DYNAMIC ECONOMETRIC MODELS
Vol. 12 (2012) 105–110

Submitted October 25, 2012 ISSN
Accepted December 20, 2012 1234-3862

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The Formula of Unconditional Kurtosis of Sign-Switching GARCH(p,q,1) Processes

A b s t r a c t. In the paper we argue that a general formula for the unconditional kurtosis of sign-switching GARCH(p,q,k) processes proposed by Thavaneswaran and Appadoo (2006) does not give correct results. To show that we revised the original theorem given by Thavaneswaran and Appadoo (2006) for the special case of the GARCH(p,q,k) process, i.e. GARCH(p,q,1). We show that the formula for the unconditional kurtosis basing on the original theorem and the revised version is different.

K e y w o r d s: Kurtosis, sign-switching GARCH models.

J E L Classification: C22.

Introduction

In the article „Properties of a New Family of Volatility Sing Models" Thavaneswaran and Appadoo (2006) proposed a general formula for the unconditional kurtosis of the sign-switching GARCH(p,q,k) process (Fornari, Mele, 1997). Unfortunately, the proposed general formula of kurtosis does not give correct results. The formula for the unconditional kurtosis of the process derived from the Theorem 2.1 a) in Thavaneswaran and Appadoo (2006) is not the same as the formula obtained without using this theorem (see equation 9 in Fornari and Mele (1997) or equation 27 in Górka (2008)).

1. Introductory Remarks

The general sign-switching GARCH(p,q,k) model is described by equations (Fornari, Mele, 1997):

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$$y_t = \sigma_t \varepsilon_t, \tag{1}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \sum_{l=1}^{k} \Phi_l s_{t-l}, \tag{2}$$

where $\varepsilon_t \sim i.i.d.(0,1)$, $\omega > 0, \alpha_i \geq 0, \beta_j \geq 0$, $\sum |\Phi_l| \leq \omega$,

$$s_t = \begin{cases} 1 & \text{for} \quad y_t > 0 \\ 0 & \text{for} \quad y_t = 0 \\ -1 & \text{for} \quad y_t < 0 \end{cases}.$$

If $u_t = y_t^2 - \sigma_t^2$ is the martingale difference with variance $\mathrm{var}(u_t) = \sigma_u^2$, the model (1)–(2) can be interpreted as ARMA(m,q) with the sign function for the y_t^2 and can be written as:

$$y_t^2 = \omega + \sum_{i=1}^{m} (\alpha_i + \beta_i) y_{t-i}^2 - \sum_{j=1}^{p} \beta_j u_{t-j} + \sum_{l=1}^{k} \Phi_l s_{t-l} + u_t, \tag{3}$$

or

$$\phi(B) y_t^2 = \omega + \beta(B) u_t + \sum_{l=1}^{k} \Phi_l s_{t-l}, \tag{4}$$

where $\phi(B) = 1 - \sum_{i=1}^{m} (\alpha_i + \beta_i) B^i = 1 - \sum_{i=1}^{m} \phi_i B^i$, $\beta(B) = 1 - \sum_{j=1}^{p} \beta_j B^j$, $m = \max\{p, q\}$, $\alpha_i = 0$ for $i > q$ and $\beta_i = 0$ for $j > p$.

The stationarity assumptions for y_t^2 specified by (4) are the following (Thavaneswaran, Appadoo, 2006):

(Z.1) All roots of the polynomial $\phi(B) = 0$ lie outside the unit circle.

(Z.2) $\sum_{i=0}^{\infty} \psi_i^2 < \infty$, where the ψ_i are coefficients of the polynomial $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$ satisfying the equation $\psi(B)\phi(B) = \beta(B)$.

Assumptions (Z.1)–(Z.2) ensure that the variance of u_t is finite and that the y_t^2 process is weakly stationary.

Assume that $k = 1$. Then the equation (4) has the form:

$$\phi(B) y_t^2 = \omega + \beta(B) u_t + \Phi_1 s_{t-1}. \tag{5}$$

If the assumptions (Z.1)–(Z.2) are satisfied, then the above equation can be converted to the form:

$$y_t^2 = \pi(B)\omega + \psi(B) u_t + \pi(B) \Phi_1 s_{t-1}, \tag{6}$$

where $\psi(B)=1+\sum_{i=1}^{\infty}\psi_i B^i$ satisfies the equation $\psi(B)\phi(B)=\beta(B)$, and $\pi(B)=1+\sum_{i=1}^{\infty}\pi_i B^i$ satisfies the equation $\pi(B)\phi(B)=1$.

2. Author's Results

The theorem presented below is the revised version of the part a) of the Theorem 2.1 presented in Thavaneswaran and Appadoo (2006) but for the special case of the GARCH(p,q,k) process, i.e GARCH(p,q,1).

Theorem. Suppose the y_t is a sign-switching GARCH(p,q,1) process specified by (1)–(2) and satisfying the assumptions (Z.1)–(Z.2), with a finite fourth moment and a symmetric distribution of ε_t. Then the unconditional kurtosis of the process y_t is given by:

$$K=\frac{\left[E\left(\sigma_t^2\right)\right]^2+\Phi_1^2\sum_{i=0}^{\infty}\pi_i^2}{\left[E\left(\sigma_t^2\right)\right]^2}\cdot\frac{E\left(\varepsilon_t^4\right)}{E\left(\varepsilon_t^4\right)-\left[E\left(\varepsilon_t^4\right)-1\right]\sum_{i=0}^{\infty}\psi_i^2}. \tag{7}$$

Proof. A kurtosis of the process y_t described by equations (1)–(2) can be written as:

$$K=\frac{E\left(y_t^4\right)}{\left[E\left(y_t^2\right)\right]^2}=\frac{E\left(\varepsilon_t^4\sigma_t^4\right)}{\left[E\left(\varepsilon_t^2\sigma_t^2\right)\right]^2}=E\left(\varepsilon_t^4\right)\frac{E\left(\sigma_t^4\right)}{\left[E\left(\sigma_t^2\right)\right]^2}. \tag{8}$$

We note that by definition of the u_t ($u_t=y_t^2-\sigma_t^2$) it follows that:

$$\begin{aligned}
&E\left(u_t\right)=0,\\
&\operatorname{var}\left(u_t\right)=\sigma_u^2=E\left(u_t^2\right)-\left[E\left(u_t\right)\right]^2\\
&\qquad=E\left(y_t^4\right)-E\left(\sigma_t^4\right)=E\left(\varepsilon_t^4\sigma_t^4\right)-E\left(\sigma_t^4\right)\\
&\qquad=E\left(\sigma_t^4\right)E\left(\varepsilon_t^4\right)-E\left(\sigma_t^4\right)\\
&\qquad=E\left(\sigma_t^4\right)\left[E\left(\varepsilon_t^4\right)-1\right].
\end{aligned}$$

Let us indicate that the variance of the process y_t^2, satisfying the assumptions of the Theorem and described by the equation (6), is given by:

$$\begin{aligned}\operatorname{var}\left(y_t^2\right) &= \sigma_u^2 \sum_{i=0}^{\infty} \psi_i^2 + \Phi_1^2 \sum_{i=0}^{\infty} \pi_i^2 \\ &= E\left(\sigma_t^4\right)\left[E\left(\varepsilon_t^4\right)-1\right]\sum_{i=0}^{\infty} \psi_i^2 + \Phi_1^2 \sum_{i=0}^{\infty} \pi_i^2 .\end{aligned} \tag{9}$$

On the other hand, this variance can be calculated from the equation (1). We get then

$$\begin{aligned}\operatorname{var}\left(y_t^2\right) &= E\left(y_t^4\right) - \left[E\left(y_t^2\right)\right]^2 \\ &= E\left(\varepsilon_t^4 \sigma_t^4\right) - \left[E\left(\varepsilon_t^2 \sigma_t^2\right)\right]^2 \\ &= E\left(\sigma_t^4\right) E\left(\varepsilon_t^4\right) - \left[E\left(\sigma_t^2\right)\right]^2 .\end{aligned} \tag{10}$$

Comparing the results of (9) and (10) we receive:

$$E\left(\sigma_t^4\right)\left[E\left(\varepsilon_t^4\right)-1\right]\sum_{i=0}^{\infty} \psi_i^2 + \Phi_1^2 \sum_{i=0}^{\infty} \pi_i^2 = E\left(\sigma_t^4\right) E\left(\varepsilon_t^4\right) - \left[E\left(\sigma_t^2\right)\right]^2 .$$

Hence,

$$E\left(\sigma_t^4\right)\left\{E\left(\varepsilon_t^4\right) - \left[E\left(\varepsilon_t^4\right)-1\right]\sum_{i=0}^{\infty} \psi_i^2\right\} = \Phi_1^2 \sum_{i=0}^{\infty} \pi_i^2 + \left[E\left(\sigma_t^2\right)\right]^2,$$

$$\frac{E\left(\sigma_t^4\right)}{\left[E\left(\sigma_t^2\right)\right]^2}\left\{E\left(\varepsilon_t^4\right) - \left[E\left(\varepsilon_t^4\right)-1\right]\sum_{i=0}^{\infty} \psi_i^2\right\} = \frac{\Phi_1^2 \sum_{i=0}^{\infty} \pi_i^2 + \left[E\left(\sigma_t^2\right)\right]^2}{\left[E\left(\sigma_t^2\right)\right]^2},$$

$$\frac{E\left(\sigma_t^4\right)}{\left[E\left(\sigma_t^2\right)\right]^2} = \frac{\left[E\left(\sigma_t^2\right)\right]^2 + \Phi_1^2 \sum_{i=0}^{\infty} \pi_i^2}{\left[E\left(\sigma_t^2\right)\right]^2} \cdot \frac{1}{E\left(\varepsilon_t^4\right) - \left[E\left(\varepsilon_t^4\right)-1\right]\sum_{i=0}^{\infty} \psi_i^2} . \tag{11}$$

Substituting (11) to (8) we obtain:

$$K = \frac{\left[E\left(\sigma_t^2\right)\right]^2 + \Phi_1^2 \sum_{i=0}^{\infty} \pi_i^2}{\left[E\left(\sigma_t^2\right)\right]^2} \cdot \frac{E\left(\varepsilon_t^4\right)}{E\left(\varepsilon_t^4\right) - \left[E\left(\varepsilon_t^4\right)-1\right]\sum_{i=0}^{\infty} \psi_i^2} .$$

□

If $\Phi_1 = 0$, then the formula (7) of the unconditional kurtosis process is reduced to the formula of the unconditional kurtosis processes generated by appropriate GARCH models (see the Theorem 2.1 in Thavaneswaran et al., (2005)).

Example. The example concerns the sign-switching GARCH(1,1,1) model with normal distribution, i.e.

$$\begin{aligned} y_t &= \sigma_t \varepsilon_t, \\ \sigma_t^2 &= \omega + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \Phi_1 s_{t-1}. \end{aligned} \tag{12}$$

If $u_t = y_t^2 - \sigma_t^2$ is the martingale difference with variance $\mathrm{var}(u_t) = \sigma_u^2$, the model (12) is following

$$y_t^2 = \omega + (\alpha_1 + \beta_1) y_{t-1}^2 + u_t - \beta_1 u_{t-1} + \Phi_1 s_{t-1}. \tag{13}$$

Then the polynomials (see the equation (5)) have the form: $\phi(B) = 1 - (\alpha_1 + \beta_1)B$, $\beta(B) = 1 - \beta_1 B$. The individual weights ψ are following: $\psi_1 = \alpha_1$, $\psi_2 = \alpha_1(\alpha_1 + \beta_1)$, …, $\psi_i = \alpha_1(\alpha_1 + \beta_1)^{i-1}$, …. The weights π are: $\pi_1 = \alpha_1 + \beta_1$, $\pi_2 = (\alpha_1 + \beta_1)^2$, …, $\pi_i = (\alpha_1 + \beta_1)^i$, …. If condition (Z.2) is satisfied, then $(\alpha_1 + \beta_1)^2 < 1$ and then:

$$\sum_{i=0}^{\infty} \psi_i^2 = 1 + \alpha_1^2 + \alpha_1^2(\alpha_1 + \beta_1)^2 + \alpha_1^2(\alpha_1 + \beta_1)^4 + \ldots = 1 + \tfrac{\alpha_1^2}{1-(\alpha_1+\beta_1)^2},$$

$$\sum_{i=0}^{\infty} \pi_i^2 = 1 + (\alpha_1 + \beta_1)^2 + (\alpha_1 + \beta_1)^4 + (\alpha_1 + \beta_1)^6 + \ldots = \tfrac{1}{1-(\alpha_1+\beta_1)^2}.$$

Assuming that $\varepsilon_t \sim N(0,1)$ and substituting into (7) we obtain:

$$\begin{aligned} K &= \frac{\left(\frac{\omega}{1-\alpha_1-\beta_1}\right)^2 + \frac{\Phi_1^2}{1-(\alpha_1+\beta_1)^2}}{\left(\frac{\omega}{1-\alpha_1-\beta_1}\right)^2} \cdot \frac{3}{3 - 2\left(1 + \frac{\alpha_1^2}{1-(\alpha_1+\beta_1)^2}\right)} \\ &= \frac{\omega^2\left[1-(\alpha_1+\beta_1)^2\right] + \Phi_1^2\left(1-\alpha_1-\beta_1\right)^2}{\omega^2} \cdot \frac{3}{1-(\alpha_1+\beta_1)^2 - 2\alpha_1^2} \\ &= \frac{3\left\{\omega^2\left[1-(\alpha_1+\beta_1)^2\right] + \Phi_1^2\left(1-\alpha_1-\beta_1\right)^2\right\}}{\omega^2\left[1 - 2\alpha_1\beta_1 - \beta_1^2 - 3\alpha_1^2\right]}. \end{aligned} \tag{14}$$

This result is the same like the formula of the unconditional kurtosis obtained by Fornari and Mele (1997) and by Górka (2008) but it is different from the result obtained by Thavaneswaran and Appadoo (2006). Nonetheless, in each case, if $\Phi_1 = 0$ then the formula (14) reduces to a formula for the unconditional kurtosis of the GARCH (1,1) process.

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Wzór na bezwarunkową kurtozę procesu generowanego przez model sign-switching GARCH(p,q,1)

Z a r y s t r e ś c i. W artykule zauważono, że na podstawie wzoru na bezwarunkową kurtozę procesu GARCH(p,q,k) zaproponowanego przez Thavaneswarana i Appadoo (2006) nie otrzymujemy poprawnych wyników. Dlatego też w niniejszej pracy przedstawiono poprawioną formułę twierdzenia Thavaneswarana i Appadoo (2006) dla szczególnego przypadku procesu GARCH(p,q,k), tzn. GARCH(p,q,1). Wykazano, że formuła na bezwarunkową kurtozę procesu generowanego przez model sign-switching GARCH(1,1,1) bazująca na oryginalnym twierdzeniu i poprawionej wersji jest inna.

S ł o w a k l u c z o w e: Kurtoza, model sign-switching GARCH.