Algebr Represent Theor (2015) 18:881–894
DOI 10.1007/s10468-015-9520-4

Krull Dimension of Tame Generalized Multicoil Algebras

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Received: 2 September 2014 / Accepted: 20 January 2015 / Published online: 7 February 2015

Abstract We determine the Krull dimension of the module category of finite dimensional tame generalized multicoil algebras over an algebraically closed field, which are domestic.

Keywords Krull dimension · Tame algebra · Generalized multicoil algebra · Auslander-Reiten component

Mathematics Subject Classification (2010) 16G60 · 16G70 · 16P60

1 Introduction and the Main Results

Let A be a finite-dimensional k-algebra over a fixed algebraically closed field k. We denote by mod A the category of finitely generated A-modules. Here, we are interested in the Krull dimension K-dim(mod A) of the category $\mathcal{F} = \mathcal{F}(\text{mod}\, A)$ of all finitely presented contravariant functors from mod A into the category $\mathcal{A}b$ of abelian groups. Following [10] the Krull-Gabriel filtration $(\mathcal{F}_\alpha)_\alpha$ of $\mathcal{F}$ is defined recursively as follows: $\mathcal{F}_{-1} = 0$ and $\mathcal{F}_0$ is the Serre subcategory of all objects of finite length in $\mathcal{F}$. In the case when α is an ordinal number of the form $\beta + 1$ then $\mathcal{F}_\alpha$ is defined to be the Serre subcategory of all objects in $\mathcal{F}$ which become of finite length in $\mathcal{F}/\mathcal{F}_\beta$. In the case when α is a limit ordinal, then $\mathcal{F}_\alpha$ is the union of all $\mathcal{F}_\beta$ with $\beta < \alpha$. If there exists an ordinal α with $\mathcal{F}_\alpha = \mathcal{F}$, then the smallest ordinal with this property is called the Krull dimension of $\mathcal{F}$.

By a result of Auslander [5], we know that K-dim(mod A) $= 0$ if and only if A is representation-finite. Moreover, K-dim(mod A) does not exist when A is wild hereditary [6], and K-dim(mod A) $= 2$ when A is representation-infinite tame hereditary [10]. Our interest in the Krull dimension of $\mathcal{F}$ is also motivated by the fact that the filtration $(\mathcal{F}_\alpha)_\alpha$ of

Presented by Jon F. Carlson.

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$\mathcal{F}$ leads to a hierarchy of exact sequences in mod A, where the Auslander-Reiten sequences form the lowest level (see [10]). It is expected that the existence of K-dim(mod A) implies that A is domestic, that is, there is a common bound for the numbers of one-parameter families of indecomposable A-modules of any fixed dimension.

We would like to mention that the generalized multicoil algebras (respectively, tame generalized multicoil algebras) form a prominent class of algebras of global dimension at most 3, containing the class of quasitilted algebras of canonical type [14, 30] (respectively, tame quasitilted algebras of canonical type), and are obtained by sophisticated gluings of concealed canonical algebras (respectively, tame concealed algebras) using admissible algebra operations (see Section 3 for details). Moreover, recently the tame generalized multicoil algebras showed to be important in describing the structure of the module category indΛ of an arbitrary cycle-finite algebra Λ (see [18, Theorems 7.1 and 7.2] or [19, Theorem 1.8]). We also refer to the article [24] for the Hochschild cohomology of generalized multicoil algebras.

The following theorem is the main result of the paper.

Theorem 1.1 *Let A be a tame generalized multicoil algebra. The following statements are equivalent:*

(i) K-dim(mod A) $= 2$.
(ii) K-dim(mod A) *exists.*
(iii) *A is domestic.*

In the representation theory of algebras a prominent role is played by the algebras with a separating family of components in the following sense. A family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of the Auslander-Reiten quiver Γ_A of an algebra A is called *separating* in mod A if the modules in *ind* A split into three disjoint classes $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$ such that:

(S1) $\mathcal{C}^A$ is a sincere generalized standard family of components;

(S2) $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\mathrm{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;

(S3) any morphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ factors through the additive category add $\mathcal{C}^A$ of $\mathcal{C}^A$.

We then say that $\mathcal{C}^A$ *separates* $\mathcal{P}^A$ from $\mathcal{Q}^A$ and write $\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A$. We note that then $\mathcal{P}^A$ and $\mathcal{Q}^A$ are uniquely determined by $\mathcal{C}^A$ (see [4, (2.1)] or [26, (3.1)]). Moreover, $\mathcal{C}^A$ is called *sincere* if any simple A-module occurs as a composition factor of a module in $\mathcal{C}^A$, and *generalized standard* if $\mathrm{rad}^\infty(X, Y) = 0$ for all modules X and Y from $\mathcal{C}^A$. We refer also to the survey article [23] for the structure of arbitrary algebras with separating families of Auslander-Reiten components.

Frequently, we may recover A completely from the shape and categorical behaviour of the separating family $\mathcal{C}^A$ of components of Γ_A. For example, the tilted algebras [12, 26], or more generally double tilted algebras [25], are determined by their (separating) connecting components. Further, it was proved in [13] that the class of algebras with a separating family of stable tubes coincides with the class of concealed canonical algebras. This was extended in [21] to a characterization of algebras with a separating family of almost cyclic coherent Auslander-Reiten components. Recall that a component Γ of an Auslander-Reiten quiver Γ_A is called *almost cyclic* if all but finitely many modules in Γ lie on oriented cycles contained entirely in Γ. Moreover, a component Γ of Γ_A is said to be *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$ (that is, $X_i \neq \tau_A X_{i+2}$ for any $i \geq 1$) in Γ.

(C2) For each injective module I in Γ there is an infinite sectional path $\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ (that is, $Y_{j+2} \neq \tau_A Y_j$ for any $j \geq 1$) in Γ.

It has been proved in [21, Theorem A] that the Auslander-Reiten quiver Γ_A of an algebra A admits a separating family of almost cyclic coherent components if and only if A is a generalized multicoil enlargement of a (possibly decomposable) concealed canonical algebra C. Moreover, for such an algebra A, we have that A is triangular, gl.dim $A \leq 3$, and $\mathrm{pd}_A X \leq 2$ or $\mathrm{id}_A X \leq 2$ for any module X in ind A (see [21, Corollary B and Theorem E]).

As an immediate consequence of Theorem 1.1, Theorem 3.1, the definition of separating family of components of the Auslander-Reiten quiver Γ_A of an algebra A, and [17, Theorem 1.1] we obtain the following fact.

Corollary 1.2 *Let A be a tame algebra with a separating family of almost cyclic coherent Auslander-Reiten components. The following statements are equivalent:*

(i) K-dim(mod A) = 2.
(ii) K-dim(mod A) *exists.*
(iii) *A is domestic.*
(iv) $(\mathrm{rad}^{\infty}(\mathrm{mod}\, A))^3 = 0$.

2 Preliminaries

Throughout this paper, k will denote a fixed algebraically closed field. An algebra A will always mean a basic, connected (unless otherwise specified), associative finite dimensional k-algebra with an identity. Thus there exists a connected bound quiver (Q_A, I_A) and an isomorphism $A \cong kQ_A/I_A$. Equivalently, $A \cong kQ_A/I_A$ may be considered as a k-linear category, of which the object class A_0 is the set of points of Q_A, and the set of morphisms $A(x, y)$ from x to y is the quotient of the k-vector space $kQ_A(x, y)$ of all formal linear combinations of paths in Q_A from x to y by the subspace $I_A(x, y) = kQ_A(x, y) \cap I_A$ (see [7]). An algebra A with Q_A acyclic (without oriented cycles) is said to be *triangular*. A full subcategory C of A is said to be *convex* if any path in Q_A with source and target in Q_C lies entirely in Q_C.

By an A-module is meant a finitely generated right A-module. We denote by mod A the category of A-modules, by ind A the full subcategory consisting of a complete set of representatives of the isomorphism classes of indecomposable A-modules, by Γ_A the Auslander-Reiten quiver of A and by τ_A the Auslander-Reiten translation in Γ_A. We shall agree to identify the vertices of Γ_A with the corresponding modules in ind A, and the components of Γ_A with the corresponding full subcategories of ind A. A component $\mathcal{P}$ of Γ_A is called *postprojective* if $\mathcal{P}$ is acyclic and every module in $\mathcal{P}$ lies in the τ_A-orbit of a projective module. Dually, a component $\mathcal{Q}$ of Γ_A is called *preinjective* if $\mathcal{Q}$ is acyclic and every module in $\mathcal{Q}$ lies in the τ_A-orbit of an injective module. Recall also that the *Jacobson radical* rad(mod A) of the module category mod A is the ideal of mod A generated by all noninvertible morphisms in ind A. Then the *infinite radical* $\mathrm{rad}^{\infty}(\mathrm{mod}\, A)$ of mod A is the intersection of all powers $\mathrm{rad}^i(\mathrm{mod}\, A)$, $i \geq 1$, of rad(mod A).

Let A be an algebra and $\mathcal{Q}$ be an infinite preinjective component of Γ_A. Let $\mathcal{S}$ be a set of indecomposable representatives of each infinite τ_A-orbit of modules from $\mathcal{Q}$. Moreover, assume that for any indecomposable module M from $\mathcal{S}$ there exist an indecomposable N from $\mathcal{S}$ and an irreducible morphism $M \to N$ or $N \to M$. Then we say that $\mathcal{S}$ is *left stable quasi-section* of $\mathcal{Q}$.

Let A be an algebra and $k[x]$ the polynomial algebra in one variable. Following [9] A is said to be *tame* if, for any dimension d, there exists a finite number of $k[x]$–A-bimodules M_i, $1 \leq i \leq n_d$, which are finitely generated and free as left $k[x]$-modules, and all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $k[x]/(x-\lambda) \otimes_{k[x]} M_i$ for some $\lambda \in k$ and some $i \in \{1, \ldots, n_d\}$. Let $\mu_A(d)$ be the least number of $k[x]$–A-bimodules satisfying the above conditions for d. Then A is said to be *domestic* if there exists a positive integer m such that $\mu_A(d) \leq m$ for any $d \geq 1$. From the validity of the second Brauer-Thrall conjecture we know that $\mu_A(d) = 0$ for any $d \geq 1$ if and only if A is representation-finite. Recall that an algebra A is said to be *representation-finite* if ind A admits only a finite number of pairwise nonisomorphic modules. Otherwise, we say that A is *representation-infinite*.

Let $\mathcal{C}$ be an abelian category. A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is called a *Serre subcategory* if it is closed under subobjects, quotients and extensions. If $\mathcal{C}' \subseteq \mathcal{C}$ is a Serre subcategory, then one defines the *quotient category* $\mathcal{C}/\mathcal{C}'$ as follows. The objects of $\mathcal{C}/\mathcal{C}'$ coincide with the objects of $\mathcal{C}$, and if X and Y are objects of $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}/\mathcal{C}'}(X, Y) := \varinjlim \mathrm{Hom}_{\mathcal{C}}(X', Y/Y')$, where X' and Y' run through all subobjects of X and Y , respectively, such that X/X' and Y' belong to $\mathcal{C}'$. Again $\mathcal{C}/\mathcal{C}'$ is an abelian category and the *quotient functor* $T : \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ is exact.

Let $\mathcal{C}$ be a small abelian category. The *Krull-Gabriel filtration* $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ is defined as follows: $\mathcal{C}_{-1} = 0$, $\mathcal{C}_0$ is the Serre subcategory of all objects of finite length in $\mathcal{C}$. In the case when α is an ordinal number of the form $\beta + 1$ then $\mathcal{C}_\alpha$ is defined to be the Serre subcategory of all objects in $\mathcal{C}$ which become of finite length in $\mathcal{C}/\mathcal{C}_\beta$. If α is a limit ordinal, then $\mathcal{C}_\alpha$ is the union of all $\mathcal{C}_\beta$ with $\beta < \alpha$. If there exists an ordinal α with $\mathcal{C}_\alpha = \mathcal{C}$, then the smallest ordinal with this property is called the *Krull dimension* of $\mathcal{C}$, denoted by K-dim $\mathcal{C}$. We shall also denote by T_0 and T_1 the quotient functors $T_0 : \mathcal{C} \to \mathcal{C}/\mathcal{C}_0$ and $T_1 : \mathcal{C} \to \mathcal{C}/\mathcal{C}_1$, respectively.

Let $\mathcal{D}$ be a subcategory of mod A for some algebra A. Denote by $\mathcal{F}(\mathcal{D})$ the category of finitely presented contravariant functors from $\mathcal{D}$ to the category $\mathcal{A}b$ of abelian groups. Assume that $\mathcal{F}(\mathcal{D})$ is abelian. Then K-dim $\mathcal{D}$ is by definition the Krull dimension of $\mathcal{F}(\mathcal{D})$.

The following result from [32, Lemma 2.1] will be applied.

Lemma 2.1 *Let A be an algebra and $\mathcal{Q}$ be an infinite preinjective component of Γ_A having a left stable quasi-section $\mathcal{S}$ of Euclidean type. Assume also that any indecomposable module M in $\mathcal{S}$ does not belong to a path of irreducible morphisms $N \to \cdots \to M$, where N is indecomposable and the τ_A-orbit of N is finite. Then* K-dim $\mathcal{Q} = 2$.

In the proof of our main result we need also the following fact.

Lemma 2.2 *Let A be an algebra, M, N, U, V modules in* mod A*, and $M \oplus U \xrightarrow{[f,g]} V$ and $N \xrightarrow{h} U$ monomorphisms. Then the morphism $M \oplus N \xrightarrow{[f,gh]} V$ is a monomorphism.*

Proof The morphism $M \oplus N \xrightarrow{[f,gh]} V$ is a monomorphism as the composition of the following two monomorphisms $M \oplus N \xrightarrow{\left[\begin{smallmatrix} id_M & 0 \\ 0 & h \end{smallmatrix}\right]} M \oplus U \xrightarrow{[f,g]} V$. □

For basic background on the representation theory of algebras applied in the paper, we refer to the books [1, 26–28].

3 Tame Generalized Multicoil Algebras

In this section we introduce and exhibit basic properties of the class of tame generalized multicoil algebras, playing the fundamental role in our proof of Theorem 1.1. This is the class of tame algebras among the class of all algebras having a separating family of almost cyclic coherent components investigated in [21, 22]. Recall that a module X in mod A is called a *brick* if $\mathrm{End}_A(X) \cong k$.

It has been proved in [20, Theorem A] that a connected component Γ of an Auslander-Reiten quiver Γ_A is almost cyclic and coherent if and only if Γ is a generalized multicoil, obtained from a family of stable tubes by a sequence of operations called *admissible*. We recall the latter and simultaneously define the corresponding enlargements of algebras.

We start with the one-point extensions and one-point coextensions of algebras. Let A be an algebra and M be a module in mod A. Then the *one-point extension* of A by M is the matrix algebra

$$A[M] = \begin{bmatrix} A & 0 \\ M & k \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ m & \lambda \end{bmatrix};\ \lambda \in k,\ a \in A,\ m \in M \right\}$$

with the usual addition and multiplication. The quiver $Q_{A[M]}$ of $A[M]$ contains the quiver Q_A of A as a convex subquiver, and there is an additional (extension) vertex which is a source. The $A[M]$-modules are usually identified with the triples (V, X, φ), where V is a k-vector space, X an A-module and $\varphi : V \to \mathrm{Hom}_A(M, X)$ is a k-linear map. An $A[M]$-linear map $(V, X, \varphi) \to (W, Y, \psi)$ is then identified with a pair (f, g), where $f : V \to W$ is k-linear, $g : X \to Y$ is A-linear and $\psi f = \mathrm{Hom}_A(M, g)\varphi$. Dually, one defines also the *one-point coextension* of A by M as the matrix algebra

$$[M]A = \begin{bmatrix} k & 0 \\ D(M) & A \end{bmatrix}.$$

For $r \geq 1$, we denote by $T_r(k)$ the $r \times r$-lower triangular matrix algebra

$$\begin{bmatrix} k & 0 & 0 & \dots & 0 & 0 \\ k & k & 0 & \dots & 0 & 0 \\ k & k & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & k & \dots & k & 0 \\ k & k & k & \dots & k & k \end{bmatrix}$$

Given a generalized standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X, -)\,|_\Gamma$ is the R-linear category defined as follows [3]. Let $\mathcal{H}_X$ denote the full subcategory of Γ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X, M) \neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X, f) = 0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}_X/\mathcal{I}_X$. Following the above convention, we usually identify the R-linear category $\mathcal{S}(X)$ with its quiver.

From now on let A be an algebra and Γ be a family of generalized standard infinite components of Γ_A. For an indecomposable brick X in Γ, called the *pivot*, one defines five admissible operations (ad 1)-(ad 5) and their duals (ad 1*)-(ad 5*) modifying the translation quiver $\Gamma = (\Gamma, \tau)$ to a new translation quiver (Γ', τ') and the algebra A to a new algebra A', depending on the shape of the support $\mathcal{S}(X)$ (see [20, Section 2] for the figures illustrating the modified translation quivers Γ').

(ad 1) Let $t \in \mathbb{N}$ and assume $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

If $t \geq 1$ then $D = T_t(k)$ and $Y_1, Y_2, \ldots, Y_t$ denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective D-module. We define the *modified algebra* A' of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \begin{bmatrix}1\\1\end{bmatrix}\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau' X'_0 = Y_t, \tau' X'_i = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_D, respectively.

Finally, if $t = 0$ we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified translation quiver Γ' to be the translation quiver obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $i \geq 0$.

The non-negative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have that Γ' is a generalized standard family of components of $\Gamma_{A'}$.

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation. Actually, the above operation is, in this case, the tube insertion as considered in [8].

(ad 2) Suppose that $\mathcal{S}(X)$ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

where $t \geq 1$. In particular, X is necessarily injective. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \begin{bmatrix}1\\1\end{bmatrix}\right)$ for $i \geq 1$, $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: X'_0 is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{1j} = Y_{j-1}$ if $j \geq 2, \tau' X'_i = Z_{i-1,t}$ if $i \geq 2, \tau' X'_1 = Y_t, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

The integer $t \geq 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have that Γ' is a generalized standard family of components of $\Gamma_{A'}$.

(ad 3) Assume $\mathcal{S}(X)$ is the mesh-category of two parallel sectional paths:

$$\begin{array}{ccccccccc} Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & \\ \uparrow & & \uparrow & & & & \uparrow & & \\ X = X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t \to \cdots \end{array}$$

with the upper sectional path finite and $t \geq 2$. In particular, X_{t-1} is necessarily injective. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ for $i \geq 1$, $1 \leq j \leq t$, $j \leq i$, and $X'_i = (k, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: X'_0 is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $2 \leq j \leq t$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' X'_i = Y_i$ if $1 \leq i \leq t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq t+1$, $\tau' Y_j = X'_{j-2}$ if $2 \leq j \leq t$, $\tau'(\tau^{-1} X_i) = X'_i$, if $i \geq t$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ^*. We note that X'_{t-1} is injective.

The integer $t \geq 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t+1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have that Γ' is a generalized standard family of components of $\Gamma_{A'}$.

(ad 4) Suppose that $\mathcal{S}(X)$ consists an infinite sectional path, starting at X

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

and

$$Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

with $t \geq 1$, be a finite sectional path in Γ_A. Let $r \in \mathbb{N}$. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. For $r = 0$ we define the *modified algebra* A' of A to be the one-point extension $A' = A[X \oplus Y]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*.

For $r \geq 1$, let $G = T_r(k)$, $U_{1,t+1}$, $U_{2,t+1}$, …, $U_{r,t+1}$ denote the indecomposable projective G-modules, $U_{r,t+1}$, $U_{r,t+2}$, …, $U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective-injective G-module. We define the *modified algebra* A' of A to be the triangular matrix algebra of the form:

$$A' = \begin{bmatrix} A & 0 & 0 & \ldots & 0 & 0 \\ Y & k & 0 & \ldots & 0 & 0 \\ Y & k & k & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & k & k & \ldots & k & 0 \\ X \oplus Y & k & k & \ldots & k & k \end{bmatrix}$$

with $r+2$ columns and rows and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangles consisting of the modules $U_{sl} = \left(k, Y_l \oplus U_{s,t+1}, \begin{bmatrix}1\\1\end{bmatrix}\right)$ for $1 \le s \le r$, $1 \le l \le t$, and $Z_{ij} = \left(k, X_i \oplus U_{rj}, \begin{bmatrix}1\\1\end{bmatrix}\right)$ for $i \ge 0$, $1 \le j \le t+r$, and $X'_i = (k, X_i, 1)$ for $i \ge 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \ge 1$, $j \ge 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \ge 1$, $\tau' Z_{0j} = U_{r,j-1}$ if $2 \le j \le t+r$, Z_{01}, U_{k1}, $1 \le k \le r$ are projective, $\tau' U_{kl} = U_{k-1,l-1}$ if $2 \le k \le r$, $2 \le l \le t+r$, $\tau' U_{1l} = Y_{l-1}$ if $2 \le l \le t+1$, $\tau' X'_0 = U_{r,t+r}$, $\tau' X'_i = Z_{i-1,t+r}$ if $i \ge 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*, or Γ_G, respectively.

We note that the quiver $Q_{A'}$ of A' is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r+1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

The integers $t \ge 1$ and $r \ge 0$ are such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangles equals $t+r+1$. We call $t+r$ the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have that Γ' is a generalized standard family of components of $\Gamma_{A'}$.

For the definition of the next admissible operation we need also the finite versions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), which we denote by (fad 1), (fad 2), (fad 3) and (fad 4), respectively. In order to obtain these operations we replace all infinite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots$ (in the definitions of (ad 1), (ad 2), (ad 3), (ad 4)) by the finite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots \to X_s$. For the operation (fad 1) $s \ge 0$, for (fad 2) and (fad 4) $s \ge 1$, and for (fad 3) $s \ge t-1$. In all above operations X_s is injective (see [20] or [21] for the details).

(ad 5) We define the *modified algebra* A' of A to be the iteration of the extensions described in the definitions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), and their finite versions corresponding to the operations (fad 1), (fad 2), (fad 3) and (fad 4). The *modified translation quiver* Γ' of Γ is obtained in the following three steps: first we are doing on Γ one of the operations (fad 1), (fad 2) or (fad 3), next a finite number (possibly zero) of the operation (fad 4) and finally the operation (ad 4), and in such a way that the sectional paths starting from all the new projective vertices have a common cofinite (infinite) sectional subpath.

Since Γ is a generalized standard family of components of Γ_A, we then have that Γ' is a generalized standard family of components of $\Gamma_{A'}$.

Finally, together with each of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4) and (ad 5), we consider its dual, denoted by (ad 1*), (ad 2*), (ad 3*), (ad 4*) and (ad 5*). These ten operations are called the *admissible operations*. Following [20] a connected translation quiver Γ is said to be a *generalized multicoil* if Γ can be obtained from a finite family $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s$ of stable tubes by an iterated application of admissible operations (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3), (ad 3*), (ad 4), (ad 4*), (ad 5) or (ad 5*). If $s = 1$, such a translation quiver Γ is said to be a *generalized coil*. The admissible operations of types (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*) and (ad 3*) have been introduced in [2–4], the admissible operations (ad 4) and (ad 4*) for $r = 0$ in [16], and the admissible operations (ad 4), (ad 4*) for $r \ge 1$, (ad 5) and (ad 5*) in [20, 21].

Observe that any stable tube is trivially a generalized coil. A tube is a generalized coil having the property that each admissible operation in the sequence defining it is of the form (ad 1) or (ad 1*). Moreover, if we apply only operations of type (ad 1) (respectively, of type (ad 1*)) then such a generalized coil is a ray tube (respectively, a coray tube). Observe that a generalized coil without injective (respectively, projective) vertices is a ray tube (respectively, a coray tube). A *quasi-tube* is a generalized coil having the property that each of the admissible operations in the sequence defining it is of type (ad 1), (ad 1*), (ad 2) or (ad 2*). Finally, following [3] a coil is a generalized coil having the property that each of the admissible operations in the sequence defining it is one of the forms (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*). We note that any generalized multicoil Γ is a coherent translation quiver with trivial valuations and its cyclic part ${}_c\Gamma$ (the translation subquiver of Γ obtained by removing from Γ all acyclic vertices and the arrows attached to them) is infinite, connected and cofinite in Γ, and so Γ is almost cyclic.

Let C be the product $C_1 \times \ldots \times C_m$ of a family $C_1, \ldots, C_m$ of tame concealed algebras and $\mathcal{T}^C$ the disjoint union $\mathcal{T}^{C_1} \cup \ldots \cup \mathcal{T}^{C_m}$ of $\mathbb{P}_1(k)$-families $\mathcal{T}^{C_1}, \ldots, \mathcal{T}^{C_m}$ of pairwise orthogonal generalized standard stable tubes of $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$, respectively. Following [21], we say that an algebra A is a *generalized multicoil enlargement* of $C_1, \ldots, C_m$ if A is obtained from C by an iteration of admissible operations of types (ad 1)-(ad 5) and (ad 1*)-(ad 5*) performed either on stable tubes of $\mathcal{T}^C$ or on generalized multicoils obtained from stable tubes of $\mathcal{T}^C$ by means of the operations done so far. It follows from [21, Corollary B] that then A is a triangular algebra. In fact, in [21] generalized multicoil enlargements of finite families of arbitrary concealed canonical algebras (*generalized multicoil algebras*) have been introduced and investigated. But in the tame case we may restrict to the generalized multicoil enlargements of tame concealed algebras. Namely, we have the following consequence of [21, Theorems A and F].

Theorem 3.1 *Let A be an algebra. The following statements are equivalent:*

(i) *A is tame and Γ_A admits a separating family of almost cyclic coherent components.*
(ii) *A is a tame generalized multicoil enlargement of a finite family of tame concealed algebras.*

From now on, by a *tame generalized multicoil algebra* we mean a connected tame generalized multicoil enlargement of a finite family of tame concealed algebras. As a consequence of [21, Theorems C and F] and the proof of [21, Theorem C] we obtain the following fact.

Theorem 3.2 *Let A be a tame generalized multicoil algebra obtained from a family $C_1, \ldots, C_m$ of tame concealed algebras. There are full convex subcategories $A^{(l)} = A_1^{(l)} \times \ldots \times A_m^{(l)}$ and $A^{(r)} = A_1^{(r)} \times \ldots \times A_m^{(r)}$ of A such that the following statement hold:*

(i) *For each $i \in \{1, \ldots, m\}$, $A_i^{(l)}$ and $A_i^{(r)}$ are representation-infinite tilted algebras of Euclidean type or tubular algebras.*
(ii) *A can be obtained from $A^{(l)}$ by a sequence of admissible operations of types (ad 1), (ad 2), (ad 3), (ad 4) or (ad 5).*
(iii) *A can be obtained from $A^{(r)}$ by a sequence of admissible operations of types (ad 1*), (ad 2*), (ad 3*), (ad 4*) or (ad 5*).*
(iv) *The Auslander-Reiten quiver Γ_A of A is of the form*

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A,$$

where $\mathcal{C}^A$ is a family of generalized multicoils separating $\mathcal{P}^A$ from $\mathcal{Q}^A$ such that:

(a) *$\mathcal{C}^A$ is obtained from the $\mathbb{P}_1(k)$-families $\mathcal{T}^{C_1}, \ldots, \mathcal{T}^{C_m}$ of stable tubes of $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$ by admissible operations corresponding to the admissible operations leading from $C_1, \ldots, C_m$ to A;*
(b) *$\mathcal{P}^A$ is the disjoint union $\mathcal{P}^{A_1^{(l)}} \cup \ldots \cup \mathcal{P}^{A_m^{(l)}}$, where, for each $i \in \{1, \ldots, m\}$, $\mathcal{P}^{A_i^{(l)}}$ is either the postprojective component of $\Gamma_{A_i^{(l)}}$, if $A_i^{(l)}$ is a representation-infinite tilted algebra of Euclidean type, or $\mathcal{P}^{A_i^{(l)}} = \mathcal{P}_0^{A_i^{(l)}} \cup \mathcal{T}_0^{A_i^{(l)}} \cup \left(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{A_i^{(l)}}\right)$, if $A_i^{(l)}$ is a tubular algebra;*
(c) *$\mathcal{Q}^A$ is the disjoint union $\mathcal{Q}^{A_1^{(r)}} \cup \ldots \cup \mathcal{Q}^{A_m^{(r)}}$, where, for each $i \in \{1, \ldots, m\}$, $\mathcal{Q}^{A_i^{(r)}}$ is either the preinjective component of $\Gamma_{A_i^{(r)}}$, if $A_i^{(r)}$ is a representation-infinite tilted algebra of Euclidean type, or $\mathcal{Q}^{A_i^{(r)}} = \left(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{A_i^{(r)}}\right) \cup \mathcal{T}_\infty^{A_i^{(r)}} \cup \mathcal{Q}_\infty^{A_i^{(r)}}$, if $A_i^{(r)}$ is a tubular algebra.*

Remark 3.3 From the proof of [21, Theorem C] we know that $A^{(l)} = A_1^{(l)} \times \ldots \times A_m^{(l)}$ (respectively, $A^{(r)} = A_1^{(r)} \times \ldots \times A_m^{(r)}$) is a unique maximal convex truncated branch coextension (respectively, extension) of $C = C_1 \times C_2 \times \cdots \times C_m$ inside A, that is, $A_i^{(l)}$ (respectively, $A_i^{(r)}$) is a unique maximal convex truncated branch coextension (respectively, extension) of C_i inside A, $i \in \{1, \ldots, m\}$.

It follows from [29, Theorem 4.1] and Theorem 3.2 that, if A is tame generalized multicoil algebra, then A is cycle-finite (see Section 5 for the definition). Applying now [29, Theorem 5.1], we obtain the following fact.

Corollary 3.4 *Let A be a tame generalized multicoil algebra and $\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A$ the canonical decomposition of Γ_A. The following statements are equivalent:*

(i) *A is domestic.*
(ii) *$A^{(l)}$ and $A^{(r)}$ are products of representation-infinite tilted algebras of Euclidean type.*
(iii) *$\mathcal{P}^A$ is a disjoint union of postprojective components of Euclidean type and $\mathcal{Q}^A$ is a disjoint union of preinjective components of Euclidean type.*

4 Proof of Theorem 1.1

Clearly, (i) implies (ii). Let A be a tame generalized multicoil algebra obtained from a family $C_1, \ldots, C_m$ of tame concealed algebras. Assume that A is non-domestic. Then it follows from Theorem 3.2 (i) that there are full convex subcategories $A^{(l)} = A_1^{(l)} \times \ldots \times A_m^{(l)}$, $A^{(r)} = A_1^{(r)} \times \ldots \times A_m^{(r)}$ of A and $i, j \in \{1, \ldots, m\}$ such that at least one of the algebras $A_i^{(l)}$ or $A_j^{(r)}$ is tubular. By [11, Theorem 1.6 and Proposition 4.1] the Krull dimension of a tubular algebra does not exist, and hence at least one of the dimensions K-dim(mod $A^{(l)}$) or K-dim(mod $A^{(r)}$) does not exist. Moreover, applying [11, Theorem 2.1], we infer that K-dim(mod A) does not exist too. This proves that the statement (ii) implies the statement (iii). It remains to show that (iii) implies (i). We will apply arguments similar to those applied in the proof of [32, Proposition 2.2].

By Theorem 3.2 and Remark 3.3 there exists a unique maximal convex truncated branch coextension $A^{(l)} = A_1^{(l)} \times \ldots \times A_m^{(l)}$ of $C = C_1 \times C_2 \times \cdots \times C_m$ which is a full convex subcategory of A. Moreover, A is obtained from $A^{(l)}$ by a sequence of admissible operations of types (ad 1)-(ad 5). From Corollary 3.4 we know that $A^{(l)}$ is a product of representation-infinite tilted algebras $A_1^{(l)}, \ldots, A_m^{(l)}$ of Euclidean type. By [11, Theorem 3.4] we get that, for each $i \in \{1, \ldots, m\}$, K-dim(mod $A_i^{(l)}$) $= 2$, and hence K-dim(mod $A^{(l)}$) $= \sup_{i=1,\ldots,m}\{$K-dim(mod $A_i^{(l)}$)$\} = 2$. Let Λ be the first modified algebra of $A^{(l)}$ made to obtain A. Let M be an indecomposable Λ-module. If M belongs to a postprojective component of Γ_Λ then M belongs to the postprojective component of $\Gamma_{A_i^{(l)}}$ for some $i = 1, \ldots, m$ and the functor $\mathrm{Hom}_\Lambda(-, M)$ is of finite length. Assume that M belongs to a generalized multicoil $\mathcal{C}$. Since different generalized multicoils in mod Λ are pairwise orthogonal, it follows from [11], that if $\mathcal{C}$ is a coray tube of $\Gamma_{A^{(l)}}$, then $T_1\,\mathrm{Hom}_\Lambda(-, M) = 0$. Thus we may assume that M is a module from the generalized multicoil of Γ_Λ different from the coray tubes of $\Gamma_{A^{(l)}}$. If M is a directing Λ-module which is not an $A^{(l)}$-module then again $\mathrm{Hom}_\Lambda(-, M)$ is of finite length. If M is a non-directing Λ-module which is not an $A^{(l)}$-module then we have the following three cases to consider.

(a) If Λ is a modified algebra of $A^{(l)}$ obtained by applying the admissible operation of type (ad 1), (ad 2), (ad 3) or (ad 4) with $r = 0$ then M is isomorphic to Z_{ij} or X_i' (see Section 3). Assume first that $M \cong Z_{ij}$. Then we have an obvious monomorphism $X_i \oplus Y_j \to Z_{ij}$, which induces a monomorphism of functors $\alpha : \mathrm{Hom}_\Lambda(-, X_i \oplus Y_j) \to \mathrm{Hom}_\Lambda(-, Z_{ij})$. It follows from the description of generalized multicoils that the set $\mathcal{S}_\alpha$ of all indecomposable modules N such that $\mathrm{coker}\,\alpha(N) \neq 0$ is finite. Indeed, we have:

- For (ad 1) and (ad 4) with $r = 0$, $\mathcal{S}_\alpha = \{Z_{kl} \mid 0 \le k \le i, 1 \le l \le j\}$.
- For (ad 2), $\mathcal{S}_\alpha = \{X_0', Z_{kl} \mid 1 \le k \le i, 1 \le l \le j\}$.
- For (ad 3), $\mathcal{S}_\alpha = \{X_0', \ldots, X_{j-1}', Z_{kl} \mid 1 \le k \le i, 1 \le l \le j, l \ge k\}$.

Moreover, $\mathrm{coker}\,\alpha$ is finitely generated. Therefore, we get that $\mathrm{coker}\,\alpha$ is of finite length and $T_0\,\mathrm{Hom}_\Lambda(-, X_i \oplus Y_j) \cong T_0\,\mathrm{Hom}_\Lambda(-, Z_{ij})$. Assume now that $M \cong X_i'$. Again, we have an obvious monomorphism $X_i \to X_i'$, which induces a monomorphism of functors $\beta : \mathrm{Hom}_\Lambda(-, X_i) \to \mathrm{Hom}_\Lambda(-, X_i')$ and the set $\mathcal{S}_\beta$ of all indecomposable modules N such that $\mathrm{coker}\,\beta(N) \neq 0$ is finite. In this subcase we get:

- For (ad 1) and (ad 4) with $r = 0$, $\mathcal{S}_\beta = \{X_k', Z_{kl} \mid 0 \le k \le i, 1 \le l \le t\}$.
- For (ad 2), $\mathcal{S}_\beta = \{X_0', X_k', Z_{kl} \mid 1 \le k \le i, 1 \le l \le t\}$ and $\mathcal{S}_\beta = \{X_0'\}$ when $i = 0$.

Note that in the above two subcases t denotes the parameter of the suitable admissible operation.

- For (ad 3), $\mathcal{S}_\beta = \{X_0', X_k', Z_{kl} \mid 1 \le k \le i, 1 \le l \le k\}$ and $\mathcal{S}_\beta = \{X_0'\}$ when $i = 0$.

Hence $\mathrm{coker}\,\beta$ is of finite length since, moreover, it is finitely generated. Thus $T_0\,\mathrm{Hom}_\Lambda(-, X_i) \cong T_0\,\mathrm{Hom}_\Lambda(-, X_i')$.

(b) If Λ is a modified algebra of $A^{(l)}$ obtained by applying the admissible operation of type (ad 4) with $r \ge 1$ then M is isomorphic to U_{kl} for $1 \le k \le r$, $1 \le l \le t$, Z_{ij} for $i \ge 0$, $1 \le j \le t + r$, or X_i' for $i \ge 0$ (see Section 3). Assume first that $M \cong U_{kl}$, $1 \le k \le r$, $1 \le l \le t$, where $t + r$ is the parameter of (ad 4). Then we have a monomorphism $Y_l \to U_{kl}$, which induces a monomorphism of functors $\gamma : \mathrm{Hom}_\Lambda(-, Y_l) \to \mathrm{Hom}_\Lambda(-, U_{kl})$ and the set $\mathcal{S}_\gamma$ of all indecomposable modules

N such that $\operatorname{coker}\gamma(N) \neq 0$ is finite. Indeed, $\mathcal{S}_\gamma = \{U_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq l\}$. Again, $\operatorname{coker}\gamma$ is finitely generated. Therefore, we get that $\operatorname{coker}\gamma$ is of finite length and $T_0 \operatorname{Hom}_\Lambda(-, Y_l) \cong T_0 \operatorname{Hom}_\Lambda(-, U_{kl})$. Assume now that $M \cong Z_{ij}$, $i \geq 0$, $1 \leq j \leq t + r$. We consider two subcases.

- For $i \geq 0$, $1 \leq j \leq t$ we have monomorphisms $X_i \oplus U_{rj} \to Z_{ij}$ and $Y_j \to U_{rj}$. Hence, by Lemma 2.2, we infer that $X_i \oplus Y_j \to Z_{ij}$ is a monomorphism. Again, we get the induced monomorphism of functors $\delta : \operatorname{Hom}_\Lambda(-, X_i \oplus Y_j) \to \operatorname{Hom}_\Lambda(-, Z_{ij})$, $\mathcal{S}_\delta = \{U_{pl}, Z_{kl} \mid 1 \leq p \leq r, 1 \leq l \leq j, 0 \leq k \leq i\}$, and $T_0 \operatorname{Hom}_\Lambda(-, X_i \oplus Y_j) \cong T_0 \operatorname{Hom}_\Lambda(-, Z_{ij})$.
- For $i \geq 0$, $t + 1 \leq j \leq t + r$ we have a monomorphism $X_i \oplus U_{rj} \to Z_{ij}$, where $U_{r,t+1}$, $U_{r,t+2}$, $\ldots$, $U_{r,t+r}$ are given indecomposable injective $T_r(k)$-modules. Again, we get the induced monomorphism of functors $\varepsilon : \operatorname{Hom}_\Lambda(-, X_i \oplus U_{rj}) \to \operatorname{Hom}_\Lambda(-, Z_{ij})$, $\mathcal{S}_\varepsilon = \{U_{r,t+1}, \ldots, U_{r,t+j}, Z_{kl} \mid 0 \leq k \leq i, 1 \leq l \leq j\}$, and $T_0 \operatorname{Hom}_\Lambda(-, X_i \oplus U_{rj}) \cong T_0 \operatorname{Hom}_\Lambda(-, Z_{ij})$.

  Finally, assume that $M \cong X'_i$, $i \geq 0$. Again, we have an obvious monomorphism $X_i \to X'_i$, the induced monomorphism of functors $\zeta : \operatorname{Hom}_\Lambda(-, X_i) \to \operatorname{Hom}_\Lambda(-, X'_i)$, $\mathcal{S}_\zeta = \{X'_k, Z_{kl} \mid 0 \leq k \leq i, 1 \leq l \leq t + r\}$, and $T_0 \operatorname{Hom}_\Lambda(-, X_i) \cong T_0 \operatorname{Hom}_\Lambda(-, X'_i)$.

(c) Λ is a modified algebra of $A^{(l)}$ obtained by applying the admissible operation of type (ad 5). Since in the definition of (ad 5) we use the finite versions (fad 1), (fad 2), (fad 3), (fad 4) of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4) and the admissible operation (ad 4), we conclude that the required statements follows from the above considerations.

Therefore, we may assume that M is in fact an $A^{(l)}$-module. Let $F = \operatorname{Hom}_\Lambda(-, M)$ and $G = \operatorname{Hom}_\Lambda(-, M)|_{\operatorname{mod} A^{(l)}}$. Let I be the simple Λ-module corresponding to the extension vertex of $A^{(l)}[X]$, where X is the pivot of the suitable admissible operation. Since $\operatorname{Hom}_\Lambda(I, M) = 0$ then for any A-module Z we get $F(Z) = G(Z')$, where Z' is the restriction of Z to $A^{(l)}$. Moreover, the category $\operatorname{mod} A^{(l)}$ is contained in the obvious way into the category $\operatorname{mod}\Lambda$. From this we conclude that, if $T'_1 G = 0$, then $T_1 F = 0$, where

$$T'_1 : \mathcal{F}(\operatorname{mod} A^{(l)}) \to \mathcal{F}(\operatorname{mod} A^{(l)})/\mathcal{F}_1(\operatorname{mod} A^{(l)})$$

and

$$T_1 : \mathcal{F}(\operatorname{mod}\Lambda) \to \mathcal{F}(\operatorname{mod}\Lambda)/\mathcal{F}_1(\operatorname{mod}\Lambda)$$

are the canonical quotient functors. By [11] we have $T'_1 G = 0$, and hence $T_1 F = 0$. Since F is not of finite length, we get that $T_0 F \neq 0$.

Let $\mathcal{X}$ be the full subcategory of $\operatorname{mod}\Lambda$ generated by all indecomposable modules from the generalized multicoils and the postprojective components, and let $\mathcal{Y}$ be the full subcategory of $\operatorname{mod}\Lambda$ generated by all indecomposable modules from the preinjective components. Note that, since the projective cover of any finitely presented functor is a functor $\operatorname{Hom}_\Lambda(-, N)$ for some module N, it is enough to check only Hom-functors. Therefore, by the above arguments we have $\text{K-dim}\,\mathcal{X} = 1$. Moreover, using Lemma 2.1 we get that $\text{K-dim}\,\mathcal{Y} = 2$. Hence, applying [11, Theorem 2.6] we obtain

$$\text{K-dim}(\operatorname{mod}\Lambda) = \max(\text{K-dim}\,\mathcal{X} + 1, \text{K-dim}\,\mathcal{Y}) = 2.$$

Finally, we can complete the proof by an obvious induction on the number of admissible operations leading from $A^{(l)}$ to A.

5 Concluding Remarks

Since the tame quasitilted algebras of canonical type form a distinguished special class of tame generalized multicoil algebras, we obtain the following fact.

Corollary 5.1 *Let A be a tame quasitilted algebra of canonical type. The following statements are equivalent:*

(i) K-dim(mod A) $= 2$.
(ii) K-dim(mod A) *exists.*
(iii) *A is domestic.*

Let A be an algebra. Recall that a *cycle* in a module category mod A is a sequence $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_r} X_r = X_0$ of nonzero nonisomorphisms in ind A, and the cycle is said to be *finite* if $f_i \notin \mathrm{rad}^\infty(\mathrm{mod}\,A)$ for any $1 \le i \le r$. If every cycle in mod A is finite then A is said to be *cycle-finite*. Recall also that a component $\mathcal{C}$ of Γ_A is called *semiregular* if $\mathcal{C}$ does not contain both a projective and an injective module. It has been proved in [15] that a semiregular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a ray tube or coray tube (see remarks after definitions of admissible operations).

As an immediate consequence of Corollary 5.1 and [31, Theorem 5.1] we obtain the following fact.

Corollary 5.2 *Let A be a cycle-finite algebra such that every component of* Γ_A *is semiregular, and* $\mathrm{pd}_A X \le 1$ *or* $\mathrm{id}_A X \le 1$ *for all but finitely many isomorphism classes of modules X in ind A. Then the following statements are equivalent:*

(i) K-dim(mod A) $= 2$.
(ii) K-dim(mod A) *exists.*
(iii) *A is domestic.*

Acknowledgments This work was supported by the research grant DEC-2011/02/A/ST1/00216 of the Polish National Science Center.

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