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THE NOTION OF THE DIAMETER OF MEREOLOGICAL BALL IN TARSKI'S GEOMETRY OF SOLIDS

Abstract. In [3] Gruszczyński and Pietruszczak have obtained the full development of Tarski's geometry of solids that was sketched in [15, 16]. In this paper[1] we introduce in Tarski's theory the notion of *congruence of mereological balls* and then the notion of *diameter of mereological ball*. We prove many facts about these new concepts, e.g., we give a characterization of mereological balls in terms of its *center* and *diameter* and we prove that the set of all diameters together with the relation of *inequality of diameters* is the dense linearly ordered set without the least and the greatest element.

Keywords: Tarski's geometry of solids; mereology; diameter of mereological ball; congruence of mereological balls; point-free geometry

Introduction

Alfred Tarski in his paper [15, 16] proposed a method of axiomatization of geometry without using the notion of point as primitive. As it was shown in [3], with the adequate systems of Tarski's postulates, three different classes of relational structures can be distinguished. They are called, according to [3], T*-structures, T$'$-structures and T-structures. These structures are related to Euclidean geometry on different levels.

Considerations that are presented in this paper are based on the theory of T*-structures, which we call *Tarski's geometry of solids*. The aim of Tarski's theory of solids is fulfilled to the greatest extent by ge-

[1] This is an English version of the first part of my PhD thesis [14], whose supervisor was prof. Andrzej Pietruszczak.

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ometrical notions defined in these structures. For a detailed analysis of different structures related to Tarski's paper, the reader should see [3, 4]. However, for our purposes, we give in Section 1 a sketch of the theory of T*-structures and we recall its main characteristics. The main result concerning Tarski's theory is that the notions of the mereological solid, of the mereological ball and of the part-whole relation are isomorphic, respectively, to the notions of the regular open set of the open ball and of the relation of inclusion, which are defined in the point-based Euclidean geometry. In this paper we expand this particular result by adding the definition of the notion of *congruence of mereological balls*.

Before we proceed to the proper constructions, we present in Section 2 two important theorems which we call *constructional theorems*. These theorems characterise relations **IT** and **ET** in terms of operations $\mathbb{int}$ and $\mathbb{fr}$. With the use of these theorems we move consideration from mereological balls to Euclidean geometry. Since all constructions in this section (and generally, in this entire paper) are done in Hilbert's geometry, we describe in Appendix "Basic facts from geometry" the most important axioms and facts of geometry. We hope that this will make it easy for the reader to follow geometrical constructions that are presented during all of proofs.

The most important and new definitions are introduced in Section 3. As we show there, beside the notion of point, the relation of *concentricity of mereological balls* defined by Tarski allows us also to define the binary relation of *congruence of mereological balls*. According to this definition, two balls are congruent iff both of them are between a pair of concentric balls, being tangentially embraced by them. We show that such a relation is an equivalence relation in the set of all mereological balls. In consequence, for any mereological ball, the equivalence classes of such a relation we call *a diameter* of this ball. About the diameter of ball defined in such a way we prove numerous facts, most important of which are expressed in Theorem 4.1. There is also a counterpart of one of the Euclidean axioms which states that in any point there is a mereological ball of a given diameter. This fact entails, in addition, that any point allows us to generate a whole class of diameters, whereas any diameter allows us to generate a whole set of points. At the end of the Section 3, we give a characterization of mereological balls by the notion of a diameter of a ball and the notion of a point.

In Section 4 we consider the set of equivalence classes of the relation of congruence of mereological balls itself. For this purpose, we introduce two binary relations that allow us to compare diameters. These relations are defined in terms of the relations of *being a part* and *being an ingrediens* and they are called *relations of inequalities of diameters*. These relations have many properties following from the relation of inequality and sharp inequality of segments, determined in the set of points. Thanks to these properties, we can formulate theorem which characterizes the set of all diameters. It states that this set, together with the relation of inequality of diameters, is the dense linearly ordered set without greatest and smallest element.

1. Tarski's geometry of solids. T*-structures

The universe of the discourse of Tarski's geometry of solids is made of *space* and its «pieces» which we call *mereological solids*. Among solids we distinguish specific types which we call *mereological balls* (simply *solids* and *balls* in the case it follows from the context that we refer to elements of $\mathbb{S}$ and $\mathbb{B}$, respectively). The relations between solids can be described in terms of a binary relation which we will call, after Leśniewski [8], *ingrediens relation*[2]. The notion of *solid*, *ball* and *being an ingrediens* are the only primitive notions in geometry of solids. Let $\mathbb{S}$ be the set of all solids, $\mathbb{B}$ be the set of all balls, and $\sqsubseteq$ be the relation of *being an ingrediens*. We accepted that the relation $\sqsubseteq$ partially orders the set $\mathbb{S}$, i.e., it is included in $\mathbb{S}\times\mathbb{S}$ and it is reflexive, antisymmetrical, and transitive:

$$\forall_{x\in\mathbb{S}}\ x\sqsubseteq x\,, \tag{$\mathrm{r}_\sqsubseteq$}$$

$$\forall_{x,y\in\mathbb{S}}(x\sqsubseteq y\wedge y\sqsubseteq x\Longrightarrow x=y), \tag{$\mathrm{antis}_\sqsubseteq$}$$

$$\forall_{x,y,z\in\mathbb{S}}(x\sqsubseteq y\wedge y\sqsubseteq z\Longrightarrow x\sqsubseteq z). \tag{$\mathrm{t}_\sqsubseteq$}$$

By the primitive relation $\sqsubseteq$ we introduce in the set $\mathbb{S}$ three auxiliary binary relations: $\sqsubset$, $\bigcirc$, and $\wr$, which we call, respectively, *being a (proper) part*, *overlapping*, and *disjointness*:

$$x\sqsubset y\overset{\mathrm{df}}{\iff} x\sqsubseteq y\wedge x\neq y\,, \tag{df $\sqsubset$}$$

$$x\bigcirc y\overset{\mathrm{df}}{\iff} \exists_{z\in\mathbb{S}}(z\sqsubseteq x\wedge z\sqsubseteq y), \tag{df $\bigcirc$}$$

$$x\wr y\overset{\mathrm{df}}{\iff} \neg\exists_{z\in\mathbb{S}}(z\sqsubseteq x\wedge z\sqsubseteq y). \tag{df $\wr$}$$

[2] One object is an ingrediens of another iff it is either its (proper) part or is identical with it (see [7, 8]). The relation of *being an ingrediens* is often called a *part relation*, where the term 'part' allows for so called improper parts, i.e. whole objects.

The relation $\sqsubset$ is irreflexive, asymmetric and transitive in $\mathbb{S}$, i.e., we have:

$$\neg\exists_{x\in\mathbb{S}}\; x\sqsubset x\,, \tag{$\mathrm{r}_\sqsubset$}$$

$$\neg\exists_{x,y\in\mathbb{S}}(x\sqsubset y\wedge y\sqsubset x), \tag{$\mathrm{as}_\sqsubset$}$$

$$\forall_{x,y,z\in\mathbb{S}}(x\sqsubset y\wedge y\sqsubset z\Longrightarrow x\sqsubset z). \tag{$\mathrm{t}_\sqsubset$}$$

Moreover, the relation $\bigcirc$ is reflexive and symmetric, the relation $\wr$ is irreflexive and symmetric (see e.g. [9, 10, 11, 13]), and we have the following connections between relation $\sqsubseteq$ and relations $\bigcirc$, $\wr$, and $\sqsubset$; for all $x,y,z\in\mathbb{S}$:

$$x\sqsubseteq y\iff x\sqsubset y\vee x=y\,, \tag{1}$$

$$x\sqsubset y\iff x\sqsubseteq y\wedge y\not\sqsubseteq x\,,$$

$$x\sqsubseteq y\wedge y\wr z\implies x\wr z\,, \tag{2}$$

$$x\sqsubseteq y\wedge x\bigcirc z\implies y\bigcirc z\,.$$

Moreover, from theory of mereological structures we use the following binary relation **sum** included in $\mathbb{S}\times 2^{\mathbb{S}}$:

$$x\ \mathbf{sum}\ S\overset{\mathrm{df}}{\iff}\forall_{s\in S}\ s\sqsubseteq x\wedge\forall_{y\in\mathbb{S}}\big(y\sqsubseteq x\Longrightarrow\exists_{s\in S}\ s\bigcirc y\big). \tag{df $\mathbf{sum}$}$$

If x **sum** S, then we say that the solid x is a *mereological sum* (or *collective set*) of all members of the (distributive) set S. From (df **sum**) and ($\mathrm{r}_\sqsubseteq$) we obtain:

$$\neg\exists_{x\in\mathbb{S}}\ x\ \mathbf{sum}\ \emptyset\,.$$

The pair $\langle\mathbb{S},\sqsubseteq\rangle$ is a mereological structure in Tarski's sense[3], i.e., it satisfies the following condition:[4]

$$\forall_{S\in 2^{\mathbb{S}}\setminus\{\emptyset\}}\exists^1_{x\in\mathbb{S}}\ x\ \mathbf{sum}\ S\,,$$

which says that for any non-empty set S of solids there exists exactly one its mereological sum.

Notice that in all mereological structures, for any subset S of $\mathbb{S}$ and any solid $x\in\mathbb{S}$ we have:

$$x\ \mathbf{sum}\ S\iff S\neq\emptyset\wedge x=\sup\nolimits_\sqsubseteq S\,.$$

Thus, there exists exactly one solid which is the mereological sum (and so also supremum) of the set $\mathbb{S}$. This solid we denote by $\mathbbm{s}$ and we call

[3] For detailed discussion of mereological structures, see e.g. [5, 6, 9, 10, 11, 12, 13].

[4] A formula of the form $\ulcorner\exists^1_{x\in S}\,\varphi(x)\urcorner$ says that in a set S there exists exactly one solid x such that $\varphi(x)$. This formula is an abbreviation for the following: $\ulcorner\exists_{x\in S}\,\varphi(x)\wedge\forall_{x,y\in S}\big(\varphi(x)\wedge\varphi(x/y)\Rightarrow x=y\big)\urcorner$.

it *space*:[5]

$$\mathbb{s} := (\iota x)\ x\ \mathbf{sum}\ \mathbb{S} = \sup_{\sqsubseteq} \mathbb{S}. \tag{df $\mathbb{s}$}$$

In [15, 16] Tarski implicitly (see [3, pp. 483–484]) assumed the following relationship between solids and balls:

$$\forall_{x\in\mathbb{S}}\exists_{S\in 2^{\mathbb{B}}}\ x\ \mathbf{sum}\ S, \tag{$\star$}$$

i.e., every solid is a mereological sum of some non-empty set of balls.

The other postulates are related to the notions of geometry defined in universe of solids. The notion of *point* is defined with the use of the relation of *concentricity of balls*. To define this relation, we introduced in the set $\mathbb{B}$ two auxiliary binary relations: the relation **ET** *of external tangency of balls* and the relation **IT** *of internal tangency of balls*:

$$\begin{aligned} a\,\mathbf{ET}\,b \overset{\text{df}}{\iff}\ & a \wr b\ \wedge \\ & \forall_{x,y\in\mathbb{B}}(a \sqsubseteq x \wr b \wedge a \sqsubseteq y \wr b \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x), \end{aligned} \tag{df $\mathbf{ET}$}$$

$$\begin{aligned} a\,\mathbf{IT}\,b \overset{\text{df}}{\iff}\ & a \sqsubset b\ \wedge \\ & \forall_{x,y\in\mathbb{B}}(a \sqsubseteq x \sqsubseteq b \wedge a \sqsubseteq y \sqsubseteq b \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x). \end{aligned} \tag{df $\mathbf{IT}$}$$

Next, with the use of relations **ET** and **IT**, we define in the set $\mathbb{B}$ two ternary relations: the relation **EDT** *of external diametrical tangency of balls* and the relation **IDT** *of internal diametrical tangency of balls*:

$$\begin{aligned} ab\,\mathbf{EDT}\,c \overset{\text{df}}{\iff}\ & a\,\mathbf{ET}\,c \wedge b\,\mathbf{ET}\,c\ \wedge \\ & \forall_{x,y\in\mathbb{B}}(a \sqsubseteq x \wr c \wedge b \sqsubseteq y \wr c \Longrightarrow x \wr y), \end{aligned} \tag{df $\mathbf{EDT}$}$$

$$\begin{aligned} ab\,\mathbf{IDT}\,c \overset{\text{df}}{\iff}\ & a\,\mathbf{IT}\,c \wedge b\,\mathbf{IT}\,c \wedge \\ & \forall_{x,y\in\mathbb{B}}(x \wr c \wedge y \wr c \wedge a\,\mathbf{ET}\,x \wedge b\,\mathbf{ET}\,x \Longrightarrow x \wr y). \end{aligned} \tag{df $\mathbf{IDT}$}$$

With the use of the relations defined above, in $\mathbb{B}$ we can introduce the binary relation $\circledcirc$ of *concentricity* of balls:

$$\begin{aligned} & a \circledcirc b \overset{\text{df}}{\iff} \big[a = b\ \vee \\ & \big(a \sqsubset b \wedge \forall_{x,y\in\mathbb{B}}(xy\,\mathbf{EDT}\,a \wedge x\,\mathbf{IT}\,b \wedge y\,\mathbf{IT}\,b \Longrightarrow xy\,\mathbf{IDT}\,b)\big) \vee \\ & \big(b \sqsubset a \wedge \forall_{x,y\in\mathbb{B}}(xy\,\mathbf{EDT}\,b \wedge x\,\mathbf{IT}\,a \wedge y\,\mathbf{IT}\,a \Longrightarrow xy\,\mathbf{IDT}\,a)\big)\big]. \end{aligned} \tag{df $\circledcirc$}$$

Directly from (df $\circledcirc$) it follows that the relation $\circledcirc$ is reflexive and symmetric in $\mathbb{B}$. Using the relation $\circledcirc$ we define the notion of *point* as the

[5] The Greek letter 'ι' stands for the standard description operator. The expression $\ulcorner(\iota x)\,\varphi(x)\urcorner$ is read "the only object x which satisfies the condition $\varphi(x)$". To use 'ι' we first have to ensure both existence and uniqueness of the object that satisfies φ, i.e., we have: $\exists^1_{x\in S}\,\varphi(x)$.

set of these balls that are concentric with a given ball. Let us denote the set of all points by Π. Then, for any $\alpha \in 2^{\mathbb{B}}$ we have:

$$\alpha \in \Pi \overset{\mathrm{df}}{\iff} \exists_{y \in \mathbb{B}}\ \alpha = \{x \in \mathbb{B} : x \circledcirc y\}\,. \tag{df Π}$$

In the set of all points Π following Tarski we define the ternary relation Δ of *equidistance of two points from a third one*. This relation allows us to compare distances between points. For all points $\alpha, \beta, \gamma \in \Pi$ we put:

$$\alpha\beta \mathrel{\Delta} \gamma \overset{\mathrm{df}}{\iff} \alpha = \beta = \gamma\ \vee\ \exists_{c \in \gamma} \neg \exists_{a \in \alpha \cup \beta} (a \sqsubseteq c \vee a \wr c). \tag{df Δ}$$

The first specific postulate of the geometry of solids claims that:

$$\langle \Pi, \Delta \rangle \text{ is a Pieri's structure.} \tag{P1}$$

According to [2], all Pieri's structures are models of three-dimensional Euclidean geometry in terms of *point* and *equidistance relation*. Thus, in $\langle \Pi, \Delta \rangle$ we can introduce the natural topology of Euclidean space. Let $\mathbf{BO}_\Pi$ by the family of all open balls and $\mathbf{RO}_\Pi^+$ be the family of all non-empty regular open sets in this topology. Of course, $\mathbf{BO}_\Pi \subsetneq \mathbf{RO}_\Pi^+$.

The other specific postulates of the geometry of solids establish the relation between solids and regular open sets in Euclidean topology. For this purpose, we introduce an operation $\mathrm{int}\colon \mathbb{S} \to 2^\Pi$ which assigns to every solid the set of its *interior points*. For any $x \in \mathbb{S}$ we put:

$$\mathrm{int}(x) := \{\alpha \in \Pi : \exists_{a \in \alpha}\ a \sqsubseteq x\}. \tag{df int}$$

Note that from (df int) and ($\mathrm{t}_\sqsubseteq$) we obtain:

$$\forall_{x,y \in \mathbb{S}} (x \sqsubseteq y \implies \mathrm{int}(x) \subseteq \mathrm{int}(y)).$$

With the use of the operation int we can formulate two final postulates of the geometry of solids. The first claims that the interior points of each mereological solid is a non-empty regular open set in Euclidean topology

$$\forall_{x \in \mathbb{S}}\ \mathrm{int}(x) \in \mathbf{RO}_\Pi^+. \tag{P2}$$

The second postulate says that each regular open set is an interior of some solid:

$$\forall_{U \in \mathbf{RO}_\Pi^+} \exists_{x \in \mathbb{S}}\ \mathrm{int}(x) = U. \tag{P3}$$

DEFINITION 1.1. A structure $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq \rangle$ is a $\mathrm{T}^\star$-structure iff $\langle \mathbb{S}, \sqsubseteq \rangle$ is a mereological structure and $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq \rangle$ satisfies ($\star$) and (P1)–(P3).

In any T*-structure $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq \rangle$ for all $x, y \in \mathbb{S}$ we obtain:

$$x \sqsubseteq y \iff \mathbb{int}(x) \subseteq \mathbb{int}(y), \tag{3}$$

$$x \wr y \iff \mathbb{int}(x) \cap \mathbb{int}(y) = \emptyset\,. \tag{4}$$

In [3] it is proved that the relation $\circledcirc$ is transitive in any T*-structure $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq \rangle$. Thus, $\circledcirc$ is an equivalence relation in the set $\mathbb{B}$. So all points can be identified with equivalence classes of the relation $\circledcirc$, i.e.:

$$\Pi := \mathbb{B}/_{\circledcirc}\,. \tag{def$'$ Π}$$

By reflexivity of the relation $\circledcirc$, for a given ball b we can consider a point, whose element is b. Such a point is the equivalence class $\|b\|_{\circledcirc}$. This class we will denote by π_b and called “the point generating by b”. So for any $b \in \mathbb{B}$ we put:

$$\pi_b := \{a \in \mathbb{B} : a \circledcirc b\} =: \|b\|_{\circledcirc}. \tag{df π_b}$$

From reflexivity, symmetry and transitivity of $\circledcirc$ it follows that:

$$\forall_{b \in \mathbb{B}}\; b \in \pi_b\,, \tag{5}$$

$$\forall_{a,b \in \mathbb{B}} (a \circledcirc b \iff \pi_a = \pi_b), \tag{6}$$

$$\forall_{\alpha \in \Pi} \forall_{b \in \mathbb{B}} (b \in \alpha \iff \alpha = \pi_b). \tag{7}$$

Moreover, we can also introduce an operation $\mathbb{fr} \colon \mathbb{S} \to 2^{\Pi}$ which ascribes to each solid the set of its *fringe points*. For any $x \in \mathbb{S}$ we put:

$$\mathbb{fr}(x) := \{\alpha \in \Pi \mid \forall_{a \in \alpha} (a \not\sqsubseteq x \;\wedge\; a \bigcirc x)\}. \tag{df $\mathbb{fr}$}$$

Directly from (df $\mathbb{int}$) and (df $\mathbb{fr}$) we obtain that for any $x \in \mathbb{S}$:

$$\mathbb{int}(x) \cap \mathbb{fr}(x) = \emptyset. \tag{8}$$

For arbitrary different points α and β from Π we put:

$$\mathbb{B}_\alpha^\beta := \{b \in \mathbb{B} : b \in \alpha \wedge \beta \in \mathbb{fr}(b)\}\,,$$

i.e., $\mathbb{B}_\beta^\alpha$ is the set of all mereological balls being elements of α and having β as its fringe point. Moreover, we put

$$\mathrm{S}_\alpha^\beta := \{\gamma \in \Pi : \gamma\beta \,\Delta\, \alpha\},$$

i.e., S_α^β is the sphere in $\langle \Pi, \Delta \rangle$ such that α is the center point of S_α^β and β is its element. Finally, for any open ball $B \in \mathbf{BO}_\Pi$ let $\mathrm{Fr}(B)$ be its fringe. Then we put:

$$\mathrm{B}_\alpha^\beta := (\iota\, B)\; (B \in \mathbf{BO}_\Pi \wedge \mathrm{Fr}(B) = \mathrm{S}_\alpha^\beta),$$

i.e., B_α^β is the open ball from $\mathbf{BO}_\Pi$ such that S_α^β is its surface, in other words, α is the center of B_α^β and β lies on the surface of B_α^β.

The most important properties, which are crucial for this paper, are expressed in the following facts which are proved in [3] and which will be used in this paper.

Fact 1.1 ([3, pp. 509]). *For any different points α and β from Π for some $b \in \mathbb{B}$ we have $\mathbb{B}_\alpha^\beta = \{b\}$ and $\mathtt{int}(b) = \mathrm{B}_\alpha^\beta$.*

Fact 1.2 ([3, p. 510]). *For any $b \in \mathbb{B}$ we have $\mathtt{int}(b) \in \mathbf{BO}_\Pi$ and there is $\beta \in \Pi$ such that $\beta \neq \pi_b$, $\mathbb{B}_{\pi_b}^\beta = \{b\}$, $\mathtt{fr}(b) = \mathrm{S}_{\pi_b}^\beta$ and $\mathtt{int}(b) = \mathrm{B}_{\pi_b}^\beta$.*

Fact 1.3 ([3, p. 510]). *For any Euclidean ball $B \in \mathbf{BO}_\Pi$ there exists exactly one mereological ball $b \in \mathbb{B}$ such that $\mathtt{int}(b) = B$.*

Fact 1.4 ([3, pp. 511 and 518]). *The mapping $\mathtt{int}\colon \mathbb{S} \to \mathbf{RO}_\Pi^+$ is an isomorphism from $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq\rangle$ onto $\langle \mathbf{RO}_\Pi^+, \mathbf{BO}_\Pi, \subseteq\rangle$; so the mapping $\mathtt{int}|_{\mathbb{B}}$ a bijection from $\mathbb{B}$ onto $\mathbf{BO}_\Pi$.*

On the basis of the above facts, the operation $\mathtt{int}$ transforms any given mereological ball a into the open Euclidean ball $\mathrm{B}_{\pi_a}^\alpha$, while the operation $\mathtt{fr}$ transforms a into the Euclidean sphere $\mathrm{S}_{\pi_a}^\alpha$.

2. Constructional theorems

Below we will prove two theorems characterising relations **IT** and **ET** in terms of operations $\mathtt{int}$ and $\mathtt{fr}$. We call these theorems *constructional theorems*, since they are the basis of the proper constructions that will be done in further part of this paper.

First, we will show that the interior points of mereological balls that are internally tangent include themselves appropriately, and sets of their fringe points have exactly one common point.

Theorem 2.1. *For any $a, b \in \mathbb{B}$:*

$$a \,\mathbf{IT}\, b \iff \mathtt{int}(a) \subseteq \mathtt{int}(b) \wedge \exists^1_{\gamma\in\Pi}\big(\gamma \in \mathtt{fr}(a) \cap \mathtt{fr}(b)\big).$$

Proof. "⇒" Let a and b be any mereological balls such that $a\,\mathbf{IT}\,b$. Let $S_a := \mathtt{fr}(a)$, $B_a := \mathtt{int}(a)$, $S_b := \mathtt{fr}(b)$ and $B_b := \mathtt{int}(b)$. By (df **IT**), $a \sqsubset b$, hence $\mathtt{int}(a) \subseteq \mathtt{int}(b)$, by (3).

Suppose towards a contradiction that $S_a \cap S_b = \emptyset$ (see Figure 1). Let $L(\pi_a, \pi_b)$ be a straight line crossing centers of spheres S_a and S_b. Straight line $L(\pi_a, \pi_b)$ intersects the sphere S_a in points α, α' and inter-

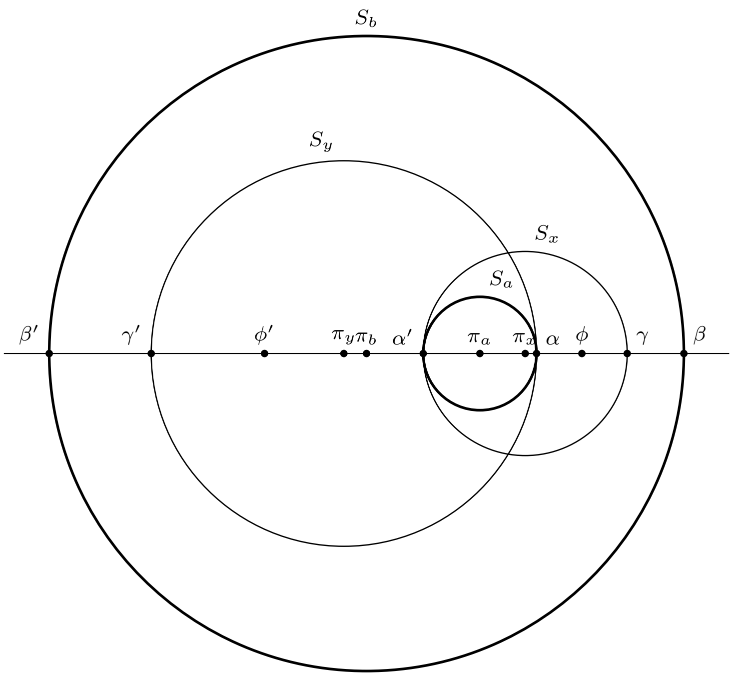

Figure 1. Assumption in the proof of Theorem 2.1

sect the sphere S_b in points β, β'. Since $(B_a \cup S_a) \subseteq B_b$, so $\mathbf{B}(\beta'\,\alpha\,\beta)$ and $\mathbf{B}(\beta'\,\alpha'\,\beta)$, where $\mathbf{B}$ is a ternary relation of *betweenness* (see Appendix, p. 557). Furthermore, suppose that $\mathbf{B}(\alpha'\,\alpha\,\beta)$. By assumption points α and β are distinct. Let γ be an arbitrary point such that $\mathbf{B}(\alpha\,\gamma\,\beta)$. Let $\pi_x := \mathrm{mid}(\alpha', \gamma)$. Since $\mathbf{B}(\alpha'\,\alpha\,\beta)$ and $\mathbf{B}(\alpha\,\gamma\,\beta)$. Then we have $\mathbf{B}(\alpha'\,\alpha\,\gamma)$, by Axiom O8 in Appendix on p. 557. Hence, by (df $<$) from Appendix, we have $\alpha'\,\alpha < \alpha'\,\gamma$. Moreover, by Fact A.4, we obtain $\alpha'\,\pi_a < \alpha'\,\pi_x$, which means that $\mathbf{B}(\alpha'\,\pi_a\,\pi_x)$, by (df $<$).

Let us consider the Euclidean ball $B^{\gamma}_{\pi_x}$. We will show that $B_a \subseteq B^{\gamma}_{\pi_x}$. Let ϕ be an arbitrary point such that $\phi \in B_a$. Then we have $\pi_a\,\phi < \pi_a\,\alpha'$, thus $[\pi_a\,\phi] < [\pi_a\,\alpha']$ and, by ($\mathrm{mon}_<$) in Fact A.5 in Appendix, we have: $[\pi_x\,\pi_a]+[\pi_a\,\phi] < [\pi_x\,\pi_a]+[\pi_a\,\alpha']$. Since $\mathbf{B}(\alpha'\,\pi_a\,\pi_x)$, so $[\pi_x\,\pi_a]+[\pi_a\,\alpha'] = [\pi_x\,\alpha']$. By triangle inequality, for points π_x, π_a and ϕ we have: $[\pi_x\,\phi] \leqslant [\pi_x, \pi_a] + [\pi_a\,\phi]$. Thus: $[\pi_x\,\phi] \leqslant [\pi_x, \pi_a] + [\pi_a\,\phi] < [\pi_x\,\pi_a] + [\pi_a\,\alpha'] = [\pi_x\,\alpha']$. Hence, by transitivity of the relation $\leqslant$ it follows that $[\pi_x\,\phi] < [\pi_x\,\phi]$, thus $\pi_x\,\phi < \pi_x\,\phi$ and consequently $\phi \in B^{\gamma}_{\pi_x}$. Thus (i): $B_a \subseteq B^{\gamma}_{\pi_x}$.

Now we will show that $B^{\gamma}_{\pi_x} \subseteq B_b$. We will prove that $\mathbf{B}(\beta'\,\pi_x\,\beta)$. Since $\mathbf{B}(\alpha'\,\alpha\,\beta)$ and $\mathbf{B}(\alpha\,\gamma\,\beta)$, so $\mathbf{B}(\alpha'\,\gamma\,\beta)$, by (OB) on p. 558. Hence, using $\mathbf{B}(\alpha'\pi_x\,\gamma)$, we have $\mathbf{B}(\alpha'\,\pi_x\,\beta)$, by (OD) on p. 558. Since $\mathbf{B}(\beta'\,\alpha'\,\beta)$, so $\mathbf{B}(\beta'\,\pi_x\,\beta)$, by (OB). We will consider two logical possibilities.

First, suppose that $\pi_x = \pi_b$. Then, since $\mathbf{B}(\alpha'\,\gamma\,\beta)$ and $\mathbf{B}(\beta'\,\alpha'\,\beta)$, so $\alpha'\,\gamma < \alpha'\,\beta$ and $\alpha'\,\beta < \beta\,\beta'$, by (df $<$). Hence $\alpha'\,\gamma < \beta\,\beta'$, by ($\mathrm{t}_<$) on p. 559. Since $\pi_x = \mathrm{mid}(\alpha', \gamma)$ and $\pi_b = \mathrm{mid}(\beta, \beta')$, so $\pi_x\,\gamma < \pi_b\,\beta$, by Fact A.4. Let ϕ be an arbitrary point such that $\phi \in B^{\gamma}_{\pi_x}$. Then $\pi_x\,\phi < \pi_x\,\gamma$ and $\pi_x\,\phi < \pi_b\,\beta$, by ($\mathrm{t}_<$), because $\pi_x\,\gamma < \pi_b\,\beta$. Moreover, since $\pi_x = \pi_b$, so $\pi_b\,\phi < \pi_b\,\beta$. Thus, $\phi \in B_b$.

Second, suppose that $\pi_b \neq \pi_x$. Then, since $\mathbf{B}(\beta'\,\pi_b\,\beta)$ and $\mathbf{B}(\beta'\,\pi_x\,\beta)$, so either $\mathbf{B}(\beta'\,\pi_x\,\pi_b)$ or $\mathbf{B}(\pi_b\,\pi_x\,\beta)$, by (OC) on p. 558. Let $\phi \in B^{\gamma}_{\pi_x}$ and suppose that $\mathbf{B}(\beta'\,\pi_x\,\pi_b)$. By triangle inequality for points π_b, ϕ and π_x we have: $[\pi_b\,\phi] \leqslant [\pi_b\,\pi_x] + [\pi_x\,\phi]$. Since $\phi \in B^{\gamma}_{\pi_x} = B^{\alpha'}_{\pi_x}$, so also $\pi_x\,\phi < \pi_x\,\alpha'$. Hence, by ($\mathrm{mon}_<$), we obtain: $[\pi_b\,\pi_x] + [\pi_x\,\phi] < [\pi_b\,\pi_x] + [\pi_x\,\alpha']$.

We will show that $\mathbf{B}(\alpha'\,\pi_x\,\pi_b)$ and then we will be able to get $[\pi_b\,\pi_x] + [\pi_x\,\alpha'] = [\pi_b\,\alpha']$. We have $\mathbf{B}(\beta'\,\alpha\,\beta)$ and $\mathbf{B}(\beta'\,\alpha'\,\beta)$. Since $\alpha \neq \alpha'$, so either $\mathbf{B}(\beta'\,\alpha'\,\alpha)$ or $\mathbf{B}(\alpha\,\alpha'\,\beta)$, by (OC). Since $\mathbf{B}(\alpha'\,\alpha\,\beta)$, so $\neg\mathbf{B}(\alpha\,\alpha'\,\beta)$, by Axiom O3 in Appendix. Thus, we have $\mathbf{B}(\beta'\,\alpha'\,\alpha)$. But $\pi_a = \mathrm{mid}(\alpha', \alpha)$, hence $\mathbf{B}(\alpha'\,\pi_a\,\alpha)$ and $\mathbf{B}(\beta'\,\alpha'\,\pi_a)$, by Axiom O8. Hence $\mathbf{B}(\beta'\,\alpha'\,\pi_x)$, by Axiom O7, since $\mathbf{B}(\alpha'\,\pi_a\,\pi_x)$. Next, because $\mathbf{B}(\beta'\,\pi_x\,\pi_b)$. So $\mathbf{B}(\beta'\,\alpha'\,\pi_b)$, by (OD). Thus, $\mathbf{B}(\beta'\,\alpha'\,\pi_b)$ and $\mathbf{B}(\beta'\,\pi_x\,\pi_b)$, but $\pi_x \neq \alpha'$. So $\mathbf{B}(\beta'\,\pi_x\,\alpha')$ or $\mathbf{B}(\alpha'\,\pi_x\,\pi_b)$, by (OC). Suppose that $\mathbf{B}(\beta'\,\pi_x\,\alpha')$. Then $\mathbf{B}(\beta'\,\pi_x\,\alpha')$, by Axiom O2. Since $\mathbf{B}(\beta'\,\alpha'\,\pi_x)$, so $\mathbf{B}(\pi_x\,\alpha'\,\beta')$, by Axiom O2. Therefore $\neg\mathbf{B}(\alpha'\,\pi_x\,\beta')$, by Axiom O3. So we have a contradiction. Thus, we finally get that $\mathbf{B}(\alpha'\,\pi_x\,\pi_b)$. Then for points π_b, ϕ, and π_x, by triangle inequality we have $[\pi_b\,\phi] \leqslant [\pi_b\,\pi_x] + [\pi_x\,\phi] < [\pi_b\,\pi_x] + [\pi_x\,\alpha'] = [\pi_b\,\alpha']$. Hence $[\pi_b\,\phi] < [\pi_b\,\alpha']$, by ($\mathrm{t}_<$), which means that $\pi_b\,\phi < \pi_b\,\alpha'$. Since $\mathbf{B}(\beta'\,\alpha'\,\pi_b)$, so $\pi_b\,\alpha' < \pi_b\,\beta'$, by (df $<$). And according to previous inequalities we have $\pi_b\,\phi < \pi_b\,\beta'$, by ($\mathrm{t}_<$). Moreover, because $B^{\beta'}_{\pi_b} = B_b$, so we finally get (ii): $B^{\gamma}_{\pi_x} \subseteq B_b$.

Now, choosing any point γ' such that $\mathbf{B}(\beta'\,\gamma'\,\alpha')$ and then the point $\pi_y = \mathrm{mid}(\gamma', \alpha)$, we can construct the Euclidean ball $B^{\gamma'}_{\pi_y}$ which includes the ball B_a and which is included in the ball B_b. Proof of these facts, by symmetry of the construction with respect to the point π_a, is analogous to the proof that $B_a \subseteq B^{\alpha'}_{\pi_x} \subseteq B_b$. Thus, we immediately have (c): $B_a \subseteq B^{\gamma'}_{\pi_y}$; and (d): $B^{\gamma'}_{\pi_y} \subseteq B_b$.

Now, we will show that $B^{\alpha'}_{\pi_x} \nsubseteq B^{\gamma'}_{\pi_y}$ and $B^{\gamma'}_{\pi_y} \nsubseteq B^{\alpha'}_{\pi_x}$ (we will show only that $B^{\alpha'}_{\pi_x} \nsubseteq B^{\gamma'}_{\pi_y}$: the proof of the second fact is analogous). Let ϕ and ϕ' be arbitrary points such that: $\phi = \mathrm{mid}(\alpha, \gamma)$ and $\phi' = \mathrm{mid}(\alpha', \gamma')$.

First, we will show that $\phi \in B^{\alpha'}_{\pi_x}$. Since $\phi = \mathrm{mid}(\alpha, \gamma)$, so $\mathbf{B}(\alpha\,\phi\,\gamma)$. Hence $\mathbf{B}(\alpha'\,\phi\,\gamma)$, by (OB), since $\mathbf{B}(\alpha'\,\alpha\,\gamma)$. So $\phi \in B^{\alpha'}_{\pi_x}$, since $\alpha', \gamma \in (B^{\alpha'}_{\pi_x} \cup S^{\alpha'}_{\pi_x})$ and by the fact that any closed Euclidean ball is a convex set. From $\mathbf{B}(\beta\,\alpha\,\alpha')$ and $\mathbf{B}(\alpha\,\pi_a\,\alpha')$ it follows that $\mathbf{B}(\beta\,\alpha\,\pi_a)$, by Axiom O8. So $\mathbf{B}(\pi_a\,\alpha\,\beta)$, by Axiom O2. Hence $\mathbf{B}(\pi_a\,\alpha\,\gamma)$, by Axiom O8, since $\mathbf{B}(\alpha\,\gamma\,\beta)$. Therefore $\mathbf{B}(\pi_a\,\alpha\,\phi)$, by Axiom (08), because $\mathbf{B}(\alpha\,\phi\,\gamma)$. Next, we have $\mathbf{B}(\alpha\,\alpha'\,\beta)$, by Axiom O2, since $\mathbf{B}(\alpha\,\alpha'\,\beta')$. Hence $\mathbf{B}(\alpha\,\alpha'\,\gamma')$, by Axiom O8, because $\mathbf{B}(\alpha'\,\gamma'\,\beta')$. Therefore $\alpha\,\alpha' < \alpha\,\gamma'$, (df $<$). Moreover, since $\pi_a = \mathrm{mid}(\alpha, \alpha')$ and $\pi_y = \mathrm{mid}(\alpha, \gamma')$, so $\alpha\,\pi_a < \alpha\,\pi_y$, by Fact A.4. Hence $\mathbf{B}(\alpha\,\pi_a\,\pi_y)$, by (df $<$). So $\mathbf{B}(\pi_y\,\pi_a\,\alpha)$, by Axiom O2. Since $\mathbf{B}(\pi_y\,\pi_a\,\alpha)$ and $\mathbf{B}(\pi_a\,\alpha\,\phi)$, so $\mathbf{B}(\pi_y\,\alpha\,\phi)$, by (OA) on p. 558. Hence $\pi_y\,\alpha < \pi_y\,\phi$, by (df $<$). Finally, since $B^{\gamma'}_{\pi_y} = B^{\alpha}_{\pi_y}$, so $\phi \notin B^{\gamma'}_{\pi_y}$.

Thus, we have (a): $B^{\alpha'}_{\pi_x} \nsubseteq B^{\gamma'}_{\pi_y}$; and, by symmetry of the construction, we also have (b): $B^{\gamma'}_{\pi_y} \nsubseteq B^{\alpha'}_{\pi_x}$. By Fact 1.4, we have mereological balls x and y such that $\mathbb{int}(x) = B^{\alpha'}_{\pi_x}$ and $\mathbb{int}(y) = B^{\gamma'}_{\pi_y}$. So, by (i) and (ii), $B_a \subseteq B^{\gamma}_{\pi_x} \subseteq B_b$, and, by (c) and (d), $B_a \subseteq B^{\gamma'}_{\pi_y} \subseteq B_b$. Hence $a \sqsubseteq x \sqsubseteq b$ and $a \sqsubseteq y \sqsubseteq b$. But from (a) and (b) it follows that $x \not\sqsubseteq y$ and $y \not\sqsubseteq x$, which is a contradiction with $a\,\mathbf{IT}\,b$. Thus $\mathbb{fr}(a) \cap \mathbb{fr}(b) \neq \emptyset$.

To prove that there is exactly one point γ such that $\gamma \in \mathbb{fr}(a) \cap \mathbb{fr}(b)$ let us suppose towards a contradiction that there exists point $\delta \neq \gamma$ such that $\delta \in \mathbb{fr}(a) \cap \mathbb{fr}(b)$. Then, by Fact A.10 we have $B_a \backslash B_b \neq \emptyset$ and $B_b \backslash B_a \neq \emptyset$, so $B_a \nsubseteq B_b$ and $B_b \nsubseteq B_a$. Hence $a \not\sqsubseteq b$ and $b \not\sqsubseteq a$, which is a contradiction with $a \sqsubseteq b$. Thus, we finally get $\mathbb{int}(a) \subseteq \mathbb{int}(b)$ and there is exactly one $\gamma \in \Pi$ such that $\gamma \in \mathbb{fr}(a) \cap \mathbb{fr}(b)$.

"$\Leftarrow$" Suppose that $\mathbb{int}(a) \subseteq \mathbb{int}(b)$ and there is exactly one $\gamma \in \Pi$ such that $\gamma \in \mathbb{fr}(a) \cap \mathbb{fr}(b)$. Let $S_a = \mathbb{fr}(a)$, $S_b = \mathbb{fr}(b)$, $B_a = \mathbb{int}(a)$, and $B_b = \mathbb{int}(b)$. Let x and y be arbitrary mereological balls and suppose that $a \sqsubseteq x \sqsubseteq b$ and $a \sqsubseteq y \sqsubseteq b$. We put $S_x := \mathbb{fr}(x)$, $S_y := \mathbb{fr}(y)$, $B_x := \mathbb{int}(x)$, and $B_y := \mathbb{int}(y)$.

First, we will show that points γ, π_a, π_b, π_x, and π_y are collinear and then we will show how they are ordered on a common straight line. By assumption we have $S_a \cap S_b = \{\gamma\}$. Let us suppose that $\gamma \notin S_a \cap S_x$. Then either $\gamma \notin S_a$ or $\gamma \notin S_a$. Since $\gamma \in S_a$, so by (df $\mathbb{fr}$) there exists a mereological ball $c \in \gamma$ such that $c \wr x$ or $c \sqsubseteq x$. Suppose that $c \wr x$. Since

$a \sqsubseteq x$, so $a \sqsubseteq x \wr c$ and $a \wr c$, by (2). Hence it follows that $\gamma \notin \mathbb{fr}(a)$, by (df $\mathbb{fr}$). So we obtain a contradiction. Now, suppose that $c \sqsubseteq x$. Then $c \sqsubseteq b$, by transitivity of $\sqsubseteq$, since $x \sqsubseteq b$. Hence $\gamma \in \mathbb{int}(a)$, by (df $\mathbb{int}$). Thus, $\gamma \in \mathbb{fr}(a) \cap \mathbb{int}(a)$, which is a contradiction with (8). Therefore $\gamma \in S_a \cap S_x$. We can repeat analogous reasoning for spheres S_a and S_y and as a result we will obtain that $\gamma \in S_a \cap S_y$. Now, suppose that there exists a point $\delta \neq \gamma$ such that $\delta \in S_a \cap S_x$. Then, $B_a \nsubseteq B_x$ and $B_x \nsubseteq B_a$, by Fact A.10. Thus, $a \not\sqsubseteq x$ and $x \not\sqsubseteq a$, which is a contradiction with $a \sqsubseteq x$. Therefore $S_a \cap S_x = \{\gamma\}$. Analogous reasoning we can repeat for spheres S_a and S_y and as a result we will obtain that $S_a \cap S_y = \{\gamma\}$. Thus, $S_a \cap S_b = \{\gamma\}$, $S_a \cap S_x = \{\gamma\}$, and $S_a \cap S_y = \{\gamma\}$. By definition of a straight line in Pieri's structure, for some straight lines L, K, and M we have that: $\pi_a, \pi_b, \gamma \in L$, $\pi_b, \pi_x, \gamma \in K$, and $\pi_b, \pi_y, \gamma \in M$.

Thus, $\pi_b, \gamma \in L$, $\pi_b, \gamma \in K$, and $\pi_b, \gamma \in M$. Hence $K = L = M$, by Fact A.1. Therefore points π_a, γ, π_b, π_x, and π_y are collinear. By assumptions that $a \sqsubseteq x \sqsubseteq b$ and $a \sqsubseteq y \sqsubseteq b$, and by (3), we have $B_a \subseteq B_x \subseteq B_b$ and $B_a \subseteq B_y \subseteq B_b$. Hence $\gamma\,\pi_a < \gamma\,\pi_x$ and $\gamma\,\pi_a < \gamma\,\pi_y$, since $\gamma \in S_a \cap S_x \cap S_y \cap S_b$. So $\mathbf{B}(\gamma\,\pi_a\pi_x)$ and $\mathbf{B}(\gamma\,\pi_a\,\pi_y)$, by (df $<$). Since $\gamma \neq \pi_a$, so either $\mathbf{B}(\pi_a\,\pi_x\,\pi_y)$ or $\mathbf{B}(\pi_a\,\pi_y\,\pi_x)$, by (OE) on p. 558. So taking into consideration that $\mathbf{B}(\gamma\,\pi_a\pi_x)$ and $\mathbf{B}(\gamma\,\pi_a\,\pi_y)$ we have: $\mathbf{B}(\gamma\,\pi_x\,\pi_y)$ or $\mathbf{B}(\gamma\,\pi_y\,\pi_x)$, by (OA). Now suppose that $\mathbf{B}(\gamma\,\pi_x\,\pi_y)$. Then, $\gamma\,\pi_x < \gamma\,\pi_y$, by (df $<$). We will show that $B_x \subseteq B_y$.

Let ϕ be an arbitrary point in B_x. Then $\pi_x\,\phi < \pi_x\,\gamma$. Hence $[\pi_x\,\phi] < [\pi_x\,\gamma]$ and $[\pi_y\,\pi_x] + [\pi_x\,\phi] < [\pi_y\,\pi_x] + [\pi_x\,\gamma]$, by (mon$_<$). For π_y, ϕ, and π_x, by triangle inequality we have: $[\pi_y\,\phi] \leqslant [\pi_y\,\pi_x] + [\pi_x\,\phi]$. Hence $[\pi_y\,\phi] < [\pi_y\,\pi_x] + [\pi_x\,\gamma]$, by previous inequality and by (t$_<$). Since $\mathbf{B}(\gamma\,\pi_x\,\pi_y)$, so $[\pi_y\,\pi_x] + [\pi_x\,\gamma] = [\pi_y\,\gamma]$. In consequence $[\pi_y\,\phi] < [\pi_y\,\gamma]$. So $\pi_y\,\phi < \pi_y\,\gamma$. Hence $\phi \in B_y$. Using reasoning analogous to that which was used for $\mathbf{B}(\gamma\,\pi_y\,\pi_x)$', we will get $B_y \subseteq B_x$. Thus, either $B_x \subseteq B_y$ or $B_y \subseteq B_x$. So finally we get either $x \sqsubseteq y$ or $y \sqsubseteq x$, which proves that $a\,\mathbf{IT}\,b$. $\square$

Now, we will show that sets of interior points of mereological balls that are externally tangent are disjoint and the sets of their fringe points have exactly one common point.

Theorem 2.2. *For any $a, b \in \mathbb{B}$:*

$$a\,\mathbf{ET}\,b \iff \mathbb{int}(a) \cap \mathbb{int}(b) = \emptyset \wedge \exists^1_{\gamma \in \Pi}(\gamma \in \mathbb{fr}(a) \cap \mathbb{fr}(b)).$$

Proof. "$\Rightarrow$" Let a and b be any mereological balls such that $a\,\mathbf{ET}\,b$. By (df $\mathbf{ET}$) we have $a \wr b$. Hence $(*)$: $\mathbb{int}(a) \cap \mathbb{int}(b) = \emptyset$, by (4).

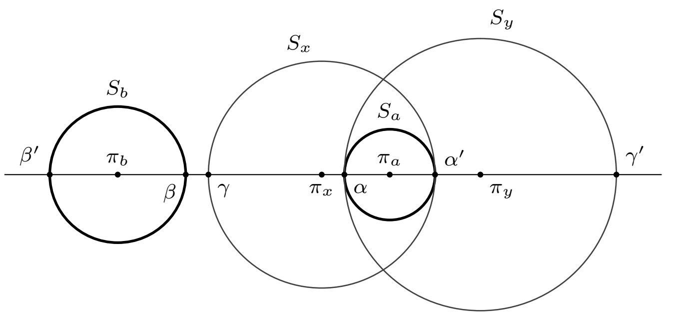

Figure 2. Assumption in the proof of Theorem 4.8

Now, we will show that $\mathbb{fr}(a) \cap \mathbb{fr}(b) \neq \emptyset$. Suppose towards a contradiction that $\mathbb{fr}(a) \cap \mathbb{fr}(b) = \emptyset$ (see Figure 2). Let $L(\pi_a, \pi_b)$ be a straight line crossing points π_a and π_b. Straight line $L(\pi_a, \pi_b)$ crossing the sphere $S_a := \mathbb{fr}(a)$ in points α and α' and the sphere $S_b := \mathbb{fr}(b)$ in points β and β'. Moreover, suppose that $\mathbf{B}(\pi_a\,\alpha\,\pi_b)$ and $\mathbf{B}(\pi_a\,\beta\,\pi_b)$. Since $S_a \cap S_b = \emptyset$, so $\alpha \neq \beta$. Let γ be an arbitrary point such that $\mathbf{B}(\beta\,\gamma\,\alpha)$ and let γ' be an arbitrary point such that $\mathbf{B}(\pi_a\,\alpha'\,\gamma)$. Then, let $\pi_x := \mathrm{mid}(\gamma, \alpha')$ and $\pi_y := \mathrm{mid}(\alpha, \gamma')$. In an analogous way as in the proof of Theorem 2.1, for Euclidean balls B_a, $\mathrm{B}^{\gamma}_{\pi_x}$, and $\mathrm{B}^{\gamma'}_{\pi_y}$ we can prove that (i): $B_a \subsetneq \mathrm{B}^{\gamma}_{\pi_x}$; (ii): $B_a \subsetneq \mathrm{B}^{\gamma'}_{\pi_y}$; (iii) $\mathrm{B}^{\gamma}_{\pi_x} \not\subseteq \mathrm{B}^{\gamma'}_{\pi_y}$; (iv): $\mathrm{B}^{\gamma'}_{\pi_y} \not\subseteq \mathrm{B}^{\gamma}_{\pi_x}$.

We will show that also (v): $B_b \cap \mathrm{B}^{\gamma}_{\pi_a} = \emptyset$; and (vi) $B_b \cap B^{\delta}_{\gamma'} = \emptyset$. Indeed, to prove (v) suppose, towards a contradiction, that there exists a point χ such that $\chi \in B_b$ and $\chi \in \mathrm{B}^{\gamma}_{\pi_x}$. Then $\pi_x\,\chi < \pi_x\,\gamma$ and $\pi_b\,\chi < \pi_b\,\beta$. Since $\mathbf{B}(\pi_b\,\beta\,\gamma)$ then, by (df $<$), we have $\pi_b\,\beta < \pi_b\,\gamma$ and by ($\mathrm{t}_<$) we obtain $\pi_b\,\chi < \pi_b\,\gamma$. Thus, we have: $[\pi_x\,\chi] < [\pi_x\,\gamma]$ and $[\pi_b\,\chi] < [\pi_b\,\gamma]$. Hence, by (a2), we have: $[\pi_x\,\chi] + [\pi_b\,\chi] < [\pi_x\,\gamma] + [\pi_b\,\gamma]$. By triangle inequality for points π_x, π_b and χ we have: $[\pi_x\,\pi_b] \leqslant [\pi_x\,\chi] + [\chi\,\pi_b]$, hence, be previous inequality and by ($\mathrm{t}_<$) it follows that: $[\pi_x\,\pi_b] < [\pi_x\,\gamma] + [\pi_b\,\gamma]$. From $\mathbf{B}(\pi_a\,\gamma\,\pi_b)$, by (df $+$), it follows that $[\pi_x\,\gamma] + [\pi_b\,\gamma] = [\pi_x\,\pi_b]$. Hence, we have: $[\pi_x\,\pi_b] < [\pi_x\,\pi_b]$, which is a contradiction with ($\mathrm{irr}_<$). In an analogous way we can prove (vi).

Summarizing, by (i), (iii), (v) we have (\$): $B_a \subsetneq \mathrm{B}^{\gamma}_{\pi_x}$ and $\mathrm{B}^{\gamma}_{\pi_x} \cap B_b = \emptyset$, but $\mathrm{B}^{\gamma}_{\pi_x} \not\subseteq \mathrm{B}^{\gamma'}_{\pi_y}$. And, by (ii), (iv) and (vi), we have (\$\$): $B_a \subsetneq \mathrm{B}^{\gamma'}_{\pi_y}$ and $\mathrm{B}^{\gamma'}_{\pi_y} \cap B_b = \emptyset$, but $\mathrm{B}^{\gamma'}_{\pi_y} \not\subseteq \mathrm{B}^{\gamma}_{\pi_x}$.

Now let $c_1, c_2 \in \mathbb{B}$ be any mereological balls such that $\mathbb{int}(c_1) = \mathrm{B}^{\gamma}_{\pi_a}$ and $\mathbb{int}(c_2) = \mathrm{B}^{\delta}_{\gamma'}$. Then, by (\$) and (\$\$), $a \sqsubset c_1 \wr b$, but $c_1 \not\sqsubseteq c_2$ and $a \sqsubset$

$c_2 \wr b$, but $c_2 \not\sqsubseteq c_1$. Hence $\neg\, a\, \mathbf{ET}\, b$, by (df **ET**), which is a contradiction with the main assumption. Hence, we finally get $\exists^1_{\gamma\in\Pi}(\gamma \in \mathbb{fr}(a) \cap \mathbb{fr}(b))$.

"$\Leftarrow$" Suppose that $\mathbb{int}(a) \cap \mathbb{int}(b) = \emptyset$ and $\exists^1_{\gamma\in\Pi}(\gamma \in \mathbb{fr}(a) \cap \mathbb{fr}(b))$. Let $S_a := \mathbb{fr}(a)$, $S_b := \mathbb{fr}(b)$ and $B_a := \mathbb{int}(a)$, $B_b := \mathbb{int}(b)$. Since $\mathbb{int}(a) \cap \mathbb{int}(b) = \emptyset$, so $a \wr b$, by (4).

Now, let x and y be any mereological balls and suppose that (i) $b \sqsubseteq x \wr a$ and (ii) $b \sqsubseteq y \wr a$. Let $S_x := \mathbb{fr}(x)$, $S_y := \mathbb{fr}(y)$ and $B_x := \mathbb{int}(x)$ and $B_y := \mathbb{int}(y)$. First, we will show that points $\gamma, \pi_b, \pi_x, \pi_y$ are collinear. By assumption we have $\gamma \in S_a \cap S_b$. We will show that $\gamma \in S_x$. Suppose towards a contradiction that $\gamma \notin S_x$. Then, by (df $\mathbb{fr}$), there exists a mereological ball $c \in \gamma$ such that $c \wr x$ or $c \sqsubseteq x$. If $c \wr x$, then by assumption (i) we have $b \sqsubseteq x \wr c$, then, by (2) we have $c \wr b$. Hence, again by (df $\mathbb{fr}$), we obtain $\gamma \notin S_b$, which is a contradiction with assumption. So, suppose that $c \sqsubseteq x$. Then, by assumption (i) we have $c \sqsubseteq x \wr a$ and by (2) we obtain $c \wr a$. Hence, by (df $\mathbb{fr}$), it follows that $\gamma \notin S_a$, which is also a contradiction with assumption. In an analogous way we obtain for the sphere S_y that $\gamma \in S_y$.

So $\gamma \in S_b \cap S_x$ and $\gamma \in S_b \cap S_y$. Moreover, $S_b \cap S_x = \{\gamma\} = S_b \cap S_y$. Indeed, to prove the first equality suppose, towards a contradiction, that there exists a point δ such that $\delta \neq \gamma$ and $\delta \in S_b \cap S_x$. By the fact that any closed Euclidean ball is a convex set, it follows that for any ϕ such that $\mathbf{B}(\delta\,\phi\,\gamma)$: $\phi \in B_b$ and $\phi \in B_x$. Hence $B_a \cap B_x \neq \emptyset$, and then $a \bigcirc x$, which is a contradiction with assumption. We can repeat analogous reasoning for spheres S_b and S_y. Hence $S_a \cap S_b = S_b \cap S_x = S_b \cap S_y = \{\gamma\}$. So, by definition of a straight line in Pieri's structure, for some straight lines L, K, M: $\pi_a, \pi_b, \gamma \in L$, $\pi_b, \pi_x, \gamma \in K$, and $\pi_b, \pi_y, \gamma \in M$.

Since $\pi_b, \gamma \in L$, $\pi_b, \gamma \in K$, and $\pi_b, \gamma \in M$, so $K = L = M$, by Fact A.1 and consequently we have that points π_a, γ, π_b, π_x, and π_y are collinear. Now, we will show how they are ordered on a common straight line.

Since $\gamma \neq \pi_x \neq \pi_b$, so either $\mathbf{B}(\gamma\,\pi_x\,\pi_b)$, or $\mathbf{B}(\pi_x\,\pi_b\,\gamma)$, or $\mathbf{B}(\pi_x\,\gamma\,\pi_b)$, by Axiom O4. Suppose that $\mathbf{B}(\gamma\,\pi_x\,\pi_b)$. Then $\pi_b\,\pi_x < \pi_b\,\gamma$ and as a consequence we have $\pi_x \in B_b$. Let $\phi \in B_x$ be an arbitrary point such that $\phi \neq \pi_x$. Then, for points π_b, π_x and ϕ by triangle inequality we have: $[\pi_b\,\phi] \leqslant [\pi_b\,\pi_x] + [\pi_x\,\phi]$. Since $\gamma \in S_x$, so $[\pi_x\,\phi] < [\pi_x\,\gamma]$. Hence $[\pi_x\,\phi] + [\pi_b\,\pi_x] < [\pi_x\,\gamma] + [\pi_b\,\pi_x]$, by (mon$_<$). So $[\pi_b\,\pi_x] + [\pi_x\,\phi] < [\pi_b\,\pi_x] + [\pi_x\,\gamma]$, by (comm$_+$) on p. 560. Since $\mathbf{B}(\gamma\,\pi_x\,\pi_b,)$, so by (df $+$) it follows that $[\pi_b\,\pi_x] + [\pi_x\,\gamma] = [\pi_b\,\gamma]$. Thus, $[\pi_b\,\pi_x] + [\pi_x\,\gamma] < [\pi_b\,\phi] \leqslant$

$[\pi_b\,\pi_x] + [\pi_x\,\phi] < [\pi_b\,\gamma]$. Hence, by triangle inequality for points π_b, π_x, and ϕ and by ($t_<$), it follows that $[\pi_b\,\phi] < [\pi_b\,\gamma]$. Thus, $\pi_b\,\phi < \pi_b\,\gamma$ and $\phi \in B_b$, which means that $B_x \subsetneq B_b$. Hence $x \sqsubseteq b$ and $x \bigcirc b$, which is a contradiction with assumption.

So, suppose that $\mathbf{B}(\pi_x\,\gamma\,\pi_b)$. Since we have $\mathbf{B}(\pi_a\,\gamma\,\pi_b)$, so $\mathbf{B}(\pi_b\,\gamma\,\pi_x)$ and $\mathbf{B}(\pi_b\,\gamma\,\pi_a)$, by Axiom O2. Hence either $\mathbf{B}(\gamma\,\pi_x\,\pi_a)$ or $\mathbf{B}(\gamma\,\pi_a\,\pi_x)$, by (OE). Suppose that $\mathbf{B}(\gamma\,\pi_x\,\pi_a)$. Then we have $\pi_a\,\pi_x < \pi_a\,\gamma$ and it follows that $\pi_x \in B_a$, and consequently we have $x \bigcirc a$, which is a contradiction with the main assumption. Now, if $\mathbf{B}(\gamma\,\pi_a\,\pi_x)$, then $\pi_x\,\pi_a < \pi_x\,\gamma$ and consequently $\pi_a \in B_x$ and we again have $x \bigcirc a$: a contradiction. Thus, we finally get $\mathbf{B}(\pi_x\,\pi_b\,\gamma)$ and, by Axiom O2, we have $\mathbf{B}(\gamma\,\pi_b\,\pi_x)$. Similarly, since $\pi_b \neq \pi_y \neq \gamma$, so either $\mathbf{B}(\gamma\,\pi_y\,\pi_b)$ or $\mathbf{B}(\pi_y\,\pi_b\,\gamma)$ or $\mathbf{B}(\pi_y\,\gamma\,\pi_b)$, by Axiom O4. Thus, in an analogous way, we prove that $\mathbf{B}(\gamma\,\pi_b\,\pi_y)$.

So, we have $\mathbf{B}(\pi_x\,\pi_b\,\gamma)$ and $\mathbf{B}(\gamma\,\pi_b\,\pi_y)$. Therefore $\mathbf{B}(\pi_b\,\pi_x\,\pi_y)$ or $\mathbf{B}(\pi_b\,\pi_y\,\pi_x)$, by (OE). Since points γ, π_x, π_y are collinear, so by definition of a straight line in Pieri's structure we obtain $S_x \cap S_y = \{\gamma\}$.

Thus, in an analogous way as in the proof of the fact that $B_x \subsetneq B_b$, we get in this case following: $B_x \subsetneq B_y$ or $B_y \subsetneq B_x$. Hence, it follows that for mereological balls x and y either $x \sqsubseteq y$ or $y \sqsubseteq x$, and we finally get $a\,\mathbf{ET}\,b$. □

3. The notion of *congruence of mereological balls*

Before we defined the notion of the *congruence of mereological balls* we will make an observation leading to the way in which we can define this notion in T*-structures. For this purpose, let us consider mereological balls a and b such that $a \circledcirc b$ and $a \sqsubset b$. By Fact 1.4, the operation $\mathtt{int}$ transforms the balls a and b onto the Euclidean balls B_a and B_b which have a common center. Let us denote this center by γ. Hence $\gamma = \pi_a = \pi_b$. Since $a \sqsubset b$, we additionally have $B_a \subset B_b$, by (3). Let us consider Euclidean balls B_c, $B_{c'}$ such that: $S_c \cap S_a = \{\alpha\}$ and $S_c \cap S_b = \{\alpha'\}$, and $S_{c'} \cap S_a = \{\beta\}$ and $S_{c'} \cap S_b = \{\beta'\}$.

It is easy to show that the Euclidean balls B_c and $B_{c'}$ defined in such manner have the same diameter in geometrical sense. Indeed, we will prove something more.

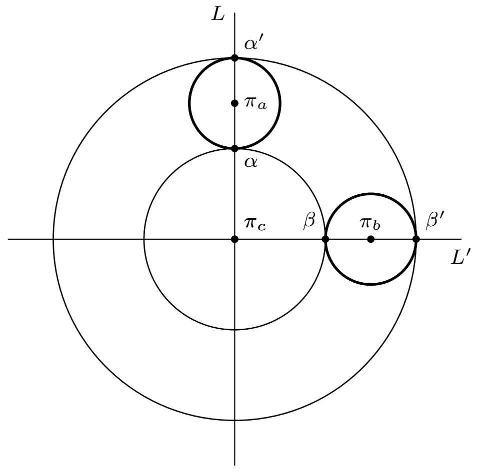

Figure 3. Construction in the proof of the Lemma 3.1

LEMMA 3.1. *Suppose that mereological balls a, b, c, and c' satisfy the following condition:*

$$c \circledcirc c' \ \wedge\ a\,\mathbf{ET}\,c \ \wedge\ b\,\mathbf{ET}\,c \ \wedge\ a\,\mathbf{IT}\,c' \ \wedge\ b\,\mathbf{IT}\,c'. \tag{$\diamond$}$$

Let $\mathbb{fr}(c) \cap \mathbb{fr}(a) = \{\alpha\}$, $\mathbb{fr}(c) \cap \mathbb{fr}(b) = \{\beta\}$, $\mathbb{fr}(c') \cap \mathbb{fr}(a) = \{\alpha'\}$ and $\mathbb{fr}(c') \cap \mathbb{fr}(b) = \{\beta'\}$. Then:

$$\pi_c\,\alpha \equiv \pi_c\,\beta,\ \pi_c\,\alpha' \equiv \pi_c\,\beta' \text{ and } \alpha\,\alpha' \equiv \beta\,\beta'.$$

PROOF. Let $S_a := \mathbb{fr}(a)$, $S_b := \mathbb{fr}(b)$, $S_c := \mathbb{fr}(c)$, $\mathrm{S}_{c'} := \mathbb{fr}(c')$, $B_c := \mathbb{int}(c)$ and $B_{c'} := \mathbb{int}(c')$ (see Figure 3). Since $\alpha \in S_c$ and $\beta \in S_c$, so $\alpha\,\beta\,\Delta\,\pi_c$, and consequently $\pi_c\,\alpha \equiv \pi_c\,\beta$. Similarly, since $\alpha' \in \mathrm{S}_{c'}$ and $\beta \in \mathrm{S}_{c'}$, so by $\pi_c = \pi_{c'}$ we obtain $\pi_c\,\alpha' \equiv \pi_c\,\beta'$. Next, since $S_c \cap S_a = \{\alpha\}$ and $S_c \cap S_b = \{\beta\}$, so by definition of a straight line in Pieri's structure, there are straight lines L and L' such that $\pi_c, \alpha, \pi_a \in L$ and $\pi_c, \beta, \pi_b \in L'$. Since $\mathrm{S}_{c'} \cap S_a = \{\alpha'\}$ and $\mathrm{S}_{c'} \cap S_b = \{\beta'\}$, so there are also straight lines K and K' such that $\pi_c, \alpha', \pi_a \in K$ and $\pi_c, \beta, \pi_b \in K'$. Hence $\pi_c, \pi_a \in L$, $\pi_c, \pi_a \in K$, $\pi_c, \pi_b \in L'$, and $\pi_c, \pi_b \in K'$.

Hence, by Fact A.1, we have $L = K$ and $L' = K'$. We also have $\mathbf{B}(\pi_c\,\alpha\,\alpha')$. Indeed, suppose towards a contradiction that $\mathbf{B}(\pi_c\,\alpha'\,\alpha)$. Then, by (df $<$) on p. 559, we have $\pi_c\,\alpha' < \pi_c\,\alpha$, which means that $B_{c'} \subseteq B_c$. Thus $c' \sqsubset c$. So we have a contradiction with ($\diamond$). In an analogous way we obtain $\mathbf{B}(\pi_c\,\beta\,\beta')$.

Thus, we have $\mathbf{B}(\pi_c\,\alpha\,\alpha')$, $\mathbf{B}(\pi_c\,\beta\,\beta')$, $\pi_c\,\alpha \equiv \pi_c\,\beta$, and $\pi_c\,\alpha' \equiv \pi_c\,\beta'$, So, by fact (C4′), we obtain $\alpha\,\alpha' \equiv \beta\,\beta'$. □

The property of the relation $\circledcirc$ expressed in Lemma 3.1 shows the way in which it can be used to define the relation of the congruence of mereological balls. For this purpose we will use as the definition of constructed relation the condition ($\diamond$) expressed in this lemma. After these considerations, let us adopt the following definition of the relation $\mathbb{diam} \subseteq \mathbb{B} \times \mathbb{B}$ of *congruence of mereological balls*:

$$a\,\mathbb{diam}\,b \overset{\text{df}}{\iff} \exists_{c,\,c'\in\mathbb{B}}(c \circledcirc c' \;\wedge\; a\,\mathbf{ET}\,c \;\wedge\; b\,\mathbf{ET}\,c \;\wedge\; a\,\mathbf{IT}\,c' \;\wedge\; b\,\mathbf{IT}\,c'). \tag{df diam}$$

According to (df diam), we will say that the mereological balls a and b are congruent iff we find a pair of concentric balls c and c' in which the balls a and b are externally tangent to ball c and internally tangent to ball c'.

Now, we will prove that the relation diam defined in such a way is an equivalence relation in the set of all mereological balls.

THEOREM 3.2. *The relation* diam *has following properties*:

(i) *for any $a \in \mathbb{B}$ we have $a\,\mathbb{diam}\,a$,*
(ii) *for all $a, b \in \mathbb{B}$, if $a\,\mathbb{diam}\,b$, then $b\,\mathbb{diam}\,a$,*
(iii) *for all $a, b, c \in \mathbb{B}$, if $a\,\mathbb{diam}\,b$ and $b\,\mathbb{diam}\,c$, then $a\,\mathbb{diam}\,c$.*

PROOF. *Ad* (i) Let $a \in \mathbb{B}$. Let $S_a := \mathbb{fr}(a)$ and $B_a := \mathbb{int}(a)$. Let α be an arbitrary point such that $\alpha \in S_a$ and let $L(\pi_a, \alpha)$ be the straight line crossing center of the sphere S_a and point α. Straight line $L(\pi_a, \alpha)$ intersects the sphere S_a in point $\alpha' \neq \alpha$, so $\mathbf{B}(\alpha'\,\pi_a\,\alpha)$. By Axiom O5, there exists a point $\gamma \in L(\pi_a, \alpha)$ such that $\mathbf{B}(\pi_a\,\alpha\,\gamma)$. Hence and by $\mathbf{B}(\alpha'\,\pi_a\,\alpha)$ we have $\mathbf{B}(\alpha'\,\alpha\,\gamma)$, by (OA). Hence and by $\mathbf{B}(\alpha'\,\pi_a\,\alpha)$ we obtain $\mathbf{B}(\alpha\,\pi_a\,\alpha')$ and $\mathbf{B}(\gamma\,\alpha\,\alpha')$, by Axiom O2, and we have $\mathbf{B}(\gamma\,\pi_a\,\alpha')$, by (OB).

First, let us consider the sphere $\mathrm{S}^{\alpha}_{\gamma}$ whose center is the point γ and going through α. Since $\mathbf{B}(\pi_a\alpha\gamma)$, we have $[\pi_a\,\gamma] = [\pi_a\,\alpha] + [\gamma\,\alpha]$, by Fact A.6. Since $\alpha \in S_a$ and $\alpha \in S^{\alpha}_{\gamma}$, so by Fact A.8(i) we have that $S_a \cap S^{\alpha}_{\gamma} = \{\alpha\}$ and $B_a \cap B^{\alpha}_{\gamma} = \emptyset$. Hence, by Theorem 4.8, we have that for mereological ball c such that $\mathbb{int}(c) = B^{\alpha}_{\gamma}$ and $\mathbb{fr}(c) = S^{\alpha}_{\gamma}$, (a): $a\,\mathbf{ET}\,c$. Now, let us consider the sphere $S^{\alpha'}_{\gamma}$. Since points α', π_a and γ are collinear, so by definition of a straight line in Pieri's structure we have that $S_a \cap S^{\alpha'}_{\gamma} = \{\alpha'\}$. Let ϕ be an arbitrary point such that $\phi \in B_a$. By triangle inequality, for points γ, π_a and ϕ we have: $[\gamma\,\phi] \leqslant [\gamma\,\pi_a]+[\pi_a\,\phi]$. Since $\alpha' \in S_a$, so $[\pi_a\,\phi] < [\pi_a\,\alpha']$ and by (mon$_<$) (see Fact A.5) we have: $[\gamma\,\pi_a]+[\pi_a\,\phi] < [\gamma\,\pi_a]+[\pi_a\,\alpha']$. From $\mathbf{B}(\gamma\,\pi_a\,\alpha')$ we have $[\gamma\,\pi_a]+[\pi_a\,\alpha'] = [\gamma\,\alpha']$ and by previous inequalities we obtain $[\gamma\,\phi] < [\gamma\,\alpha']$, by (t$_<$) (see Fact A.5). Hence $\gamma\,\phi < \gamma\,\alpha'$, which means that $\phi \in B_\gamma^{\alpha'}$. So, we have $S_a \cap S_\gamma^{\alpha'} = \{\alpha'\}$ and $B_a \subseteq B_\gamma^{\alpha'}$.

Hence, by Theorem 2.1, for mereological ball c' such that $\mathbb{int}(c') = B_\gamma^{\alpha'}$ and $\mathbb{fr}(c') = S_\gamma^{\alpha'} = \{\alpha'\}$ we have (b): $a\,\mathbf{IT}\,c'$. Since the point γ is the center of Euclidean balls B_γ^{α} and $B_\gamma^{\alpha'}$, so $\{c, c'\} \subseteq \gamma$. Hence (c): $c \circledcirc c'$. We also have $\gamma\,\alpha < \gamma\,\alpha'$, by (df $<$), since $\mathbf{B}(\gamma\,\alpha\,\alpha')$. Hence $B_\gamma^{\alpha} \subseteq B_\gamma^{\alpha'}$ and we obtain (d): $c \sqsubset c'$. By (a)–(d) and (df $\mathbb{diam}$), we finally get $a\,\mathbb{diam}\,a$.

Ad (ii) Directly from definition.

Ad (iii) Suppose that $a\,\mathbb{diam}\,b$ and $b\,\mathbb{diam}\,c$. We will show how to construct a pair of mereological balls that satisfy the condition defining the relation $\mathbb{diam}$ for balls a and c (see Figure 4). According to assumption and (df $\mathbb{diam}$), for some mereological balls x, x' we have:

$$x \circledcirc x' \wedge a\,\mathbf{ET}\,x \wedge b\,\mathbf{ET}\,x \wedge a\,\mathbf{IT}\,x' \wedge b\,\mathbf{IT}\,x',$$

and for some mereological balls y, y' we have:

$$y \circledcirc y' \wedge b\,\mathbf{ET}\,y \wedge c\,\mathbf{ET}\,y \wedge b\,\mathbf{IT}\,y' \wedge c\,\mathbf{IT}\,y'.$$

Hence, by theorems 4.8 and 2.1, for mereological balls a and b we have:

$$\mathbb{fr}(a) \cap \mathbb{fr}(x) = \{\alpha\} \text{ and } \mathbb{fr}(a) \cap \mathbb{fr}(x') = \{\alpha'\},$$
$$\mathbb{fr}(b) \cap \mathbb{fr}(x) = \{\beta\} \text{ and } \mathbb{fr}(b) \cap \mathbb{fr}(x') = \{\beta'\},$$

and for mereological balls b and c we have:

$$\mathbb{fr}(b) \cap \mathbb{fr}(y) = \{\gamma\} \text{ and } \mathbb{fr}(b) \cap \mathbb{fr}(y') = \{\gamma'\},$$
$$\mathbb{fr}(c) \cap \mathbb{fr}(y) = \{\delta\} \text{ and } \mathbb{fr}(c) \cap \mathbb{fr}(y') = \{\delta'\}.$$

By Lemma 4.1 we have $\alpha\,\alpha' \equiv \beta\,\beta'$ and $\gamma\,\gamma' \equiv \delta\,\delta'$. Let $\mathbb{fr}(a) := S_a$, $\mathbb{fr}(b) := S_b$, $\mathbb{fr}(c) := S_c$, $\mathbb{fr}(x) := S_x$, $\mathbb{fr}(x') := \mathrm{S}_{x'}$. Since points α, π_a, α' are collinear and $\{\alpha, \alpha'\} \subseteq S_a$, so by (df mid) (see Appendix) we have (i): $\pi_a := \mathrm{mid}(\alpha, \alpha')$. Analogously, for spheres S_b and S_c we have (ii): $\pi_b := \mathrm{mid}(\beta, \beta')$, (iii): $\pi_b := \mathrm{mid}(\gamma, \gamma')$, and (iv): $\pi_c := \mathrm{mid}(\delta, \delta')$.

Since $\beta, \beta', \gamma, \gamma' \in S_b$, so by (i) and (iii) we have $\beta\,\beta' \equiv \gamma\,\gamma'$. So, we obtain $\alpha\,\alpha' \equiv \beta\,\beta'$, $\beta\,\beta' \equiv \gamma\,\gamma'$, and $\gamma\,\gamma' \equiv \delta\,\delta'$. Hence $\alpha\,\alpha' \equiv \delta\,\delta'$, by (t$_\equiv$). Let us consider the segment $\pi_a\,\pi_c$ and let K be its bisector. Let

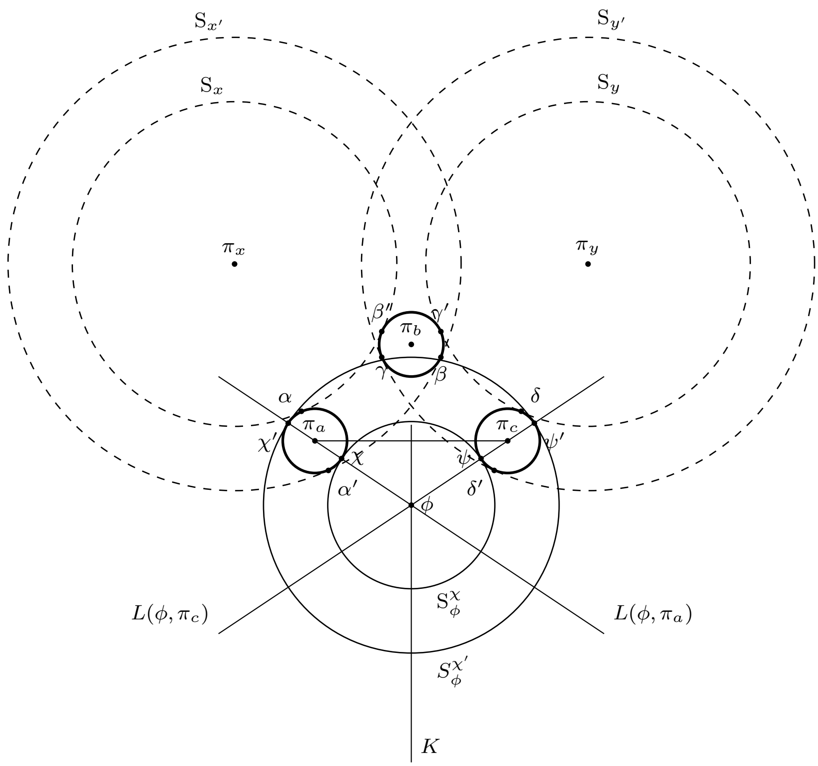

Figure 4. Construction in the proof of transitivity of the relation $\mathbb{diam}$

$\phi \in K$ be any point such that: $\phi \in B_a \cup S_a \cup B_c \cup S_c$. Let $L(\phi, \pi_a)$ be the straight line crossing ϕ and π_a and let $L(\phi, \pi_c)$ be the straight line crossing ϕ and π_c. Since the straight line $L(\phi, \pi_a)$ is crossing the center of the sphere S_a, it intersects the sphere S_a in points χ and χ'. Similarly, since the straight line $L(\phi, \pi_c)$ crossing center of the sphere S_c, so it intersects the sphere S_c in points ψ and ψ'. Moreover, suppose that $\mathbf{B}(\pi_a\, \chi\, \phi)$ and $\mathbf{B}(\pi_c\, \psi\, \phi)$. By Fact A.7, we have $\phi\, \pi_a \equiv \phi\, \pi_c$. Since $\alpha\, \alpha' \equiv \delta\, \delta'$, so by (i), (iv) and Fact A.3 we have $\pi_a\, \alpha \equiv \pi_c\, \delta$. Next, since $\chi, \alpha \in S_a$ and $\psi, \delta \in S_c$, so we also have $\pi_a\, \chi \equiv \pi_a\, \alpha$ and $\pi_c\, \psi \equiv \pi_c\, \delta$. Hence, by ($\mathrm{t}_{\equiv}$), we have: $\pi_a\, \chi \equiv \pi_c\, \psi$. Since $\phi\, \pi_a \equiv \phi\, \pi_c$, by $\mathbf{B}(\pi_a\, \chi\, \phi)$ and $\mathbf{B}(\pi_c\, \psi\, \phi)$, we have $\phi\, \chi \equiv \phi\, \psi$, by (C4′). Thus, $\chi\psi\ \Delta\ \phi$. So, by definition of a sphere in Pieri's structure, we finally get $\chi, \psi \in \mathrm{S}^{\chi}_{\phi}$.

Again, using above ordering, we have: $[\pi_a\, \phi] = [\phi\, \chi] + [\pi_a\, \chi]$ and $[\pi_c\, \psi] = [\phi\, \psi] + [\pi_c\, \psi]$. Hence, by Fact A.8(i), we have $\mathrm{S}^{\chi}_{\phi} \cap S_a = \{\chi\}$,

$\mathrm{B}_\phi^\chi \cap B_a = \emptyset$, $S_\phi^\psi \cap S_c = \{\psi\}$, and $\mathrm{B}_\phi^\chi \cap B_c = \emptyset$. By Fact 1.4, there exists exactly one mereological ball z such that $\mathbb{int}(z) = \mathrm{B}_\phi^\chi$ and $\mathbb{fr}(z) = S_\chi^\psi$. Hence, by Theorem 4.8, we have (a): $a\,\mathbf{ET}\,z$; and (b): $c\,\mathbf{ET}\,z$.

Now, we will construct a mereological ball which is concentric with ball z. From $\mathbf{B}(\pi_a\,\chi\,\phi)$ and $\mathbf{B}(\pi_c\,\psi\,\phi)$ we have $\mathbf{B}(\phi\,\chi\,\pi_a)$ and $\mathbf{B}(\phi\,\psi\,\pi_c)$, by Axiom O2. Next, since $\pi_a = \mathrm{mid}(\chi,\chi')$ and $\pi_c = \mathrm{mid}(\psi,\psi')$, so $\mathbf{B}(\chi\,\pi_a\,\chi')$ and $\mathbf{B}(\psi\,\pi_c\,\psi')$, and, by Axiom O7, we obtain $\mathbf{B}(\phi\,\chi\,\chi')$ and $\mathbf{B}(\phi\,\psi\,\psi')$. Let us observe that $\alpha\,\alpha' \equiv \chi\,\chi'$ and $\psi\,\psi' \equiv \delta\,\delta'$, which means by $\alpha\,\alpha' \equiv \delta\,\delta'$ and ($\mathrm{t}_\equiv$) that $\chi\,\chi' \equiv \psi\,\psi'$. Hence, by $\mathbf{B}(\phi\,\chi\,\chi')$ and $\mathbf{B}(\phi\,\psi\,\psi')$ we have $\phi\,\chi' \equiv \phi\,\psi'$, by (C4). Thus $\chi'\,\psi'\,\Delta\,\phi$ and, by definition of a sphere in a Pieri's structure, we have $\chi', \psi' \in \mathrm{S}_\phi^{\chi'}$. Hence, we have: $\mathrm{S}_\phi^{\chi'} \cap S_a = \{\chi'\}$, and $B_a \subseteq \mathrm{B}_\phi^{\chi'}$, $\mathrm{S}_\phi^{\chi'} \cap S_c = \{\psi'\}$, and $B_c \subseteq \mathrm{B}_\phi^{\chi'}$.

By Fact 1.4, there exists exactly one mereological ball z' such that $\mathbb{int}(z') = B_\phi^{\chi'}$ and $\mathbb{fr}(z') = S_\phi^{\chi'}$. Hence, by Theorem 2.1 we obtain (c): $a\mathbf{IT}\,z'$; and (d): $c\mathbf{IT}\,z'$. Since Euclidean balls B_ϕ^χ and $B_\phi^{\chi'}$ have a common center point ϕ, so $z, z' \in \phi$. Thus, (e): $z \circledcirc z'$.

In result, by (a)–(e), we have: $z \circledcirc z'$, $a\,\mathbf{ET}\,z$, $c\,\mathbf{ET}\,z$, $a\,\mathbf{IT}\,z'$, and $c\,\mathbf{IT}\,z'$. Hence $a\,\mathbb{diam}\,c$, by (df $\mathbb{diam}$). □

In further consideration, for any $b \in \mathbb{B}$, its equivalence class of the relation $\mathbb{diam}$ will be denoted as $\mathbb{d}_b$. So, we put:

$$\mathbb{d}_b := \|b\|_{\mathbb{diam}}. \tag{df $\mathbb{d}_b$}$$

Hence for all $a, b \in \mathbb{B}$ we have:

$$\mathbb{d}_a = \mathbb{d}_b \iff a\,\mathbb{diam}\,b\,.$$

Let $\mathbb{D}$ be the set of all equivalence classes of the relation $\mathbb{diam}$, i.e.:

$$\mathbb{D} := \{\|a\|_{\mathbb{diam}} : b \in \mathbb{B}\}. \tag{df $\mathbb{D}$}$$

Elements from the set $\mathbb{D}$ will be denote by variables '$\mathbb{x}$', '$\mathbb{y}$', '$\mathbb{z}$' etc. According to these considerations, for all $b \in \mathbb{B}$ and $\mathbb{x} \in \mathbb{D}$ we have:

$$b \in \mathbb{x} \iff \mathbb{x} = \mathbb{d}_b\,. \tag{9}$$

Now, we can adopt the following definition of the *diameter of mereological ball*:

DEFINITION 3.1. The diameter of any mereological ball is its equivalence class of the relation $\mathbb{diam}$.

The set $\mathbb{D}$ of all equivalence classes of the relation $\mathbb{diam}$ will be called *the set of diameters*.

4. Properties of the relation $\mathbb{diam}$ and the class $\mathbb{D}$

The first property of the relation $\mathbb{diam}$ described below shows the adequacy of its definition. We will show that the images under operation $\mathbb{int}$ of the mereological balls of the same diameter in the sense of the relation $\mathbb{diam}$ have the same diameter in the geometrical sense.

THEOREM 4.1. *For all $a, b \in \mathbb{B}$ and all $\alpha \in \mathbb{fr}(a)$, $\beta \in \mathbb{fr}(b)$:*

$$\mathbb{d}_a = \mathbb{d}_b \iff \pi_a\,\alpha \equiv \pi_b\,\beta.$$

PROOF. "$\Rightarrow$" Let $a, b \in \mathbb{B}$ and let $\alpha \in \mathbb{fr}(a)$ and $\beta \in \mathbb{fr}(b)$. Suppose that $\mathbb{d}_a = \mathbb{d}_b$. Then $a \,\mathbb{diam}\, b$ and, by Lemma 3.1, for $\alpha' \in \mathbb{fr}(a)$ and $\beta' \in \mathbb{fr}(b)$ we have $\pi_a\,\alpha' \equiv \pi_b\,\beta'$. Since $\alpha \in \mathbb{fr}(a)$ and $\beta \in \mathbb{fr}(b)$, so $\pi_a\,\alpha \equiv \pi_a\,\alpha'$ and $\pi_b\,\beta \equiv \pi_b\,\beta'$. Hence, by ($t_{\equiv}$), we have $\pi_a\,\alpha \equiv \pi_b\,\beta$.

"$\Leftarrow$" Let a and b be an arbitrary mereological balls and suppose that $\alpha \in \mathbb{fr}(a)$ and $\beta \in \mathbb{fr}(b)$. Moreover, suppose that $\pi_a\,\alpha \equiv \pi_b\,\beta$. Let L be a bisector of the segment $\pi_a\,\pi_b$. Let γ be a point lying on the straight line L such that $\gamma \notin S_a \cup B_a \cup S_b \cup B_b$. Then, the straight line $L(\gamma, \pi_a)$ is crossing a sphere S_a in points δ, δ' and the straight line $L(\gamma, \pi_b)$ is crossing a sphere S_b in points ϕ, ϕ'. Moreover, suppose that $\mathbf{B}(\pi_a\,\delta\,\gamma)$ and $\mathbf{B}(\pi_b\,\phi\,\gamma)$. Since $\alpha, \delta \in \mathbb{fr}(a)$ and $\beta, \phi \in \mathbb{fr}(b)$, so taking spheres S^{δ}_{γ} and $S^{\delta'}_{\phi}$ and then using an analogous construction to the one that was used to prove transitivity of the relation $\mathbb{diam}$ for mereological balls c and c' such that $\mathbb{fr}(c) = S^{\delta}_{\gamma}$ and S^{ϕ}_{γ}, we obtain: $c \circledcirc c'$, $a\,\mathbf{ET}\,c$, $b\,\mathbf{ET}\,c$, $a\,\mathbf{IT}\,c'$, and $b\,\mathbf{IT}\,c'$, which means that $a \,\mathbb{diam}\, b$, thus $\mathbb{d}_a = \mathbb{d}_b$. □

FACT 4.2. *For any $a, b \in \mathbb{B}$, if $a \sqsubset b$, then $\mathbb{d}_a \neq \mathbb{d}_b$.*

PROOF. Let $\alpha \in \mathbb{fr}(x)$ and $\beta \in \mathbb{fr}(y)$, and suppose that $a \sqsubset b$. Then, by (3) and ($\text{antis}_{\sqsubseteq}$), we have $\mathbb{int}(a) \subsetneq \mathbb{int}(b)$. Hence, of course, $\pi_a\ \alpha < \pi_b\ \beta$ and by (df $<$) we have $\pi_a\ \alpha \not\equiv \pi_b\ \beta$. Hence, by Theorem 4.1, we have $\mathbb{d}_a \neq \mathbb{d}_b$. □

According to (df Π), a point in $\mathrm{T}^{\star}$-structure is any set of mereological balls concentric to a given ball. The next fact says that in each point and for any element of the set $\mathbb{D}$ there exists a mereological ball of diameter equal to that element. This fact expresses in terms of geometry of solids the Euclidean postulate about the possibility of constructing a circle with any radius around each point.

FACT 4.3. *For all $\alpha \in \Pi$ and $\mathbb{x} \in \mathbb{D}$ there is $b \in \alpha$ such that $\mathbb{d}_b = \mathbb{x}$.*

PROOF. Let $\alpha \in \Pi$ and $\mathbb{x} \in \mathbb{D}$. Suppose that $x \in \mathbb{x}$. Then, by Fact 1.2, there is $\beta \in \Pi$ such that $\beta \neq \pi_x$ and $\{x\} = \Pi^{\beta}_{\pi_x} := \{b \in \mathbb{B} : b \in \pi_x \wedge \beta \in \mathbb{fr}(b)\}$. So $\beta \in \mathbb{fr}(x)$. Let γ be the point such that $\alpha\,\gamma \equiv \pi_x\,\beta$. Since $\beta \neq \pi_x$, so $\alpha \neq \gamma$ and, by Fact 1.1, for some $b \in \mathbb{B}$ we have $\{b\} = \mathbb{B}^{\gamma}_{\alpha}$. So $b \in \alpha$ and $\gamma \in \mathbb{fr}(b)$. But by (7), we have $\alpha = \pi_b$. Thus, by Theorem 4.1, we have $\mathbb{d}_b = \mathbb{d}_x = \mathbb{x}$. $\square$

Fact 4.3 allows to show some connection between classes Π and $\mathbb{D}$. For this purpose, for any non-empty set of mereological balls $X \in 2^{\mathbb{B}} \setminus \{\emptyset\}$ we assign the set $\mathbb{d}[X]$ of diameters of these balls, i.e., we put:

$$\mathbb{d}[X] := \{\mathbb{x} \in \mathbb{D} : \exists_{b \in X}\ \mathbb{d}_b = \mathbb{x}\}.$$

Directly from Fact 4.3 we obtain:

FACT 4.4. *For any $\alpha \in \Pi$ we have $\mathbb{d}[\alpha] = \mathbb{D}$.*

Fact 4.4 says that every point from Π «generates» whole class $\mathbb{D}$. Also every element of $\mathbb{D}$ generates the set Π. For this purpose, for any non-empty set of mereological balls $X \in 2^{\mathbb{B}} \setminus \{\emptyset\}$ we assign the set $\pi[X]$ of all points generates by these balls, i.e., we put:

$$\pi[X] := \{\alpha \in \Pi : \exists_{a \in X}\ \pi_a = \alpha\}.$$

FACT 4.5. *For any $\mathbb{x} \in \mathbb{D}$ we have $\pi[\mathbb{x}] = \Pi$.*

PROOF. Let $\alpha \in \Pi$. Then, by Fact 4.3, for some $a \in \alpha$ we have $\mathbb{d}_a = \mathbb{x}$. So $a \in \mathbb{x}$, by (9). Moreover, $\alpha = \pi_a$, by (7). Thus, $\alpha \in \pi[\mathbb{x}]$. $\square$

Let us observe that without additional assumptions it is impossible to generate the element $\mathbb{x}$ of $\mathbb{D}$ by whole set Π in manner described above. It is so because the only «information» that is included in the set Π about the class $\mathbb{x}$ is that the ball of diameter $\mathbb{x}$ exists in each point of Π, without distinguishing any point from $\mathbb{D}$.

The fact below is concerned with pairs of mereological balls which generate one another the fringe points. We will show that in such a case these balls have the same diameter.

Fact 4.6. *For any $a, b \in \mathbb{B}$, if $\pi_a \in \mathbb{fr}(b)$ and $\pi_b \in \mathbb{fr}(a)$, then $\mathbb{d}_a = \mathbb{d}_b$.*

Proof. Let $a, b \in \mathbb{B}$, $\pi_a \in \mathbb{fr}(b)$, and $\pi_b \in \mathbb{fr}(a)$. Let $\alpha \in \mathbb{fr}(a)$ and $\beta \in \mathbb{fr}(b)$. Then $\beta\,\pi_a \,\Delta\, \pi_b$ and $\alpha\,\pi_b \,\Delta\, \pi_a$. So $\beta\,\pi_b \equiv \pi_a\,\pi_b$ and $\alpha\,\pi_a \equiv \pi_b\,\pi_a$. Hence $\beta\,\pi_b \equiv \alpha\,\pi_a$, by ($t_\equiv$). Hence $\mathbb{d}_a = \mathbb{d}_b$, by Theorem 4.1. □

Fact 4.7. *For any different points $\alpha, \beta \in \Pi$ there are $a \in \alpha$ and $b \in \beta$ such that $\mathbb{d}_a = \mathbb{d}_b$ and $a\,\mathbf{ET}\,b$.*

Proof. Let α and β be any different points. Let $\gamma = \mathrm{mid}(\alpha, \beta)$ and let us consider spheres S^γ_α and S^γ_β. Since $\gamma \in \mathrm{S}^\gamma_\alpha$ and $\gamma \in \mathrm{S}^\gamma_\beta$, so by Fact A.8(i) we have $\alpha\,\gamma \equiv \beta\,\gamma$. Hence, by Theorem 4.1 it follows that for balls $a, b \in \mathbb{B}$ such that $\mathbb{fr}(a) = \mathrm{S}^\gamma_\alpha$ and $\mathbb{fr}(b) = \mathrm{S}^\gamma_\beta$ holds $\mathbb{d}_a = \mathbb{d}_b$. Since $\mathbf{B}(\alpha\,\gamma\,\beta)$, so $[\alpha\,\beta] = [\alpha\,\gamma] + [\gamma\,\beta]$ and by Fact A.8, (i) we obtain $\mathrm{S}^\gamma_\alpha \cap \mathrm{S}^\gamma_\beta = \{\gamma\}$ and $\mathrm{B}^\gamma_\alpha \cap \mathrm{B}^\gamma_\beta = \emptyset$. Hence $a\,\mathbf{ET}\,b$, by Theorem 2.2. □

Lemma 4.8. *For any $a, b \in \mathbb{B}$ there are balls $a' \in \mathbb{d}_a$ and $b' \in \mathbb{d}_b$ such that $a'\,\mathbf{ET}\,b'$.*

Proof. Let $a, b \in \mathbb{B}$. Let $\mathbb{fr}(a) = S_a$, $\mathbb{int}(a) = B_a$ and $\mathbb{fr}(b) = S_b$. Let $\alpha \in S_a$ and let $L(\pi_a, \alpha)$ be a straight line crossing center of the sphere S_a and point α. Let γ be an arbitrary point lying on straight line $L(\pi_a, \alpha)$ such that: $\mathbf{B}(\pi_a\,\alpha\,\gamma)$ and $(\gamma\,\alpha \equiv \pi_b\,\beta)$. Let us consider the sphere S^α_γ. For a construction analogous to the one in the proof of reflexivity of the relation $\mathbb{diam}$ we obtain $S_a \cap S^\alpha_\gamma = \{\alpha\}$ and $B_a \cap B^\alpha_\gamma = \emptyset$. Thus, by Fact 1.4 and Theorem 2.2, for mereological ball c such that $\mathbb{int}(c) = B^\alpha_\gamma$ we have $a\,\mathbf{ET}\,c$. Since $\gamma\,\alpha \equiv \pi_b\,\beta$, we have $\mathbb{d}_c = \mathbb{d}_b$, by Theorem 4.1. □

As the class $\mathbb{d}_a$ was considered as *diameter* of a given mereological ball a, it is natural to consider the point π_a as its *center*. It seems that these two notions are sufficient to unambiguously characterise each mereological ball in the universe of solids. Therefore we obtain:

Theorem 4.9. *For any $a, b \in \mathbb{B}$: if $\pi_a = \pi_b$ and $\mathbb{d}_a = \mathbb{d}_b$, then $a = b$.*

Proof. Let a and b be any mereological balls. Suppose that $\pi_a = \pi_b$ and $\mathbb{d}_a = \mathbb{d}_b$. Let $B_a = \mathbb{int}(a)$ and $B_b = \mathbb{int}(b)$. Let $\alpha \in \mathbb{fr}(a)$ and $\beta \in \mathbb{fr}(b)$. Since $\mathbb{d}_a = \mathbb{d}_b$, so $\pi_a\,\alpha \equiv \pi_b\,\beta$, by Theorem 4.1. From axioms of Pieri's structures it follows that: (i) $\forall_{\alpha,\beta,\gamma\in\Pi}(\alpha\beta \,\Delta\, \gamma \iff \alpha\gamma\,\mathbf{D}\,\beta\gamma)$ and (ii) $\forall_{\alpha,\beta,\gamma\in\Pi}(\alpha\beta \,\Delta\, \gamma \iff \mathrm{B}^\alpha_\gamma = \mathrm{B}^\beta_\gamma)$ (see [2, pp. 4 and 12]). By assumption we have $\pi_a\,\alpha \equiv \pi_a\,\beta$. So we have $\alpha\beta\,\Delta\,\pi_a$, by (i) and (df $\equiv$). Hence $B_a = \mathrm{B}^\alpha_{\pi_a} = \mathrm{B}^\beta_{\pi_a} = B_b$, by assumptions and (ii). Thus, $B_a = B_b$. Hence $a = b$, by (3) and ($\mathrm{antis}_\sqsubseteq$). □

By means of relations $\sqsubset$ and $\sqsubseteq$ in $\mathbb{D}$ we define two binary relations $<_{\mathbb{d}}$ and $\leq_{\mathbb{d}}$ which allows to compare diameters. For any $\mathbb{x}, \mathbb{y} \in \mathbb{D}$ we put:

$$\mathbb{x} <_{\mathbb{d}} \mathbb{y} \overset{\text{df}}{\Longleftrightarrow} \exists_{a \in \mathbb{x}} \exists_{b \in \mathbb{y}}\, a \sqsubset b\,, \tag{df $<_{\mathbb{d}}$}$$

$$\mathbb{x} \leq_{\mathbb{d}} \mathbb{y} \overset{\text{df}}{\Longleftrightarrow} \exists_{a \in \mathbb{x}} \exists_{b \in \mathbb{y}}\, a \sqsubseteq b\,. \tag{df $\leq_{\mathbb{d}}$}$$

By (df $\sqsubset$), (df $\leq_{\mathbb{d}}$), (df $<_{\mathbb{d}}$) and by Fact 4.2 it follows that:

$$<_{\mathbb{d}} \; = \; \leq_{\mathbb{d}} \setminus \mathrm{id},$$

that is, for any $\mathbb{x}, \mathbb{y} \in \mathbb{D}$:

$$\mathbb{x} <_{\mathbb{d}} \mathbb{y} \iff \mathbb{x} \leq_{\mathbb{d}} \mathbb{y} \wedge \mathbb{x} \neq \mathbb{y}. \tag{10}$$

PROOF. "$\Rightarrow$" Suppose that $\mathbb{x} <_{\mathbb{d}} \mathbb{y}$. By (df $<_{\mathbb{d}}$) there are $a \in \mathbb{x}$ and $b \in \mathbb{y}$ such that $a \sqsubset b$. Then $\mathbb{d}_a \neq \mathbb{d}_b$, by Fact 4.2. So $\mathbb{x} \neq \mathbb{y}$. Moreover, we have $a \sqsubseteq b$. Hence $\mathbb{x} \leq_{\mathbb{d}} \mathbb{y}$, by (df $\leq_{\mathbb{d}}$).

"$\Leftarrow$" Suppose that (a) $\mathbb{x} \leq_{\mathbb{d}} \mathbb{y}$ and (b) $\mathbb{x} \neq \mathbb{y}$. Then (c): $\mathbb{x} \cap \mathbb{y} = \emptyset$, by (b). Moreover, from (a) for some $a \in \mathbb{x}$ and $b \in \mathbb{y}$ we have $a \sqsubseteq b$. But from (c) we have $a \neq b$. Thus, $a \sqsubset b$. So $\mathbb{x} <_{\mathbb{d}} \mathbb{y}$, by (df $<_{\mathbb{d}}$). □

We can also show that relations $<_{\mathbb{d}}$ and $\leq_{\mathbb{d}}$ are characterized by relations of inequality of segments.

FACT 4.10. *For all $a, b \in \mathbb{B}$ and for all $\alpha \in \mathbb{fr}(a)$ and $\beta \in \mathbb{fr}(b)$:*

$$\mathbb{d}_a \leq_{\mathbb{d}} \mathbb{d}_b \iff \pi_a\, \alpha \leqslant \pi_b\, \beta.$$

PROOF. Let $a, b \in \mathbb{B}$ and suppose that $\alpha \in \mathbb{fr}(a)$ and $\beta \in \mathbb{fr}(b)$.

"$\Rightarrow$" Suppose that $\mathbb{d}_a \leq_{\mathbb{d}} \mathbb{d}_b$. By (df $\leq_{\mathbb{d}}$) for some $x, y \in \mathbb{B}$ we have $x \in \mathbb{d}_a$, $y \in \mathbb{d}_b$, and $x \sqsubseteq y$. Hence, by (3), for Euclidean balls $B_x := \mathbb{int}(x)$ and $B_y := \mathbb{int}(y)$ we have $B_x \subseteq B_y$. Let $\gamma \in S_x := \mathbb{fr}(x)$ and $\delta \in S_y := \mathbb{fr}(y)$. Then $\pi_x\, \gamma \leqslant \pi_y\, \delta$. Since $x \in \mathbb{d}_a$ and $y \in \mathbb{d}_b$, so $\mathbb{d}_x = \mathbb{d}_a$. Therefore $\mathbb{d}_b = \mathbb{d}_y$. So $\pi_x\, \gamma \equiv \pi_a\, \alpha$ and $\pi_y\, \delta \equiv \pi_b\, \beta$, by Theorem 4.1. Hence $\pi_a\, \alpha \leqslant \pi_b\, \beta$, since $\pi_x\, \gamma \leqslant \pi_y\, \delta$.

"$\Leftarrow$" Suppose that $\pi_a\, \alpha \leqslant \pi_b\, \beta$. Let γ be a point such that $\pi_a\, \gamma \equiv \pi_b\, \beta$. Then, of course, $\pi_a\, \alpha \leqslant \pi_a\, \gamma$ and consequently $B_a \subseteq B^{\gamma}_{\pi_a}$. Hence, for mereological ball c such that $\mathbb{int}(c) = B^{\gamma}_{\pi_a}$ we have $a \sqsubseteq c$. Hence, by (df $\leq_{\mathbb{d}}$), we obtain $\mathbb{d}_a \leq_{\mathbb{d}} \mathbb{d}_c$. Since $\pi_a\, \gamma \equiv \pi_b\, \beta$, so $\mathbb{d}_b = \mathbb{d}_c$, by Theorem 4.1. In consequence $\mathbb{d}_a \leq_{\mathbb{d}} \mathbb{d}_b$. □

From Fact 4.10 it follows that the relation $\leq_{\mathbb{d}}$ has all properties of the relation $\leqslant$ defined by (df $\leqslant$) (see Appendix). Thus, the relation $\leq_{\mathbb{d}}$ partially orders $\mathbb{D}$, i.e., for all $\mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathbb{D}$ we have:

$$\mathbb{x} \leq_{\mathbb{d}} \mathbb{x}, \tag{$\mathrm{r}_{\leq_{\mathbb{d}}}$}$$
$$(\mathbb{x} \leq_{\mathbb{d}} \mathbb{y} \wedge \mathbb{y} \leq_{\mathbb{d}} \mathbb{x}) \Longrightarrow \mathbb{x} = \mathbb{y}, \tag{$\mathrm{antis}_{\leq_{\mathbb{d}}}$}$$
$$(\mathbb{x} \leq_{\mathbb{d}} \mathbb{y} \wedge \mathbb{y} \leq_{\mathbb{d}} \mathbb{z}) \Longrightarrow \mathbb{x} \leq_{\mathbb{d}} \mathbb{z}\,. \tag{$\mathrm{t}_{\leq_{\mathbb{d}}}$}$$

Notice that, by (10) and Fact 4.10, we obtain:

FACT 4.11. *For any $a, b \in \mathbb{B}$ and for any $\alpha \in \mathbb{fr}(a)$ and $\beta \in \mathbb{fr}(b)$:*

$$\mathbb{d}_a <_{\mathbb{d}} \mathbb{d}_b \Longleftrightarrow \pi_a\, \alpha < \pi_b\, \beta.$$

From Fact 4.11 it follows that the relation $<_{\mathbb{d}}$ has all properties of the relation $<$ defined by (df $<$) (see Appendix). Thus, the relation $<_{\mathbb{d}}$ is irreflexive, asymmetric, and transitive in class $\mathbb{D}$, i.e., for all $\mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathbb{D}$ we have:

$$\neg\, \mathbb{x} <_{\mathbb{d}} \mathbb{x}, \tag{$\mathrm{irr}_{<_{\mathbb{d}}}$}$$
$$\mathbb{x} <_{\mathbb{d}} \mathbb{y} \Longrightarrow \neg\, \mathbb{y} <_{\mathbb{d}} \mathbb{x}, \tag{$\mathrm{as}_{<_{\mathbb{d}}}$}$$
$$(\mathbb{x} <_{\mathbb{d}} \mathbb{y} \wedge \mathbb{y} <_{\mathbb{d}} \mathbb{z}) \Longrightarrow \mathbb{x} <_{\mathbb{d}} \mathbb{z}. \tag{$\mathrm{t}_{<_{\mathbb{d}}}$}$$

Moreover, from the law of trichotomy for segments (a1) it follows that for all $\mathbb{x}, \mathbb{y} \in \mathbb{D}$ we have:

$$\mathbb{x} \leq_{\mathbb{d}} \mathbb{y} \text{ or } \mathbb{x} = \mathbb{y} \text{ or } \mathbb{y} \leq_{\mathbb{d}} \mathbb{x}\,.$$

Other properties of the relation $\leq_{\mathbb{d}}$ will be shown on the basis of some facts that are held for mereological balls from a given point. First, we will show that in every point α, for a given ball $a \in \alpha$ there is a mereological ball in the point which is a part of ball a, and there is a mereological ball, that part of it is mereological ball a, i.e.:

$$\forall_{a \in \mathbb{B}} \exists_{b \in \mathbb{B}} (a \circledcirc b \wedge b \sqsubset a), \tag{11}$$
$$\forall_{a \in \mathbb{B}} \exists_{b \in \mathbb{B}} (a \circledcirc b \wedge a \sqsubset b). \tag{12}$$

Indeed, let $a \in \mathbb{B}$. Let $\mathbb{int}(a) = B_a$ and $\mathbb{fr}(a) = S_a$. Let α be an arbitrary point such that $\alpha \in S_a$ and let $L(\pi_a, \alpha)$ be a straight line crossing π_a and α. Let γ, γ' be points such that $\mathbf{B}(\pi_a\, \gamma\, \alpha)$ and $\mathbf{B}(\pi_a\, \alpha\, \gamma')$. Then, by (df $<$) we have $\pi_a\, \gamma < \pi_a\, \alpha$ and $\pi_a\, \alpha < \pi_a\, \gamma'$. Since for each point $\phi \in S^{\gamma}_{\pi_a}$ we have $\pi_a\,\gamma \equiv \pi_a\,\phi$, and for each point $\psi \in S_a$ we have $\pi_a\,\psi \equiv \pi_a\alpha$, so $S^{\gamma}_{\pi_a} \subseteq B_a$ and $S_a \subseteq B^{\gamma'}_{\pi_a}$. Then, by definition of the set of interior points of an arbitrary sphere and by fact that Euclidean ball is a convex set, we have $B^{\gamma}_{\pi_a} \subseteq B_a$ and $B_a \subseteq B^{\gamma'}_{\pi_a}$. Hence, mereological ball a and mereological ball c such that $\mathbb{int}(c) = B^{\gamma}_{\pi_a}$ satisfy (12), and mereological ball a and mereological ball d such that $\mathbb{int}(d) = B^{\gamma'}_{\pi_a}$ satisfy (11).

Directly from (df Π) and (df $\circledcirc$) it follows that:

$$\forall_{\alpha\in\Pi}\forall_{a,b\in\alpha}(a = b \vee a \sqsubseteq b \vee b \sqsubseteq a). \tag{13}$$

The next fact is related to «density» of the set of mereological balls from a given point that are «between» two distinct balls from this point.

$$\forall_{\alpha\in\Pi}\forall_{a,b\in\alpha}(a \sqsubset b \Longrightarrow \exists_{c\in\alpha}\ a \sqsubset c \sqsubset b). \tag{14}$$

Indeed, let $\alpha \in \Pi$ be an arbitrary point and let $a, b \in \alpha$. Moreover, suppose that $a \sqsubset b$. Let $\mathbb{int}(a) = B_a$, $\mathbb{fr}(a) = S_a$, and $\mathbb{int}(b) = B_b$, $\mathbb{fr}(b) = \mathrm{S}_b$. Let $\beta, \gamma \in \Pi$ are points such that $\beta \in S_a$ and $\gamma \in S_b$. Let L be an arbitrary straight line crossing α. Let ϕ be a point lying on L such that $\alpha\,\phi \equiv \pi_a\,\beta$ and let ψ be a point such that $\alpha\,\psi \equiv \pi_b\,\gamma$. Hence, by (df $<_{\mathbb{d}}$), we have $\mathbb{d}_a <_{\mathbb{d}} \mathbb{d}_b$ and then by Fact 4.11 we obtain $\pi_a\,\beta < \pi_b\,\gamma$. Thus $\alpha\,\phi < \alpha\,\psi$ and by (df $<$) we have $\mathbf{B}(\alpha\,\phi\,\psi)$. Let δ be an arbitrary point such that $\mathbf{B}(\phi\,\delta\,\psi)$. Hence, by Axiom O8 and (OB), we have: $\mathbf{B}(\alpha\,\phi\,\delta)$ and $\mathbf{B}(\alpha\,\phi\,\psi)$. Hence, again by (df $<$) we have $\alpha\,\phi < \alpha\,\delta$ and $\alpha\,\delta < \alpha\,\psi$. Let $c \in \mathbb{B}$ be a mereological ball such that $\mathbb{int}(c) = B^{\delta}_{\alpha}$ and $\mathbb{fr}(c) = S^{\delta}_{\alpha}$. Thus, we have: $B_a \subset B_c$ and $B_c \subset B_b$. Since $\pi_a = \pi_b = \alpha$, so $a \sqsubset c \sqsubset b$, by (3).

From (df $\leq_{\mathbb{d}}$), by (11) and (12), it follows that:

$$\forall_{\mathbb{x}\in\mathbb{D}}\exists_{\mathbb{y}\in\mathbb{D}}\ \mathbb{y} \leq_{\mathbb{d}} \mathbb{x}\,, \tag{15}$$

$$\forall_{\mathbb{x}\in\mathbb{D}}\exists_{\mathbb{y}\in\mathbb{D}}\ \mathbb{x} \leq_{\mathbb{d}} \mathbb{y}\,. \tag{16}$$

Moreover, by (13) and (14), it follows that:

$$\forall_{\mathbb{x},\mathbb{y}\in\mathbb{D}}(\mathbb{x} \leq_{\mathbb{d}} \mathbb{y} \vee \mathbb{y} \leq_{\mathbb{d}} \mathbb{x}), \tag{17}$$

$$\forall_{\mathbb{x},\mathbb{y}\in\mathbb{D}}(\mathbb{x} \leq_{\mathbb{d}} \mathbb{y} \Longrightarrow \exists_{\mathbb{z}\in\mathbb{D}}\ \mathbb{x} \leq_{\mathbb{d}} \mathbb{z} \leq_{\mathbb{d}} \mathbb{y}). \tag{18}$$

Finally, on the basis of (15), (16), (17) and (18) we obtain the following characteristics of the set $\mathbb{D}$ with respect to the relation $\leq_{\mathbb{d}}$.

THEOREM 4.12. *A pair $\langle\mathbb{D}, \leq_{\mathbb{d}}\rangle$ is a density, linear order without greatest and smallest element.*

A. Appendix: Basic facts from geometry

I. AXIOMS OF INCIDENCE[6]

I1. For any line L there exist (two) distinct points α and β such that $\alpha, \beta \in L$.
I2. For any points α and β there exists at least one line L such that $\alpha, \beta \in L$.
I3. If points α and β are distinct, then there exists at most one line L such that $\alpha, \beta \in L$.
I4. For any plane S there exist (three) non-collinear points α, β, γ such that $\alpha, \beta, \gamma \in S$.
I5. For any points α, β, γ there exists at least one plane S such that $\alpha, \beta, \gamma \in S$.
I6. If points α, β, γ are not-collinear, then there exists at most one plane S such that $\alpha, \beta, \gamma \in S$.
I7. For any line L and for any plane S, if there exist two distinct points α and β such that $\alpha, \beta \in L$ and $\alpha, \beta \in S$, then $L \subset S$.
I8. For any planes R and S, if there exists a point α such that $\alpha \in R$ and $\alpha \in S$, then there exists a point β distinct from α and such that $\beta \in R$ and $\beta \in S$.
I9. There exist (four) non-coplanar points $\alpha, \beta, \gamma, \delta$.

The most important fact that follows from above axioms of incidence is following:

FACT A.1. *Just one line passes through two distinct points.*

II. AXIOMS OF ORDER for the ternary relation $\mathbf{B}$ *of betweenness* (if $\mathbf{B}(\alpha\,\beta\,\gamma)$ then we say that β lies between α and γ):

O1. If $\mathbf{B}(\alpha\,\beta\,\gamma)$, then points α, β, γ are collinear and distinct.
O2. If $\mathbf{B}(\alpha\,\beta\,\gamma)$, then $\mathbf{B}(\gamma\,\beta\,\alpha)$.
O3. If $\mathbf{B}(\alpha\,\beta\,\gamma)$, then $\neg\,\mathbf{B}(\beta\,\alpha\,\gamma)$.
O4. If points α, β, γ are collinear and distinct, then $\mathbf{B}(\alpha\,\beta\,\gamma)$ or $\mathbf{B}(\beta\,\gamma\,\alpha)$ or $\mathbf{B}(\gamma\,\alpha\,\beta)$.
O5. If points α and β are distinct, then there exists a point γ such that $\mathbf{B}(\alpha\,\beta\,\gamma)$.
O6. If points α and β are distinct, then there exists a point γ such that $\mathbf{B}(\alpha\,\gamma\,\beta)$.

[6] In this Appendix we give only a sketch of the main axioms and concepts that are necessary for this paper. For details, the reader should see [1, 2].

O7. If $\mathbf{B}(\alpha\,\beta\,\gamma)$ and $\mathbf{B}(\beta\,\gamma\,\delta)$, then $\mathbf{B}(\alpha\,\beta\,\delta)$.
O8. If $\mathbf{B}(\alpha\,\beta\,\delta)$ and $\mathbf{B}(\beta\,\gamma\,\delta)$, then $\mathbf{B}(\alpha\,\beta\,\gamma)$.

The most important fact that follows from axioms of order is following. For any $\alpha, \beta, \gamma, \delta \in \Pi$:

(OA) If $\mathbf{B}(\alpha\,\beta\,\gamma)$ and $\mathbf{B}(\beta\,\gamma\,\delta)$, then $\mathbf{B}(\alpha\,\gamma\,\delta)$.
(OB) If $\mathbf{B}(\alpha\,\beta\,\delta)$ and $\mathbf{B}(\beta\,\gamma\,\delta)$, then $\mathbf{B}(\alpha\,\gamma\,\delta)$.
(OC) If $\mathbf{B}(\alpha\,\beta\,\delta)$, $\mathbf{B}(\alpha\,\gamma\,\delta)$ and $\beta \neq \gamma$, then $\mathbf{B}(\alpha\,\gamma\,\beta)$ or $\mathbf{B}(\beta\,\gamma\,\delta)$.
(OD) If $\mathbf{B}(\alpha\,\beta\,\gamma)$ and $\mathbf{B}(\alpha\,\gamma\,\delta)$, then $\mathbf{B}(\alpha\,\beta\,\delta)$.
(OE) If $\alpha \neq \beta$ and $\mathbf{B}(\alpha\,\beta\,\gamma)$ and $\mathbf{B}(\alpha\,\beta\,\gamma')$, then $\mathbf{B}(\beta\,\gamma\,\gamma')$ or $\mathbf{B}(\beta\,\gamma'\,\gamma)$.

III. AXIOMS OF CONGRUENCE

C1. If $\mathbf{D}(\alpha\,\alpha\,\beta\,\gamma)$, then $\beta = \gamma$.
C2. $\mathbf{D}(\alpha\,\beta\,\beta\,\alpha)$.
C3. If $\mathbf{D}(\alpha\,\beta\,\gamma\,\delta)$ and $\mathbf{D}(\alpha\,\beta\,\phi\,\psi)$, then $\mathbf{D}(\gamma\,\delta\,\phi\,\psi)$.

Any non-ordered pair $\{\alpha, \beta\}$ which is formed of two distinct points α and β is called *segment*. Any segment $\{\alpha, \beta\}$ will be denoted by $\alpha\ \beta$. For any $\alpha, \beta, \gamma, \delta \in \Pi$ we put:

$$\alpha\,\beta \equiv \gamma\,\delta \overset{\text{df}}{\Longleftrightarrow} \mathbf{D}(\alpha\,\beta\,\gamma\,\delta). \tag{df $\equiv$}$$

From properties of the relation $\mathbf{D}$ it follows that the relation $\equiv$ is an equivalence relation, thus:

$$\alpha\,\beta \equiv \alpha\,\beta, \tag{$r_\equiv$}$$

$$\alpha\,\beta \equiv \gamma\,\delta \Longrightarrow \gamma\,\delta \equiv \alpha\,\beta, \tag{$s_\equiv$}$$

$$\alpha\,\beta \equiv \gamma\,\delta \wedge \gamma\,\delta \equiv \phi\,\psi \Longrightarrow \alpha\,\beta \equiv \phi\,\psi. \tag{$t_\equiv$}$$

Relation $\equiv$ is called the *congruence of segments*. According to above definition, other axioms of congruence are as follows:

C4. If $\mathbf{B}(\alpha_1\,\beta_1\,\gamma_1)$, $\mathbf{B}(\alpha_2\,\beta_2\,\gamma_2)$, $\alpha_1\,\beta_1 \equiv \alpha_2\,\beta_2$ and $\beta_1\,\gamma_1 \equiv \beta_2\,\gamma_2$, then $\alpha_1\,\gamma_1 \equiv \alpha_2\,\gamma_2$
C5. For every half-line A with origin α and for every segment $\beta\gamma$ there exists just one point $\delta \in A$ such that $\alpha\delta \equiv \beta\gamma$.

By axioms (C4) and (C5) and by fact that the relation $\equiv$ is an equivalence relation in the set of all segments it follows that:

C4′ If $\mathbf{B}(\alpha\,\beta\,\gamma)$, $\mathbf{B}(\alpha'\,\beta'\,\gamma')$, $\alpha\,\beta \equiv \alpha'\,\beta'$, and $\alpha\,\gamma \equiv \alpha'\,\gamma'$, then $\beta\,\gamma \equiv \beta'\,\gamma'$.

We have also a fact which may be said to be converse of Axiom (C4) and fact (C4′), i.e.:

Fact A.2. *If* $\mathbf{B}(\alpha_1\,\beta_1\,\gamma_1)$ *and* $\alpha_1\,\gamma_1 \equiv \alpha_2\,\gamma_2$*, then there exists a point* β_2 *such that* $\mathbf{B}(\alpha_2\,\beta_2\,\gamma_2)$, $\alpha_1\,\beta_1 \equiv \alpha_2\,\beta_2$ *and* $\beta_1\,\gamma_1 \equiv \beta_2\gamma_2$.

By means of the relation of *congruence of line segments* and the *betweenness relation* we define the relation of *inequality of segments.* For any segments $\alpha_1\,\beta_1$ and $\alpha_2\,\beta_2$ we put:

$$\alpha_1\,\beta_1 < \alpha_2\,\beta_2 \overset{\text{df}}{\iff} \exists_{\gamma\in\Pi}\big(\mathbf{B}(\alpha_2\,\gamma\,\beta_2) \wedge \alpha_1\,\beta_1 \equiv \alpha_2\,\gamma\big). \tag{df $<$}$$

The relation $<$ is asymmetric and transitive. Thus, for any segments $\alpha_1\,\beta_1$, $\alpha_2\,\beta_2$ and $\alpha_3\,\beta_3$:

$$\text{If } \alpha_1\,\beta_1 < \alpha_2\,\beta_2,\text{, then } \neg\,\alpha_2\,\beta_2 < \alpha_1\,\beta_1, \tag{as$_<$}$$

$$\text{If } \alpha_1\,\beta_1 < \alpha_2\,\beta_2 \text{ and } \alpha_2\,\beta_2 < \alpha_3\,\beta_3, \text{ then } \alpha_1\,\beta_1 < \alpha_3\,\beta_3. \tag{t$_<$}$$

From (as$<$) and (t$<$) it follows that the relation $<$ is also irreflexive. For an arbitrary line segment $\alpha\,\beta$:

$$\neg\,\alpha\,\beta < \alpha\,\beta. \tag{irr$_<$}$$

Next, for the relation $<$ also hold *laws of extensionality* expressed in terms of the relation $\equiv$. For any segments $\alpha_1\,\beta_1$ and $\alpha_2\,\beta_2$:

$$\text{If } \alpha_1\,\beta_1 < \alpha_2\,\beta_2 \text{ and } \alpha\,\beta \equiv \alpha_1\,\beta_1, \text{ then } \alpha\,\beta < \alpha_2\,\beta_2.$$
$$\text{If } \alpha_1\,\beta_1 < \alpha_2\,\beta_2 \text{ and } \alpha\,\beta \equiv \alpha_2\,\beta_2, \text{ then } \alpha_1\,\beta_1 < \alpha\,\beta\,.$$

From laws of extensionality for the relation $<$, (irr$<$) and (t$<$) follows the *law of trichotomy.* For any segments $\alpha_1\,\beta_1$ and $\alpha_2\,\beta_2$ holds exactly one of below:

$$\alpha_1\,\beta_1 < \alpha_2\,\beta_2 \;\text{ or }\; \alpha_1\,\beta_1 \equiv \alpha_2\,\beta_2 \;\text{ or }\; \alpha_2\,\beta_2 < \alpha_1\,\beta_1. \tag{a1}$$

Next, by means of relations $<$ and $\equiv$ we define the relation of *sharp inequality of segments.* For any segments $\alpha_1\,\beta_1$ and $\alpha_2\,\beta_2$ we put:

$$\alpha_1\,\beta_1 \leqslant \alpha_2\,\beta_2 \overset{\text{df}}{\iff} \alpha_1\,\beta_1 < \alpha_2\,\beta_2 \vee \alpha_1\,\beta_1 \equiv \alpha_2\,\beta_2 \tag{df $\leqslant$}$$

From properties of the relations $<$ and $\equiv$ it follows that the relation $\leqslant$ is a partial order in the set of all segments, i.e.:

$$\alpha_1\,\beta_1 \leqslant \alpha_1\,\beta_1, \tag{r$_\leqslant$}$$

$$\alpha_1\,\beta_1 \leqslant \alpha_2\,\beta_2 \wedge \alpha_2\,\beta_2 \leqslant \alpha_1\,\beta_1 \Rightarrow \alpha_1\,\beta_1 \equiv \alpha_2\,\beta_2, \tag{antis$_\leqslant$}$$

$$\alpha_1\,\beta_1 \leqslant \alpha_2\,\beta_2 \wedge \alpha_2\,\beta_2 \leqslant \alpha_3\,\beta_3 \Rightarrow \alpha_1\,\beta_1 \leqslant \alpha_3\,\beta_3. \tag{t$_\leqslant$}$$

We introduce the following operation *mid*: $\Pi \times \Pi \longrightarrow \Pi$ of *middle point* which ascribes for any pair of distinct points a point which lies exactly midway between them. For any $\alpha, \beta \in \Pi$ we put:

$$\mathrm{mid}(\alpha, \beta) := (\iota\, \gamma)(\mathbf{B}(\alpha\, \gamma\, \beta) \wedge \gamma\, \alpha \equiv \gamma\, \beta). \tag{df mid}$$

We have the following connections between the operation *mid* and the relations $\equiv$ and $<$.

FACT A.3. *If* $\alpha\, \beta \equiv \alpha'\, \beta'$ *and* $\gamma = \mathrm{mid}(\alpha, \beta)$, $\gamma' = \mathrm{mid}(\alpha', \beta')$, *then* $\alpha\, \gamma \equiv \alpha'\, \gamma'$ *and* $\gamma\, \beta \equiv \gamma'\, \beta'$.

FACT A.4. *If* $\alpha\, \beta < \alpha'\, \beta'$ *and* $\gamma = \mathrm{mid}(\alpha, \beta)$, $\gamma' = \mathrm{mid}(\alpha', \beta')$, *then* $\alpha\, \gamma < \alpha'\, \gamma'$ *and* $\gamma\, \beta < \gamma'\, \beta'$.

Since the relation $\equiv$ is an equivalence relation, the equivalence classes of the family of all segments with respect to the relation $\equiv$ will be called *free segments*. We denote free segments by $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$. A free segment with a representative $\alpha\, \beta$ will be denoted by $[\alpha\, \beta]$. Now, we introduce the relation of *inequality of free segments*:

$$\mathfrak{a} < \mathfrak{b} \overset{\text{df}}{\Longleftrightarrow} \mathfrak{a} = [\alpha_1\, \beta_1] \wedge \mathfrak{b} = [\alpha_2\, \beta_2] \wedge \alpha_1\, \beta_1 < \alpha_2\, \beta_2. \tag{df$'$ $<$}$$

Then, we introduce the operation of *the addition of free segments*:

$$\mathfrak{c} = \mathfrak{a} + \mathfrak{b} \overset{\text{df}}{\Longleftrightarrow} \exists_{\gamma \in \Pi}(\mathbf{B}(\alpha\, \gamma\, \beta) \wedge \mathfrak{a} = [\alpha\, \gamma] \wedge \mathfrak{b} = [\gamma\, \beta]). \tag{df $+$}$$

FACT A.5. *For any free segments* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *and* $\mathfrak{d}$ *we have:*

$$\textit{If } \mathfrak{a} < \mathfrak{b}, \textit{ then } \neg\, \mathfrak{b} < \mathfrak{a}, \tag{as$_<$}$$

$$\textit{If } \mathfrak{a} < \mathfrak{b} \textit{ and } \mathfrak{b} < \mathfrak{c}, \textit{ then } \mathfrak{a} < \mathfrak{c}, \tag{t$_<$}$$

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}, \tag{comm$_+$}$$

$$(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}), \tag{assoc$_+$}$$

$$\textit{If } \mathfrak{a} < \mathfrak{b}, \textit{ then } \mathfrak{a} + \mathfrak{c} < \mathfrak{b} + \mathfrak{c}, \tag{mon$_<$}$$

$$\textit{If } \mathfrak{a} < \mathfrak{b} \textit{ and } \mathfrak{c} < \mathfrak{d}, \textit{ then } \mathfrak{a} + \mathfrak{c} < \mathfrak{b} + \mathfrak{d}. \tag{a2}$$

We have the following fact which is known as the *triangle inequality*.

FACT A.6. *For any three distinct points* α, β, γ *we have:*

$$[\alpha\, \gamma] \leqslant [\alpha\, \beta] + [\beta\, \gamma],$$

and $[\alpha\, \beta] + [\beta\, \gamma] = [\alpha\, \gamma]$ *if and only if* $\mathbf{B}(\alpha\, \beta\, \gamma)$.

There also holds the following fact that characterized bisectors of segments.

FACT A.7. *For any segment* $\alpha\,\beta$ *and any plane* P *such that* $\alpha, \beta \in P$, *for the bisector* $M \subseteq P$ *of the segment* $\alpha\,\beta$ *we have:*

$$M = \{\gamma \in P : \gamma\,\alpha \equiv \gamma\,\beta\}.$$

FACT A.8. *Let* $\mathrm{S}_{\alpha}^{\beta}$ *and* $\mathrm{S}_{\gamma}^{\delta}$ *be arbitrary spheres. Then:*

1. *If* $[\alpha\,\gamma] = [\alpha\,\beta] + [\gamma\,\delta]$, *then* $\mathrm{S}_{\alpha}^{\beta} \cap \mathrm{S}_{\gamma}^{\delta} = \{\phi\}$ *and* $\mathrm{B}_{\alpha}^{\beta} \cap \mathrm{B}_{\gamma}^{\delta} = \emptyset$.
2. *If* $[\alpha\,\gamma] > [\alpha\,\beta] + [\gamma\,\delta]$, *then* $\mathrm{S}_{\alpha}^{\beta} \cap \mathrm{S}_{\gamma}^{\delta} = \emptyset$ *and* $\mathrm{B}_{\alpha}^{\beta} \cap \mathrm{B}_{\gamma}^{\delta} = \emptyset$.

FACT A.9. *For any spheres* $\mathrm{S}_{\alpha}^{\beta}$ *and* $\mathrm{S}_{\gamma}^{\delta}$, *if* $\mathrm{S}_{\alpha}^{\beta} \cap \mathrm{S}_{\gamma}^{\delta} = \{\phi\}$, *then:*

$$\mathrm{B}_{\alpha}^{\beta} \cap \mathrm{B}_{\gamma}^{\delta} = \emptyset \text{ or } \mathrm{B}_{\alpha}^{\beta} \subseteq \mathrm{B}_{\gamma}^{\delta} \text{ or } \mathrm{B}_{\gamma}^{\delta} \subseteq \mathrm{B}_{\alpha}^{\beta}.$$

FACT A.10. *For any spheres* $\mathrm{S}_{\alpha}^{\beta}$ *and* $\mathrm{S}_{\gamma}^{\delta}$, *if* $\mathrm{S}_{\alpha}^{\beta} \cap \mathrm{S}_{\gamma}^{\delta} = \{\phi, \phi'\}$, *then* $\mathrm{B}_{\alpha}^{\beta} \setminus \mathrm{B}_{\gamma}^{\delta} \neq \emptyset$ *and* $\mathrm{B}_{\gamma}^{\delta} \setminus \mathrm{B}_{\alpha}^{\beta} \neq \emptyset$.

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