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ON SOME EXTENSIONS OF THE CLASS OF MV-ALGEBRAS

Abstract. In the present paper we will ask for the lattice $L(\mathbf{MV}_{Ex})$ of subvarieties of the variety defined by the set $Ex(\mathbf{MV})$ of all externally compatible identities valid in the variety **MV** of all MV-algebras. In particular, we will find all subdirectly irreducible algebras from the classes in the lattice $L(\mathbf{MV}_{Ex})$ and give syntactical and semantical characterization of the class of algebras defined by P-compatible identities of MV-algebras.

Keywords: MV-algebra; variety; identity; P-compatible identity; equational base; subdirectly irreducible algebras

1. Introduction

As it is known J. Łukasiewicz (see [9]) introduced a 3-valued propositional calculus with one designated truth-value. Łukasiewicz and Tarski [10] generalized this construction to an m-valued propositional calculus (where m is a natural number or it equals $\aleph_0$) using matrices again with one designated truth-value. While giving an algebraic proof of the completeness of the Łukasiewicz infinite-valued sentential calculus, C. C. Chang introduced MV-algebras. As it is known Boolean algebras being used to semantically formulate the classical logic are in particular MV-algebras. Of course, the converse statement is not true, i.e. it is not the case that each MV-algebra is a Boolean algebra. Chang's aim was to adopt a method of prime ideal that had been used for Boolean algebras to the case of MV-algebras.

Let us recall that the above mentioned theorem states that for any Boolean algebra $\mathfrak{A}$ and disjoint an ideal I and a filter F in $\mathfrak{A}$, there is a prime ideal containing I, that is disjoint with F. This theorem

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being formulated in various versions (for example as a relative Lindenbaum lemma known as Łoś-Asser lemma) plays the key role in proofs of completeness theorems. Chang shows that as regards symbols of $+$, $\cdot$ and $^-$ a difference between MV-algebras understood as ordered 6-toples $\langle A, +, \cdot, ^-, 0, 1\rangle$ and Boolean algebras relies on the lack of the itempotence low for $+$, while the low of excluded middle has not to be fulfilled in a given MV-alebra.

An axiomatisation of the 3-valued logic was given by M. Wajsberg [18]. An axiomatisation of the m-valued, where $m \neq \aleph_0$, with arbitrary number of designated values had been proposed by J. B. Rosser and A. R. Turquette [16]. In [10] a hypothesis that $\aleph_0$-valued calculus is axiomatised by a system with modus ponens and substitution as sole rules of inference was given. Suggested axioms had the following form:

1. $p \to (q \to p)$
2. $(p \to q) \to ((q \to r) \to (p \to r))$
3. $((p \to q) \to q) \to ((q \to p) \to p)$
4. $((p \to q) \to (q \to p)) \to (q \to p)$
5. $(\sim p \to \sim q) \to (q \to p)$.

A. Tarski [17, s. 51] in a footnote indicates Wajserbga [19] as one who confirmed this hypothesis. Rose and Rosser gave its proof in [15]. An algebraic proof of the appropriate theorem was given be Chang [1, 2]. In [7] a description of pure implication logics containing implicational fragment of infinitely many valued Łukasiewicz logic, while in [8], overlogics of this logic where described.

In the below definition, axioms are treated as a formulation of properties of particular operations on the set A:

Definition 1.1. An MV-algebra is a system $\langle A, +, \cdot, ^-, 0, 1\rangle$, where A is a nonempty set, 0 and 1 are constants in the set A, $+$ and $\cdot$ are operations of arity two in the set A and $^-$ is a unarny operation on the set A, where the following equations are fulfilled:

Ax.1	$x + y \approx y + x$	Ax.1'.	$x \cdot y \approx y \cdot x$
Ax.2	$x + (y + z) \approx (x + y) + z$	Ax.2'	$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$
Ax.3	$x + \overline{x} \approx 1$	Ax.3'	$x \cdot \overline{x} \approx 0$
Ax.4	$x + 1 \approx 1$	Ax.4'	$x \cdot 0 \approx 0$
Ax.5	$x + 0 \approx x$	Ax.5'	$x \cdot 1 \approx x$
Ax.6	$\overline{(x + y)} \approx \overline{x} \cdot \overline{y}$	Ax.6'	$\overline{(x \cdot y)} \approx \overline{x} + \overline{y}$
Ax.7	$x \approx \overline{(\overline{x})}$	Ax.8.	$\overline{0} \approx 1$

Ax.9	$x \vee y \approx y \vee x$	Ax.9'	$x \wedge y \approx y \wedge x$
Ax.10	$x \vee (y \vee z) \approx (x \vee y) \vee z$	Ax.10'	$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$
Ax.11	$x + (y \wedge z) \approx (x + y) \wedge (x + y)$	Ax.11'	$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot y)$,

where operations $\vee$ and $\wedge$ are given for any $x, y \in A$ as follows:

$x \vee y \approx (x \cdot \overline{y}) + y$
$x \wedge y \approx (x + \overline{y}) \cdot y$

Besides we recall:

Definition 1.2. Let **MV** denote the class of all MV-algebras while *Id*(**MV**) — the set of all identities valid in **MV**.

Chang mentioned that the above axiomatisation is not very "economic". He stressed however, that it is very intuitive and it way we recall it. It is obvious that elements 0 and 1, as well as operations $+$, $\cdot$, and $\vee$ and $\wedge$ are respectively dual. Beside, one assumes that the operation $\cdot$, similarly as in arithmetics bides stronger than $+$.

This fact that this axiomatisation is not "non-economic", caused a search for more elegant axiomatisations. In [3] by an MV-algebra one understands any algebra $\mathfrak{A} = \langle A, 0, 1,^*, \odot, \oplus\rangle$ fulfilling the following conditions:

Ax.12 $x \odot (y \odot z) \approx (x \odot y) \odot z$
Ax.13 $x \odot y \approx y \odot x$
Ax.14 $x \odot 0 \approx 0$
Ax.15 $x \odot 1 \approx x$
Ax.16 $0^* \approx 1$
Ax.17 $1^* \approx 0$
Ax.18 $(x^* \odot y)^* \odot \approx (y^* \odot x)^* \odot x$
Ax.19 $x \oplus y \approx (x^* \odot y^*)^*$.

It is known, that the set *Id*(**MV**) determines a variety (a nonempty class of algebras that is closed under any subalgebras, arbitrary products and homomorphic images) and this variety is **MV**.

When considering MV-algebras as structures in the type $\langle 2, 2, 1, 0, 0\rangle$ with operations $+, \cdot, ^{-}, 0, 1$ one can formulate a notion of externally compatible identities by stipulating that:

Definition 1.3. An identity is *externally compatible* iff it is of any of the below form:

$$\varphi_1 \approx \varphi_1 \tag{1.1}$$

$$\varphi_1 + \varphi_2 \approx \psi_1 + \psi_2 \tag{1.2}$$

$$\varphi_1 \cdot \varphi_2 \approx \psi_1 \cdot \psi_2 \tag{1.3}$$

$$\overline{\varphi_1} \approx \overline{\psi_1}, \tag{1.4}$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are any terms in the type $\langle 2, 2, 1, 0, 0\rangle$.

Let us notice that some identities valid in the class of MV-algebras are externally compatible, but some are not. For example the commutative low $x+y \approx y+x$ is an externally compatible identity, while de Morgana low $\overline{(x \cdot y)} \approx \overline{x} + \overline{y}$ is not.

2. Syntax and semantics

While searching for an equational basis of the class MV_{Ex}, it is convenient to consider this class in the type $\langle 2, 2, 1\rangle$. Thus, we assume that the constant 0 can be defined for example as $x \cdot \overline{x}$. The constant 1 can be defined as well, for example as $x + \overline{x}$.

Let V a variety in the type τ fulfilling the following conditions:

(2.1) There is a non-trivial unary term $q(x)$, such that for any $f \in F$, the identity $q(f(x_0, \ldots, x_{\tau(f)-1})) \approx q(f(q(x_0), \ldots, q(x_{\tau(f)-1})))$ belongs to $Id(V)$.

(2.2) If $[f]_P$ is a nullary block (i.e., a block with only nullary operations) and $g, h \in [f]_P$, then there is a non-trivializing, unary term $q_{g,h}(x)$, such that the most external operational symbol in the term $q_{g,h}(x)$ belongs to $[f]_P$ and moreover the following identities:

$$g(x_0, \ldots, x_{\tau(g)-1}) = q_{g,h}(q(g(x_0, \ldots, x_{\tau(g)-1}))),$$

$$h(x_0, \ldots, x_{\tau(h)-1}) = q_{g,h}(q(h(x_0, \ldots, x_{\tau(h)-1})))$$

belong to $Id(V)$.

(2.3) If $[f]_P$ is a nullary block of the partition P, then for any $g \in [f]_P$ identity $f = g$ belongs to $Id(V)$.

Let $\mathbf{B}$ be an equational basis of a variety V. We define a set $\mathbf{B}^*$ of identities of the typu τ with the help of the following three conditions:

(2.4) Identities (2.1), (2.2) and (2.3) belong to $\mathbf{B}^*$.

(2.5) If $\phi = \psi$ belong to $\mathbf{B}$, then the identity $q(\phi) = q(\psi)$ belongs to $\mathbf{B}^*$.

(2.6) $\mathbf{B}^*$ includes only identities described in conditions (2.4) and (2.5).

It has been shown in [13] that the following theorem holds:

THEOREM 2.1. *If* **B** *is an equational basis of a variety* V *fulfilling the conditions* (2.1), (2.2) *and* (2.3), *then the set* $\mathbf{B}^*$ *defined by the conditions* (2.4), (2.5) *and* (2.6) *is an equational basis of the variety* V_P.

Besides, we have:

THEOREM 2.2 ([11]). *For any nontrivial variety* $V \in \mathcal{L}(\mathbf{MOL})$ *there is a lattice embedding of the lattice* $\overline{\mathbf{B}}$ *into* $\overline{V}$, *where* **B** *is a class of Boolean algebras.*

The the below theorem holds:

THEOREM 2.3. *The following identities:*

$$
\begin{array}{llll}
\text{Ax.1.} & x + y \approx y + x & \text{Ax.1}'. & x \cdot y \approx y \cdot x \\
\text{Ax.2.} & x + (y + z) \approx (x + y) + z & \text{Ax.2}'. & x \cdot (y \cdot z) \approx (x \cdot y) \cdot z \\
\text{Ax.3.} & x + \overline{x} \approx y + \overline{y} & \text{Ax.3}'. & x \cdot \overline{x} \approx y \cdot \overline{y} \\
\text{Ax.4.} & x + 1 \approx 1 & \text{Ax.4}'. & x \cdot 0 \approx 0 \\
\text{Ax.5.} & x + y + 0 \approx x + y & \text{Ax.5}'. & x \cdot y \cdot 1 \approx x \cdot y \\
 & (x + 0) \cdot y \approx x \cdot y & & (x \cdot 1) + y \approx x + y \\
 & \overline{x + 0} \approx \overline{x} & & \overline{x \cdot 1} \approx \overline{x} \\
\text{Ax.6.} & \overline{x + y} + z \approx \overline{x} \cdot \overline{y} + z & \text{Ax.6}'. & \overline{x \cdot y} + z \approx (\overline{x} + \overline{y}) + z \\
 & (\overline{x + y}) \cdot z \approx (\overline{x} \cdot \overline{y}) \cdot z & & (\overline{x \cdot y}) \cdot z \approx (\overline{x} + \overline{y}) \cdot z \\
 & \overline{\overline{x + y}} \cdot 0 \approx \overline{\overline{x} \cdot \overline{y}} & & \overline{\overline{x \cdot y}} \approx \overline{\overline{x} + \overline{y}} \\
\text{Ax.7.} & \overline{\overline{\overline{x}}} \approx \overline{x} & \text{Ax.8.} & \overline{0} + x \approx 1 + x \\
 & \overline{\overline{x}} + y \approx x + y & & \overline{0} \cdot x \approx 1 \cdot x \\
 & \overline{\overline{x}} \cdot y \approx x \cdot y & & \overline{\overline{0}} \approx \overline{1} \\
\text{Ax.9.} & x \vee y \approx y \vee x & \text{Ax.9}'. & x \wedge y \approx y \wedge x \\
\text{Ax.10.} & x \vee (y \vee z) \approx (x \vee y) \vee z & \text{Ax.10}'. & x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \\
\text{Ax.11.} & (x + (y \wedge z)) + t \approx ((x + y) \wedge (x + y)) + t & & \\
 & (x + (y \wedge z)) \cdot t \approx ((x + y) \wedge (x + y)) \cdot t & & \\
 & \overline{x + (y \wedge z)} \approx \overline{(x + y) \wedge (x + y)} & & \\
\text{Ax.11'.} & (x \cdot (y \vee z)) + t \approx (x \cdot y) \vee (x \cdot z) + t & & \\
 & (x \cdot (y \vee z)) \cdot t \approx (x \cdot y) \vee (x \cdot z) \cdot t & & \\
 & \overline{x \cdot (y \vee z)} \approx \overline{(x \cdot y) \vee (x \cdot z)} & &
\end{array}
$$

constitute an equational basis od the class $\mathbf{MV}_{Ex}$.

Schetch of the proof. Let us notice that the class $\mathbf{MV}_{Ex}$ fulfils assumptions of Theorem 2.1. The set composed of identities Ax.1–Ax.11 and Ax.1'–Ax.11' is denoted by B_1. Let B_2 denote the set of identities given by Theorem 2.1 when applied to the class $\mathbf{MV}_{Ex}$. We skip details of the proof since it comes down to showing that $\mathsf{Cn}(B_1) = \mathsf{Cn}(B_2)$ and goes in the standard way. ⊣

Let us consider algebras $\mathfrak{A} = (A; F^{\mathfrak{A}})$ and $\mathfrak{I} = (I; F^{\mathfrak{I}})$ of type τ and a partition P of the set F. The algebra $\mathfrak{A}$ is a P-dispersion of $\mathfrak{I}$ (see [6], [13]) iff there exists a partition $\{A_i\}_{i\in I}$ of A and there exists a family $\{c_{[f]_P}\}_{f\in F}$ of mappings $c_{[f]_P}: I \to A$ satisfying the following conditions:

(2.7) For each $i \in I$: $c_{[f]_P}(i) \in A_i$.

(2.8) For each $f \in F$ and for each $a_i \in A_{k_i}$, $i = 0, \ldots, \tau(f)-1$, $f^{\mathfrak{A}}(a_0, \ldots, a_{\tau(f)-1}) = c_{[f]_P}(f^{\mathfrak{I}}(k_0, \ldots, k_{\tau(f)-1}))$.

(2.9) If $f \in [g]_P$, then for each $i \in I$: $c_{[f]_P}(i) = c_{[g]_P}(i)$.

The following theorem holds:

Theorem 2.4 ([13]). *If P is a partition of a set F and V is a variety of the type τ fulfilling conditions (2.1), (2.2) and (2.3), then $\mathfrak{A}$ belongs to the class V_P iff $\mathfrak{A}$ is a P-dispersion of a ceratin algebra belonging to V.*

The following theorem is obvious:

Theorem 2.5 ([6]). *The lattice $\mathcal{L}(Ex(\tau))$ is isomorphic with the lattice $\Pi_F + 1$ of all partitions of the set F with the unit element 1.*

Theorem 2.6 ([4]). *Let V be a variety of the type τ, such that for a ceratin unary term $\phi(x)$, which is not a variable, then the identity $\phi(x) \approx x$ belongs to the set $Id(V)$. Let moreover a partition P of the set F fulfils the condition:*

$$V_P = D_P(V). \qquad (V_P)$$

Thus, lattices $\mathcal{L}(V)$ and $P^{(V)}$ are isomorphic.

Let us consider the following example.

Example 2.1. Let an algebra $\mathcal{A} = \langle\{0, \frac{1}{2}^+, \frac{1}{2}^\cdot, 1\}; +, \cdot, ^-\rangle$ be a dispersion of the following algebra $\mathcal{B} = (\{0, \frac{1}{2}, 1\}; +, \cdot, ^-)$ (see Diagram 1). Then: $c_+(k) = c_\cdot(k) = c_-(k) = k$, for $k \in \{0, 1\}$, $c_+(\frac{1}{2}) = c_-(\frac{1}{2}) = \frac{1}{2}^+$, and $c_\cdot(\frac{1}{2}) = \frac{1}{2}^\cdot$. Moreover, one can see that $\overline{\overline{\frac{1}{2}^\cdot}} = \frac{1}{2}^+$. Thus, the identity $\overline{\overline{x}} \approx x$ is not fulfilled in the algebra $\mathcal{A}$.

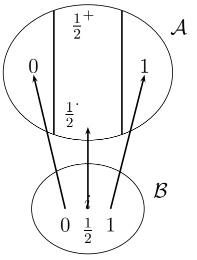


Diagram 1. Identities – algebras

It can be shown that this algebra verifies all identities externally compatible valid in the class $\mathbf{MV}_{Ex}$. It is the case since this class is fulfils assumption of Theorem 2.4. So, the next theorem follows:

THEOREM 2.7. *The class* $\mathbf{MV}_{Ex}$ *equals the class all dispersions of all* MV-*algebras.*

We have of course also a more general theorem:

THEOREM 2.8 (Characterisation of the class $\mathbf{MV}_{Ex}$). *For any partition P the class $\mathbf{MV}_P$ equals the class of all dispersions of all P-dispersions of algebras from the class* $\mathbf{MV}$.

3. Subdirectly irreducible algebras from the variety of $\mathbf{MV}_n$-algebras

In the present section we describe all subdirectly irreducible algebras from the class of MV_n-algebras.

3.1. Variety of $\mathbf{MV}_n$-algebras

In [5] R. Grigolia indicated algebras being semantical counterparts of n-valued logics for any $2 < n < \aleph_0$. The class $\mathbf{MV}_n$ of all MV_n-algebras is a subclass of the class of all MV-algebras. It is determined by the set of all identities valid in the class of all MV-algebras extended by the following identities:

Ax.12. $(n-1)x + x \approx (n-1)x$
Ax.12'. $x^{n-1} \cdot x \approx x^{n-1}$

and for $n > 3$, additionally the following axioms are added:

Ax.13. $((jx)\cdot(\bar{x}+((j-1)\cdot x)^{-}))^{(n-1)} \approx 0$

Ax.13'. $(n-1)\big(x^{j}+(\bar{x}\cdot(x^{j-1})^{-})\big) \approx 1,$

where $1 < j < n-1$ and $n-1$ is divided by j.

We obtain $\mathbf{MV}_n$ — a class of MV$_n$-algebras. Thus, each Boolean algebra is a MV$_n$-algebra for every $2 < n < \aleph_0$ and each MV$_n$-algebra for every $2 < n < \aleph_0$ is a MV-algebra.

Let $\mathcal{L}_n = \langle L_n, +, \cdot, -, 1, 0\rangle$, where $L_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ and for any $x, y \in L_n$:

- $x + y = \min(1, x + y)$,
- $x \cdot y = \max(0, x + y - 1)$,
- $\bar{x} = 1 - x$.

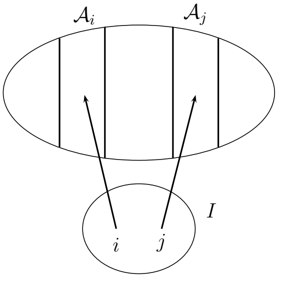
Let us recall:

Theorem 3.1 ([5]). *Each* MV$_n$*-algebra* $\mathcal{A}$ *is isomorphic to a subdirect product of algebras* $\mathcal{L}_m$*, where* $m \leqslant n$ *and* $m-1$ *divides* $n-1$*.*

Let an algebra $\mathcal{A}$ belong to the class $\mathbf{MV}_{n\,Ex}$. It is known that $\mathcal{A}$ is a dispersion of a ceratin algebras $\mathcal{I}$ from the variety $\mathbf{MV}_n$.

The following cases can occur (cf [14]):

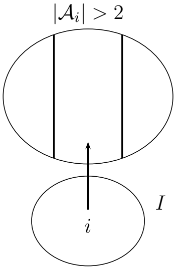
1. If $|A_i| = 1$ for every $i \in I$, then $\mathcal{A}$ belongs to the variety $\mathbf{MV}_n$, since each function c_f determines an isomorphism of algebras $\mathcal{I}$ and $\mathcal{A}$. Thus, $\mathcal{A}$ is subdirectly-irreducible iff it fulfils the condition of Theorem 3.1 concerning subdirectly-irreducible MV$_n$-algebras.
2. If $|I| = 1$ (i.e., $\mathcal{A}$ is a trivial algebra), then $\mathcal{A}$ belongs to the class determined by the externally compatible identities in the type $\langle 2, 2, 1, 0, 0\rangle$. One can easily prove that in this case the algebra $\mathcal{A}$ is subdirectly irreducible iff it is a 2-element algebra defined be all externally compatible identities in the type $\langle 2, 2, 1, 0, 0\rangle$.



3. Let $|I| > 1$ and there is $i \in I$, such that $|A_i| > 1$ (see the above figure). For any such i we define a relation R_i w $\mathcal{A}$ stipulating for $a, b \in A$ as follows:

$$aR_ib \text{ iff } a = b \text{ or } a, b \in A_i.$$

The relation R_i is a congruence that differs from Δ. Now, for any $i, j \in I$, such that $i \neq j$ and $|A_i| \neq 1 \neq |A_j|$, $\mathcal{A}$ is subdirectly irreducible. It is so since $R_i \cap R_j = \Delta$.



4. The is exactly one element $i \in I$, such that the cardinality of the set A_i bigger than 1. Without the loss of generality we can assume that is bigger than 2 (see the above diagram). Then, for every $a \in A_{i_0}$ one can define a congruence relation $R(a)$ stipulating for any x, y:

$$xR(a)y \text{ iff } x = y \text{ or } x, y \in A \setminus \{a\}.$$

Each of relations $R(a)$ is a congruence relation different from Δ and

$$\bigcap_{a \in A_{i_0}} R(a) = \Delta.$$

Thus $\mathcal{A}$ is subdirectly irreducible (see Diagram 2).

5. The is exactly one element $i \in I$, for which $A_i = \{0_1, 0_2\}$, where 0_1 is different from 0_2 and is a function c_f that is defined as follows (again see the above picture):

$$C_+(i_0) = C_.(i_0) = C_-(i_0) = O_2.$$

In this case we consider a congruence $R^{''}$ defined in the following way:

$$aR^{''}b \text{ iff } a = b \text{ or } a, b \in A \setminus \{O_1\}.$$

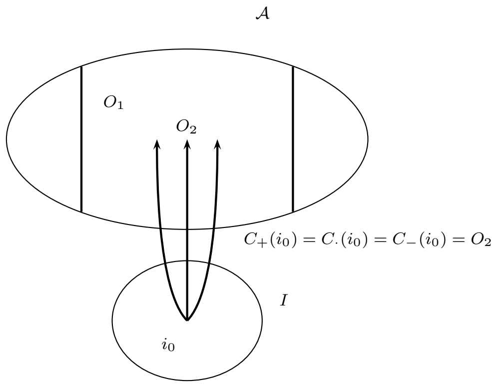


Diagram 2. Identities – algebras

One can easily check that:

$$R_{i_0} \cap R'' = \Delta.$$

Thus, $\mathcal{A}$ is subdirectly irreducible.

Obviously, among dispersions only these described below can be subdirectly irreducible algebras: there is exactly one element $i_0 \in I$, taki że $|A_{i_0}| = 2$, say $A_{i_0} = \{O_1, O_2\}$ and there is a partition $\{F_1, F_2\}$ of the set $\{+, \cdot, -\}$ with blocks $F_1, F_2 \neq \emptyset$ such that $c_f(i_0) = O_k$ for $f \in F_k$ where $k = 1, 2$.

It appears that the above mentioned dispersions are indeed subdirectly irreducible.

Thus, we have the following, main result of this part:

Theorem 3.2. *Let $\mathcal{A}$ be an algebra from the class $\mathbf{MV}_{n_{Ex}}$. The algebra $\mathcal{A}$ is subdirectly irreducible iff at least one of the following three conditions holds:*

1. *$\mathcal{A}$ belongs to the variety of MV_n-algebras and is subdirectly irreducible,*
2. *$\mathcal{A}$ is a 2-element algebra from the class defined by all externally compatible identities in the type $\langle 2, 2, 1, 0, 0\rangle$,*
3. *$\mathcal{A}$ is a dispersion of an algebra $\mathcal{I}$ from the class of MV_n-algebras and there is exactly one element $i_0 \in I$ such that $|A_{i_0}| = 2$, say $A_{i_0} = \{O_1, O_2\}$, and there is a partition $\{F_1, F_2\}$ of the set $\{+, \cdot, -\}$, where F_1, $F_2 \neq \emptyset$ and $c_f(i_0) = O_k$ for $f \in F_k$ $(k = 1, 2)$.*

4. The lattice of varieties generated by *Ex*(MV)

One can see that $Ex(\mathbf{MV})$ is a proper subset of the set $Id(\mathbf{MV})$. We conclude that the variety of MV-algebr is a proper subvariety of the variety $\mathbf{MV}_{Ex}$. Obviously, each subvariety of the class **MV** is also a proper subvariety of the variety $\mathbf{MV}_{Ex}$.

Let us stat with an analysis of the variety MV-algebr. For any variety V in the type τ we put:

$$P^{(V)} = \{K \in \mathcal{L}(V_P) : Id(K) = P(K)\}.$$

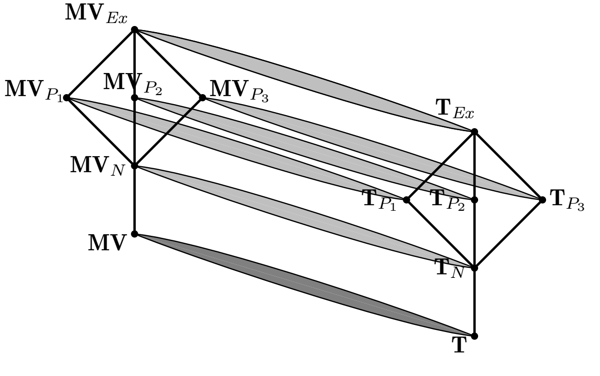
We use the following notation (see [4]):

$$P^{(\mathbf{MV})} = \{K \in \mathcal{L}(\mathbf{MOL}_P)\colon Id(K) = P(\mathbf{MV})\}.$$

The set $P^{(\mathbf{MV})}$ with the inclusion as an order is a lattice. One can say referring to the class **MV**, that it is F-normal and considering it in the w type $\langle 2, 2, 1\rangle$ we see that there are five partitions of the set of symbol of basic operations. Applying theorems 2.8, 2.5, and 2.6 we get:

Theorem 4.1. *For any partition P of the set $\{+, \cdot, ^-\}$ the lattice $P^{(\mathbf{MV})}$ is isomorphic to $\mathcal{L}(\mathbf{MV})$.*

In the below diagram we present mutual positions of lattices $P^{(\mathbf{MV})}$ in the lattice $\mathcal{L}(\mathbf{MV}_{Ex})$.



Subvariety of MV-algebras were examined by R. Grigolia, Y. Komori, A. Di Nola, and A. Lettieri. Lettieri and Di Nola [3] have given an equational basis for all **MV**-varieties, while Komori determined the lattice od subvarieties of the variety of MV-algebras (see [8]).

Following [3] we define for any natural $i > 1$ a set $\delta(i)$ as follows:

$$\delta(i) = \{n \in \mathbf{Z} : 1 \leqslant n \text{ and } n \text{ dzieli } i\}.$$

On the other hand, we any finite, nonempty set J of positive numbers, we put:

$$\Delta(i, J) = \{d \in \delta(i) \setminus \bigcup_{j \in J} \delta(j)\}$$

In the case that $J = \varnothing$, we stipulate:

$$\Delta(i, J) = \delta(i).$$

We recall the following result:

Theorem 4.2 ([3]). *Let V be a proper subvariety of the variety* **MV**. *Then there are finite sets I and J of natural numbers bigger than* 1, *such that $I \cap J \neq \varnothing$ and for any* MV*-algebra $\mathfrak{A}$, $\mathfrak{A}$ belongs to V iff $\mathfrak{A}$ fulfils the following identities:*

$$((n+1)x^n)^2 \approx 2x^{n+1}, \text{ gdzie } n = \max\{I \cup J\}; \tag{4.5}$$

$$(px^{p-1})^{n-1} \approx (n+1)x^p \tag{4.6}$$

and for any positive number p, such that $1 < p < n$ which does not divide any number from $I \cup J$;

$$(n+1)x^q \approx (n+2)x^q, \text{ for any } q \in \bigcup_{j \in J} \Delta(i, J). \tag{4.7}$$

Let us recall that the smallest proper subvariety of the variety od MV-algebras is the class of Boolean algebras. This class is characterised be a single identity $x + x \approx x$ (i.e., in this context, to determine the class of Boolean algebras it is enough to consider the identity $x + x \approx x$ and all identities fulfilled in the class **MV** and the obtained set close under the operator Cn).

Let us recall:

Theorem 4.3 ([11]). *The lattice of all nontrivial subvarieties of the variety $\mathbf{MOL}_{Ex}$,that are generated be the sum of the set $Ex(\mathbf{MOL})$ and the set of all identities of one variable in the type $\langle 2, 2, 1\rangle$, is isomorphic to the lattice $(\mathcal{L}(\mathbf{MOL}) \setminus \mathbf{T}) \times \overline{\mathbf{B}}$.*

For any class V from the lattice $\mathcal{L}(\mathbf{MV})$ we consider a set $\{K \in \mathcal{L}(V_{Ex}) : V \subseteq K \subseteq V_{Ex}\}$. Of course, this set is a lattice which is denoted by $\overline{V}$.

The following two theorems are true. We skip proofs since they are similar to proofs of theorems 2.2 and 4.3.

THEOREM 4.4. *For every nontrivial variety* $V \in L(\mathbf{MV})$ *there is a lattice embedding of the lattice* $\overline{\mathbf{B}}$ *into* $\overline{V}$*, where* $\mathbf{B}$ *is a class of Boolean algebras.*

This theorem has been illustrated on Diagram 3

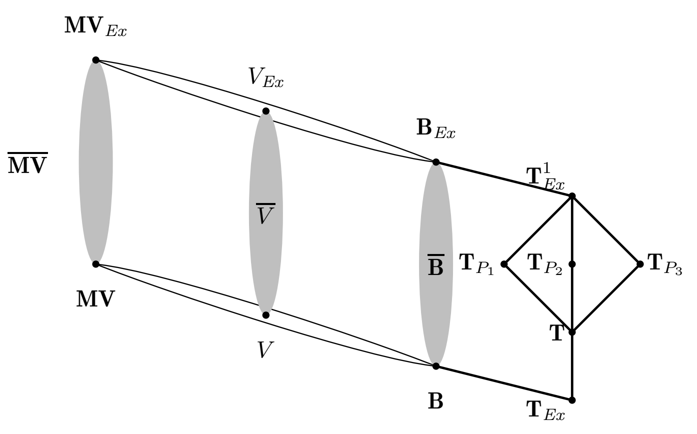


Diagram 3. The lattice of subvarieties of the variety $\mathbf{MV}_{Ex}$

Although we do not know the full description of the whole lattice $\mathcal{L}(\mathbf{MV}_{Ex})$, we do know how the sublattice of this lattice generated by identities of one variable looks like. Strictly speaking the following theorem holds:

THEOREM 4.5. *The lattice of all subvarieties of the variety* $\mathbf{MV}_{Ex}$ *that are generated by identities of one variable is isomorphic to the lattice* $\overline{T} \cup ((L(\mathbf{MV}) \backslash T) \times \overline{\mathbf{B}})$.

Having analysed structures of subdirectly irreducible algebras in the class determined be externally compatible identities of MV_n-algebras we see that there is quite a lot of them — if I may say so — of specific "types of algebras". It is connected to the fact, that the lattice $\mathcal{L}(\mathbf{MV}_{Ex})$ is also quite big and — is some sense — rather complicated. A "horizontal" analysis — selecting varieties described by Komori, Di Nola, and Lettieri, as well as a "vertical" analysis — stressing a correlation with the class of Boolean algebr, can be treated as a partial solution of the problem mentioned at the very beginning of the paper.

Finally, we have the following:

Hypothesis. In the lattice $\mathcal{L}(\mathbf{MV}_{Ex})$ there is no other elements than those predicted by Theorem 4.5.

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