{j=2^{K-1}}^{2^K-1} \gamma_X(\tau - \tau_K - j) \right\},$$

while the cross-spectral density function is as follows:

[3] Applications of other types of causal filters to examine economic dependencies across frequency bands can be found in Stawicki (1993) and Ashley and Verbrugge (2008).

$$S_{XY}(f)=S_X(f)\Bigg\{\beta_0/2^K\sum_{j=0}^{2^K-1}e^{-i2\pi f(\tau_0+j)}+\beta_1/2\left[e^{-i2\pi f\tau_1}-e^{-i2\pi f(\tau_1+1)}\right]$$
$$+\beta_2/4\left[\sum_{j=0}^{1}e^{-i2\pi f(\tau_2+j)}-\sum_{j=2}^{3}e^{-i2\pi f(\tau_2+j)}\right]+\ldots \tag{2}$$
$$+\beta_K/2^K\left[\sum_{j=0}^{2^{K-1}-1}e^{-i2\pi f(\tau_K+j)}-\sum_{j=2^{K-1}}^{2^K-1}e^{-i2\pi f(\tau_K+j)}\right]\Bigg\}.$$

Alternatively, (2) can be expressed as:

$$S_{XY}(f)=\left\{\beta_0 e^{-i2\pi f\tau_0}\tilde{G}_K(f)+\beta_1 e^{-i2\pi f\tau_1}\tilde{H}_1(f)+\ldots+\beta_K e^{-i2\pi f\tau_K}\tilde{H}_K(f)\right\}S_X(f),$$

where $\widetilde{G}_j(f)$ and $\widetilde{H}_j(f)$ denote the transfer functions of the level j scaling and wavelet filters.

To see how the frequency characteristics of the bivariate process defined via (1) look like, let us start with the simple case of the first level decomposition:

$$Y_t=\beta_0\widetilde{V}^X_{1,t-\tau_0}+\beta_1\widetilde{W}^X_{1,t-\tau_1}+\eta_t\,. \tag{3}$$

Then, the cross-spectrum reduces to:

$$S_{XY}(f)=0{,}5S_X(f)\left\{\beta_0[e^{-i2\pi f\tau_0}+e^{-i2\pi f(\tau_0+1)}]+\beta_1[e^{-i2\pi f\tau_1}-e^{-i2\pi f(\tau_1+1)}]\right\}$$

and the amplitude spectrum is:

$$A_{XY}(f)=S_X(f)\left\{\frac{\beta_0^2+\beta_1^2}{2}+\frac{\beta_0^2-\beta_1^2}{2}\cos(2\pi f)-\beta_0\beta_1\sin[2\pi f(\tau_0-\tau_1)]\sin(2\pi f)\right\}^{0,5}.$$

In the case $\tau_0=\tau_1$ the gain $G_{XY}(f)=\dfrac{A_{XY}(f)}{S_X(f)}\geq 0$ is a monotonic function with values between $|\beta_0|$ and $|\beta_1|$. It is easy to see that for all τ_0 and τ_1 the gain does not exceed $\left[\max\{|\beta_0|,|\beta_1|\}\cdot(|\beta_0|+|\beta_1|)\right]^{0,5}$ and its values at 0, $1/4$ and $1/2$ equal $|\beta_0|$, $\left\{(\beta_0^2+\beta_1^2)/2-\beta_0\beta_1\sin[\pi/2(\tau_0-\tau_1)\right\}^{0,5}$ and $|\beta_1|$, respectively.

The possibility to parsimoniously parametrise the impulse response function becomes more apparent, when further decomposition levels are considered, though the form of the theoretical amplitude spectrum of (1) is then fairly complicated, even in the 'equal lag' case. However, the values of the gain at 0, $1/4$ and $1/2$ always equal $|\beta_0|$, $\left\{(\beta_1^2+\beta_2^2)/2-\beta_1\beta_2\sin[\pi/2(\tau_2-\tau_1)\right\}^{0,5}$ and $|\beta_1|$, respectively, and to a great extent, the beta coefficients in (6.1) reveal the shape of the gain function, especially in the case of identical lags. There are basically two problems with interpreting the beta coefficients in terms of the gain. First, there is a substantial leakage associated with the Haar wavelet and scaling filters. Furthermore, if the lag parameters differ significantly across scales, the function becomes highly variable.

Figure 1 presents spectral characteristics of example bivariate processes defined via:

$$Y_t = \beta_0 \widetilde{V}^X_{2,t-\tau_0} + \beta_1 \widetilde{W}^X_{1,t-\tau_1} + \beta_2 \widetilde{W}^X_{2,t-\tau_2} + \eta_t . \quad (4)$$

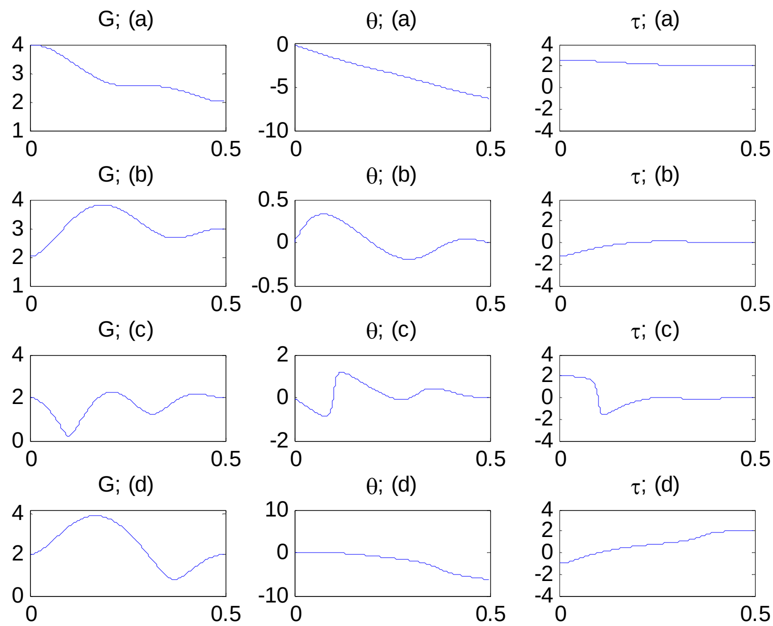

Figure 1. Spectral characteristics of the bivariate process (4). Figure presents gains (left-hand column), phase spectra (middle) and time delays defined as $\tau(f) = -\frac{\theta(f)}{2\pi f}$ (right-hand column), for the following four cases: (a) $\beta_0 = 4,\ \beta_1 = 2,\ \beta_2 = 3,\ \tau_0 = 2,\ \tau_1 = 2,\ \tau_2 = 2$, (b) $\beta_0 = 2,\ \beta_1 = 3,\ \beta_2 = 4,\ \tau_0 = 0,\ \tau_1 = 0,\ \tau_2 = 0$, (c) $\beta_0 = 2,\ \beta_1 = 2,\ \beta_2 = 2,\ \tau_0 = 2,\ \tau_1 = 0,\ \tau_2 = 0$, (d) $\beta_0 = 2,\ \beta_1 = 2,\ \beta_2 = 4,\ \tau_0 = 0,\ \tau_1 = 2,\ \tau_2 = 0$. To draw the phase spectrum we used the `unwrap` Matlab function, which converts increments greater in magnitude than or equal to π to their 2π complements.

One generalisation of the approach presented here utilises the non-decimated Haar wavelet packet transform coefficients. In order to better reflect the frequency character of the relationships under scrutiny we suggest to replace the lower level wavelet coefficients with the appropriate MODWPT coefficients from a chosen decomposition level. The MODWPT-enhanced model should enable to choose the best filters (as for their length and the frequency of oscillations captured) to describe the short term fluctuations. The best basis for the transform and the final variables left in the model can be chosen as to optimise the empirical model in terms of its parsimony (tests for equality and significance of parameters will be helpful), some goodness of fit measures and diagnostic tests.

The Haar wavelet packet filters, which produce the j-th level wavelet packet decomposition, are defined via lag polynomials of order j, whose all complex roots lie on the unit circle. For example, for j = 3 the non-decimated versions of the wavelet packet coefficients $\tilde{W}_{j,n,t}$ are obtained in the following way:

$$\tilde{W}_{3,0,t} = \tfrac{1}{8}(1+L)(1+L^2)(1+L^4)X_t, \tilde{W}_{3,1,t} = \tfrac{1}{8}(1+L)(1+L^2)(1-L^4)X_t,$$
$$\tilde{W}_{3,2,t} = \tfrac{1}{8}(1+L)(1-L^2)(1-L^4)X_t, \tilde{W}_{3,3,t} = \tfrac{1}{8}(1+L)(1-L^2)(1+L^4)X_t,$$
$$\tilde{W}_{3,4,t} = \tfrac{1}{8}(1-L)(1-L^2)(1+L^4)X_t, \tilde{W}_{3,5,t} = \tfrac{1}{8}(1-L)(1-L^2)(1-L^4)X_t,$$
$$\tilde{W}_{3,6,t} = \tfrac{1}{8}(1-L)(1+L^2)(1-L^4)X_t, \tilde{W}_{3,7,t} = \tfrac{1}{8}(1-L)(1+L^2)(1+L^4)X_t.$$

The so-called sequency ordering instead of the natural ordering is applied to the coefficients above, i.e. the index n is associated with the frequency interval $\left(\frac{n}{2^{j+1}}, \frac{n+1}{2^{j+1}}\right]$. In the case of the usual wavelet decomposition at level j = 3 we would have four coefficients of the form: $\tilde{W}_{3,0,t}$, $\tilde{W}_{3,1,t}$, $\tilde{W}_{3,2,t} + \tilde{W}_{3,3,t}$ and $\tilde{W}_{3,4,t} + \tilde{W}_{3,5,t} + \tilde{W}_{3,6,t} + \tilde{W}_{3,7,t}$. As we can see, within the MODWPT framework the hypothesis about scale dependence of the regression coefficient is just one that can be tested. Building the Haar wavelet packet regression model for forecasting purposes we expect that the best basis will be different from that including all the K-level MODWPT coefficients or that quite a big number of them will turn out to be insignificant. However, even if no reduction is possible, we still gain an interesting interpretation of the coefficients.

Turning to the specification step in building the Haar wavelet transfer function models several remarks are at place. First, let us note that the regressors in model (1) are generally not pairwise orthogonal. Though for the Haar wavelet and scaling filters both the additive decomposition and the decomposition of variance hold, i.e. for the wavelet basis, for example, we have:

$$X_t = \tilde{V}_{Kt}^X + \tilde{W}_{1t}^X + \ldots + \tilde{W}_{Kt}^X,$$
$$\mathrm{Var}(X_t) = \mathrm{Var}(\tilde{V}_{Kt}^X) + \mathrm{Var}(\tilde{W}_{1t}^X) + \ldots + \mathrm{Var}(\tilde{W}_{Kt}^X),$$

what implies also that for all decomposition levels j it holds: $\mathrm{Cov}(\tilde{V}_{jt}^X, \tilde{W}_{jt}^X) = 0$, the wavelet coefficients themselves will be generally correlated. For example, it is easy to check that:

$$\mathrm{Cov}(\tilde{W}_{1t},\tilde{W}_{2t}) = 1/8[K_X(1) - K_X(3)] = -\mathrm{Cov}(\tilde{W}_{1t},\tilde{V}_{2t}),$$

where $K_X(\cdot)$ denotes the autocovariance function of X_t. The cross-covariances can be even larger. This makes the identification of the model slightly more complicated.

A good starting point in the procedure of building the Haar wavelet model is as in the case of an ordinary transfer function model (see Box et al., 2008, Chapter XII), i.e. after differencing the series to achieve stationarity they are filtered with a prewhitening ARMA filter for the exogenous process. Then, the cross-correlation function for the filtered series is computed. The shape of this function and results of significance tests of the cross-correlation coefficients will suggest orders of lag polynomials for a transfer function model and inform whether a Haar wavelet model can be successful. The wavelet model offers a specific approach to a (relatively) parsimonious parametrisation of the impulse response function that can be applied instead of or next to the standard autoregressive structures. Furthermore, the identification stage will give also the minimal time delay for the component series and will suggest the number of decomposition levels for the additive decomposition. However, it seems sensible to start with specifying the same time delays for all component series and then consider also other models, especially if the maximal values of cross-correlations for component processes or an estimate of the phase spectrum point to the need to diversify these parameters. Several tentative models can then be considered in further steps of the Haar model building, which are exactly the same as in the case of standard transfer function models. In particular, the diagnostic checking stage includes also the inspection of the autocorrelation function of the residuals and the cross-correlation function involving the residuals and the input variable or its prewhitened version (see for details Box et al., 2008, pp. 498–501).

2. An Empirical Example

As an empirical illustration daily logarithmic returns on WIG were modelled with the help of the returns on S&P 500. In this case the level j scaling and wavelet coefficient based on the Haar wavelet are associated with $2\lambda_j$-day returns and daily increments of λ_j-day returns, accordingly. The estimation period was 2008.04.01–2010.04.16 and included 534 daily quotations. Both the Johansen and Engle-Granger approaches to cointegration pointed to the lack of long-term relationships between logarithms of prices, so we turned to examining the daily logarithmic returns. As the returns on S&P 500, when accounting for the GARCH effect, did not show any signs of autocorrelation, before examining cross-correlation patterns the two series were only corrected for volatility clustering. GARCH models with Student's t conditional distribution were estimated and the standardised residuals were used in the first step of the procedure of building the Haar wavelet transfer function models.

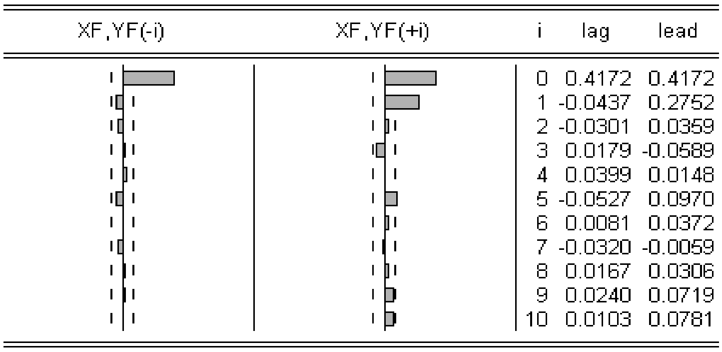

Figure 2. Estimated cross-correlation function for GARCH-filtered returns on WIG and S&P with approximate two standard error bounds computed as $\pm 2/\sqrt{N}$ (XF and YF denote GARCH-filtered S&P and WIG, accordingly)

Figure 2 presents the estimated cross-correlation function for the GARCH-adjusted returns, which shows a unidirectional character of the causal relationship and provides also a slight evidence for the presence of a longer lag distribution. Initially, we considered up to seven decomposition levels and then used Wald tests to examine equality of parameters in the wavelet models. In each case strict exogeneity of regressors was carefully investigated in order to enable a frequency characterisation of the relationship under scrutiny. However, for some of the most parsimonious representations of the data the *p*-values are sometimes still only slightly above 5%. Several transfer function models were finally chosen. In each case the noise process was parametrised as a moving average with the least possible number of parameters. According to diagnostic checks the conditional normal distribution of innovations was eventually assumed. Also some autoregressive specifications were examined, but the autoregressive terms turned out insignificant or produced worse models and forecasts.

Estimation outputs for the most interesting models are presented in Tables 1–2. The tables include also the summary of goodness-of-fit evaluation and some of the diagnostic checks. The frequency characteristics themselves are presented in Figure 3. For the high frequency components of the processes the cross-spectral measures for GARCH-filtered series gave somewhat better correspondence with the estimates in Tables 1–2 than that for the original series. Nevertheless, we decided to present estimates of the spectral characteristics for the original data as we noted that they correspond somewhat closer to the estimates of the long-term parameters in Tables 1–2. Finally, Table 3 includes a comparison of forecast accuracy of our models. Predictions were made for the next five days using forecasts of the out-of-sample values of the regressors[4].

Table 1. Estimation results of transfer function models for logarithmic returns on WIG

Variable	Coefficient	Standard error	z-statistic	p-value
Model I				
Equation for the conditional mean				
S&P	0.312	0.019	16.49	0.0000
S&P(-1)	0.235	0.020	11.65	0.0000
S&P(-2)	0.073	0.020	3.640	0.0003
MA(2)	-0.091	0.049	-1.846	0.0649
MA(6)	-0.124	0.044	-2.807	0.0050
Equation for the conditional variance				
C	1.49E-06	1.25E-06	1.195	0.2320
RESID(-1)^2	0.068	0.019	3.671	0.0002
GARCH(-1)	0.926	0.019	48.08	0.0000
adj. R^2 = 30.74%; AIC = -5.7620; SC = -5.6976; Q = 2.79 (0.43); ARCH = 1.71 (0.42); JB = 2.78 (0.25)				
Model II				
Equation for the conditional mean				
S&P	0.309	0.020	15.45	0.0000
S&P(-1)	0.229	0.020	11.52	0.0000
S&P(-2)	0.070	0.021	3.290	0.0010
S&P(-5)	0.043	0.020	2.128	0.0333
S&P(-9)	0.061	0.021	2.951	0.0032
MA(2)	-0.088	0.050	-1.768	0.0770
MA(6)	-0.137	0.045	-3.066	0.0022
Equation for the conditional variance				
C	1.33E-06	1.28E-06	1.041	0.2979
RESID(-1)^2	0.069	0.021	3.272	0.0011
GARCH(-1)	0.926	0.022	42.02	0.0000
adj. R^2 = 32.25%; AIC = -5.7659; SC = -5.6846; Q = 3.74 (0.29); ARCH = 2.45 (0.29); JB = 1.99 (0.37)				
Model III				
Equation for the conditional mean				
W1	0.077	0.026	2.914	0.0036
W2+W3+W4	0.488	0.034	14.36	0.0000
V4	0.890	0.074	11.97	0.0000
MA(2)	-0.130	0.049	-2.623	0.0087
MA(6)	-0.154	0.045	-3.459	0.0005
Equation for the conditional variance				
C	1.27E-06	1.23E-06	1.029	0.3036
RESID(-1)^2	0.070	0.019	3.593	0.0003
GARCH(-1)	0.926	0.020	45.50	0.0000
adj. R^2 = 31.35%; AIC = -5.7622; SC = -5.6966; Q = 5.18 (0.16); ARCH = 2.75 (0.25); JB = 2.42 (0.30)				

[4] A more precise evaluation of the forecast ability of our Haar wavelet transfer function models for WIG (as well as some other variables) can be found in Bruzda (2011).

Table 1. Continued

Variable	Coefficient	Standard error	z-statistic	p-value
Model IV				
Equation for the conditional mean				
W1	0.084	0.027	3.140	0.0017
W2+W3	0.471	0.041	11.36	0.0000
V3	0.724	0.058	12.43	0.0000
MA(2)	-0.111	0.049	-2.252	0.0243
MA(6)	-0.142	0.044	-3.220	0.0013
Equation for the conditional variance				
C	1.60E-06	1.35E-06	1.182	0.2372
RESID(-1)^2	0.071	0.020	3.596	0.0003
GARCH(-1)	0.922	0.021	43.16	0.0000
adj. R^2 = 30.49%; AIC = -5.7527; SC = -5.6878; Q = 3.56 (0.31); ARCH = 2.18 (0.34); JB = 3.14 (0.21)				
Model V				
Equation for the conditional mean				
V3	0.720	0.058	12.51	0.0000
W2+W3	0.480	0.040	11.86	0.0000
P6+P7	0.161	0.039	4.176	0.0000
MA(2)	-0.135	0.044	-3.084	0.0020
MA(6)	-0.104	0.049	-2.127	0.0334
Equation for the conditional variance				
C	1.53E-06	1.34E-06	1.143	0.2532
RESID(-1)^2	0.072	0.019	3.711	0.0002
GARCH(-1)	0.922	0.021	44.10	0.0000
adj. R^2 = 31.22%; AIC = -5.7648; SC = -5.6999; Q = 3.50 (0.32); ARCH = 1.89 (0.39); JB = 1.94 (0.38)				

Note: Q – Ljung-Box statistic for standardised residuals and 5 lags; ARCH – ARCH LM test statistic for 2 lags; JB – Jarque-Bera normality test; p-values in brackets; two best values of the adjusted R^2 coefficient and the information criteria are in bold; V – scaling coefficients, W – wavelet coefficients, P – wavelet packet coefficients in sequency ordering

Neither the ordinary transfer function models nor the Haar wavelet models uniformly dominated in the model building part of our analysis. However, the wavelet models produced the best forecasts of WIG and have comparable properties to the former models in terms of the fit and diagnostic checking. The best wavelet forecasts were obtained with the simplest wavelet models, while the wavelet packet-enhanced specifications resulted in a lower AIC criterion, while still providing good forecasts in terms of the RMSE.

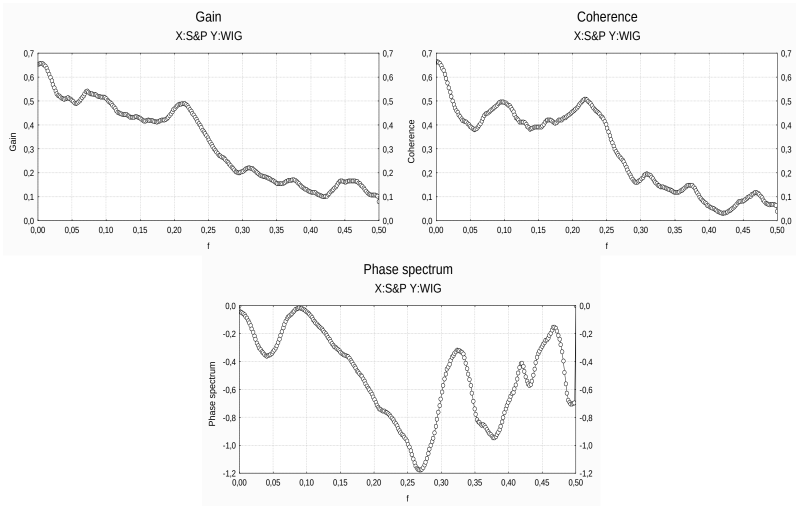

Figure 3. Gain, coherence and phase spectrum for WIG and S&P; estimates obtained via smoothing the cross-periodogram

Table 2. Estimation results of the unreduced wavelet transfer function model for logarithmic returns on WIG

Variable	Coefficient	Standard error	z-statistic	p-value
Equation for the conditional mean				
W1	0.101	0.028	3.570	0.0004
W2	0.496	0.054	9.235	0.0000
W3	0.505	0.082	6.194	0.0000
W4	0.459	0.101	4.540	0.0000
W5	0.981	0.121	8.122	0.0000
W6	0.862	0.171	5.034	0.0000
W7	0.993	0.256	3.888	0.0001
V7	0.850	0.174	4.886	0.0000
MA(2)	-0.188	0.058	-3.219	0.0013
MA(6)	-0.128	0.052	-2.464	0.0137
Equation for the conditional variance				
C	9.11E-07	1.34E-06	0.679	0.4973
RESID(-1)^2	0.080	0.024	3.320	0.0009
GARCH(-1)	0.917	0.024	38.73	0.0000
adj. R^2 = 29.81%; AIC = -5.6382; SC = -5.5100; Q = 3.87 (0.42); ARCH = 2.25 (0.33); JB = 4.05 (0.13)				

Note: See note below Table 1.

Table 3. Evaluation of forecast accuracy

	I	II	III	IV	V
Models for WIG					
RMSE	14.512	15.002	14.543	14.397	14.451
MAE	13.546	14.133	13.449	13.363	13.588

Note: RMSE – root mean squared error; MAE – mean absolute error; the mean forecast errors are multiplied by 10000; Two best results according to each criterion are in bold.

Conclusions

One of the most interesting feature of the Haar wavelet transfer function models suggested here is the interpretation of their parameters. Estimates of parameters in the Haar regressions correspond to the absolute values of the gain function. Although they do not provide precise values of the gain, they are able to recapture the shape of this function and to characterise the frequency content of a bivariate relationship. This interesting information is provided at a relatively low computational cost, as the computational complexity of the MODWT is the same as that of the well known fast Fourier transform. Besides, the models can easily be used to verify hypothesis about changes of regression coefficients across scales. It is also worth emphasizing that forecasting with the Haar wavelet transfer function models is no more complicated than in the case of standard transfer function models. Furthermore, they do not require long time series and can be easily generalised to include deterministic components and multiple exogenous variables.

The empirical analysis presented in the paper confirms that the Haar wavelet transfer function model can be quite successful in describing economic relationships and in forecasting economic variables. The approach provides an interesting insight into the frequency character of the relationships under scrutiny, being at the same time simple and parsimonious in parameters.

The causal filters applied here can also serve the purpose of band-pass filtering exogenous variables, when the causal relationship takes place in a constrained frequency range. An example of such an empirical model with an application to forecasting can be found in Bruzda (2011).

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Falkowy model funkcji transferowej oparty na falce Haara i jego zastosowania

Z a r y s t r e ś c i. W artykule proponuje się falkowy model funkcji transferowej oparty na falce Haara jako metodę konstrukcji modeli funkcji transferowej pozwalającą na oszczędną parametryzację odpowiedzi impulsowych oraz dostarczającą parametrów, które mają ciekawą interpretację częstotliwościową, dając wgląd w kształt funkcji przyrostu i spektrum fazowego procesu dwuwymiarowego. Ponadto pozwalają one na weryfikację hipotez dotyczących zmian współczynnika regresji w zależności od diadycznej skali czasu. W artykule analizuje się teoretyczne własności takich modeli i ilustruje w przykładzie empirycznym dotyczącym modelowania stóp zwrotu z indeksu WIG w zależności od stóp zwrotu z S&P 500. Interesujące jest, iż poza ciekawymi interpretacjami parametrów oszacowane falkowe modele funkcji transferowej dostarczyły także dobrych prognoz.

S ł o w a k l u c z o w e: falkowy model funkcji transferowej, falka Haara, niezdziesiątkowana dyskretna transformata falkowa.