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ON SOME SETS OF IDENTITIES SATISFIED IN ABELIAN GROUPS

Abstract. The equational theories were studied in many works (see [4], [5], [6], [7]). Let τ be a type of Abelian groups. In this paper we consider the extentions of the equational theory $Ex(\mathcal{G}^n)$ defined by so called externally compatible identities of Abelian groups and the identity $x^n \approx y^n$. The equational base of this theory was found in [3]. We prove that each equational theory $Cn(Ex(\mathcal{G}^n) \cup \{\phi \approx \psi\})$, where $\phi \approx \psi$ is an identity of type τ, is equal to the extension of the equational theory $Cn(Ex(\mathcal{G}^n) \cup E)$, where E is a finite set of one variable identities of type τ.

The notation in this paper are the same as in [1].

1. Preliminaries

Let $\tau : \{\cdot, {}^{-1}\} \to N$ be a type of Abelian groups where $\tau(\cdot) = 2$, $\tau({}^{-1}) = 1$. By $\mathcal{G}^n$ we denote the class of all Abelian groups satisfying the identity $x^n \approx y^n$, $n \geq 2$.

The identity of type τ is externally compatible (see [2]) if it is one of the form $x \approx x$ or of the form $\phi_1 \cdot \phi_2 \approx \psi_1 \cdot \psi_2$, $\phi_1^{-1} \approx \psi_1^{-1}$ for some terms $\phi_1, \phi_2, \psi_1, \psi_2$ of type τ. Let $Ex(\mathcal{G}^n)$ be a set of all externally compatible identities satisfied in $\mathcal{G}^n$. In [2] it was proved that $Ex(\mathcal{G}^n)$ is the equational theory. Let $Id(\tau)$ be a set of all identities of type τ. By $Cn(\Sigma)$, where $\Sigma \subseteq Id(\tau)$, we denote the deductive closure of Σ.

It is well known fact, that the lattice of all equational theories extending $Id(\mathcal{G}^n)$ is dually isomorphic to the lattice of all natural divisors of n with divisibility relation. It implies that $Cn(Id(\mathcal{G}^n) \cup \{\phi \approx \psi\}) = Cn(Id(\mathcal{G}^n) \cup \{\phi_1 \approx \psi_1\})$, where $\phi \approx \psi$ and $\phi_1 \approx \psi_1$ are identities of type τ and the last of them is the one variable identity. Indeed, let $\phi \approx \psi$ be an identity of type τ. So, it is equivalent to the identity of the form $x_1^{k_1} \cdot \ldots \cdot x_s^{k_s} \approx x_1^{l_1} \cdot \ldots \cdot$

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$x_s^{l_s}$, where $k_1, \ldots, k_s, l_1, \ldots, l_s \in Z_n$ and $k_i \neq l_i$ for some $i \in \{1, \ldots, s\}$.[1] Then, it is obvious that $Cn(Id(\mathcal{G}^n) \cup \{x_1^{k_1} \cdot \ldots \cdot x_s^{k_s} \approx x_1^{l_1} \cdot \ldots \cdot x_s^{l_s}\}) = Cn(Id(\mathcal{G}^n) \cup \{x_1^{k_1-l_1} \cdot \ldots \cdot x_s^{k_s-l_s} \approx x_1 \cdot x_1^{-1}\})$. Let $d = (k_1 - l_1, \ldots, k_s - l_s)$. Then $Cn(Id(\mathcal{G}^n) \cup \{x_1^{k_1-l_1} \cdot \ldots \cdot x_s^{k_s-l_s} \approx x_1 \cdot x_1^{-1}\}) = Cn(Id(\mathcal{G}^n) \cup \{x_1^d \approx x_1 \cdot x_1^{-1}\})$. Because $d = (k_1 - l_1, \ldots, k_s - l_s)$ then there exist $p_1, \ldots, p_s \in Z_n$ such that $(k_1 - l_1) \cdot p_1 + \ldots + (k_s - l_s) \cdot p_s = d$ and $Cn(Id(\mathcal{G}^n) \cup \{x_1^{k_1} \cdot \ldots \cdot x_s^{k_s} \approx x_1^{l_1} \cdot \ldots \cdot x_s^{l_s}\}) \subseteq Cn(Id(\mathcal{G}^n) \cup \{x_1^{k_1-l_1} \cdot \ldots \cdot x_s^{k_s-l_s} \approx x_1 \cdot x_1^{-1}\})$. So $Cn(Id(\mathcal{G}^n) \cup \{x_1^{k_1} \cdot \ldots \cdot x_s^{k_s} \approx x_1^{l_1} \cdot \ldots \cdot x_s^{l_s}\}) \subseteq Cn(Id(\mathcal{G}^n) \cup \{x_1^{p_1(k_1-l_1)} \cdot \ldots \cdot x_1^{p_s(k_s-l_s)} \approx x_1 \cdot x_1^{-1}\})$ and of course $Cn(Id(\mathcal{G}^n) \cup \{x_1^{k_1-l_1} \cdot \ldots \cdot x_s^{k_s-l_s} \approx x_1 \cdot x_1^{-1}\}) \subseteq Cn(Id(\mathcal{G}^n) \cup \{x_1^d \approx x_1 \cdot x_1^{-1}\})$.

For each $i \in \{1, \ldots, s\}$ we have that $d | (k_i - l_i)$, so $(x_i^{k_i-l_i} \approx x_i \cdot x_i^{-1}) \in Cn(Id(\mathcal{G}^n) \cup \{x_1^d \approx x_1 \cdot x_1^{-1}\})$. Thus $(x_1^{k_1-l_1} \cdot \ldots \cdot x_s^{k_s-l_s} \approx x_1 \cdot x_1^{-1}) \in Cn(Id(\mathcal{G}^n) \cup \{x_1^d \approx x_1 \cdot x_1^{-1}\})$.

The algorithm presented above neglects the structure of identities, and that is why it is useless in the case of extensions of the theory $Ex(\mathcal{G}^n)$.

Using the Galois connection between algebras and identities we have that the lattice of all equational theories of type τ is dually isomorphic to the lattice of all varieties of the same type. So, if we know all theories $Cn(Ex(\mathcal{G}^n) \cup \{\phi \approx \psi\})$, where ϕ and ψ are terms of type τ, we can describe all subvarieties of the variety defined by all externally compatible identities of the variety $\mathcal{G}^n$.

2. The extension of the theory $Ex(\mathcal{G}^n)$

In this paper, as in [3], by x^0 we denote $x \cdot x^{-1}$. Let us consider the following identities:

(1) $x_i \approx x_j$,
(2) $x_1^0 \cdot x_1^{k_1} \cdot \ldots \cdot x_s^{k_s} \approx x_j$,
(3) $((x_1^{k_1} \cdot \ldots \cdot x_s^{k_s})^{-1})^{-1} \approx x_j$,
(4) $x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s} \approx x_1^0 \cdot x_1^{k_1} \cdot \ldots \cdot x_s^{k_s}$,
(5) $((x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s})^{-1})^{-1} \approx ((x_1^0 \cdot x_1^{k_1} \cdot \ldots \cdot x_s^{k_s})^{-1})^{-1}$,
(6) $x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s} \approx ((x_1^0 \cdot x_1^{k_1} \cdot \ldots \cdot x_s^{k_s})^{-1})^{-1}$,

where $s \geq 2$, $i, j \in \{1, \ldots, s\}$, $l_1, \cdots, l_s, k_1, \ldots, k_s \in \{0, \ldots, n-1\}$.

It is possible to prove that every term of type τ of variables $x_1, \ldots, x_s$ $(s \geq 2)$ has one of the following canonnical forms in the variety defined by the set $Ex(\mathcal{G}^n)$: x_j, $x_1^0 \cdot x_1^{k_1} \cdot \ldots \cdot x_s^{k_s}$, $((x_1^{k_1} \cdot \ldots \cdot x_s^{k_s})^{-1})^{-1}$, where $j \in \{1, \ldots, s\}$, $k_1, \ldots, k_s \in \{0, \ldots, n-1\}$. It implies that each identity of type τ is equivalent one of the identities (1)–(6).

[1] If $k_1 = l_1, \ldots, k_s = l_s$, then it is obvious that $Cn(Id(\mathcal{G}^n) \cup \{\phi \approx \psi\}) = Id(\mathcal{G}^n)$.

Let us consider the identity (1). The following lemma is obvious.

LEMMA 1. (a) *If* $i = j$, *then* $Cn(Ex(\mathcal{G}^n) \cup \{(1)\}) = Ex(\mathcal{G}^n)$.

(b) *If* $i \neq j$, *then* $Cn(Ex(\mathcal{G}^n) \cup \{(1)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_i \approx x_i^0\})$. ∎

Now, we study the identity (2).

LEMMA 2. (a) *If* $k_j = 0$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(2)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_j^0 \approx x_j\}).$$

(b) *If* $k_j = 1, k_1 = k_2 = \ldots = k_{j-1} = k_{j+1} = \ldots = k_s = 0$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(2)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_j^0 \cdot x_j \approx x_j\}).$$

(c) *If* $k_j = 1, k_1^2 + \ldots + k_{j-1}^2 + k_{j+1}^2 + \ldots + k_s^2 > 0$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(2)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_j \approx x_j^0 \cdot x_j^{d+1}\}),$$

where $d = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_s)$.

(d) *If* $k_j \geq 2$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(2)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_j^0 \cdot x_j^{d+1} \approx x_j\}),$$

where $d = (k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_s)$.

Proof. Without losing generality we can assume that $j = 1$. Let $S_1 = Cn(Ex(\mathcal{G}^n) \cup \{(2)\})$.

(a) Let $S_2 = Cn(Ex(\mathcal{G}^n) \cup \{x_1^0 \approx x_1\})$. If we put $x_j = x_1^0, j = 2, \ldots, s$ we get $S_2 \subseteq S_1$. From the fact that $(x^0 \approx y^0) \in Ex(\mathcal{G}^n)$ we get $(x \approx y) \in S_2$. From this we obtain immediately $S_1 \subseteq S_2$.

(b) Let $S_2 = Cn(Ex(\mathcal{G}^n) \cup \{x_1 \cdot x_1^0 \approx x_1\})$. Because $k_1 - 1 = k_2 = \ldots = k_s = 0$ then $S_1 = S_2$ is obvious.

(c) Let $S_2 = Cn(Ex(\mathcal{G}^n) \cup \{x_1 \approx x_1^0 \cdot x_1^{d+1}\})$. Putting $x_j = x_1^0$ for $j \geq 2$ in the identity (1) we get $(x_1 \approx x_1 \cdot x_1^0) \in S_1$. Let the sequence $p_2, \ldots, p_s$ of integers be a solution of the equation $k_2 \cdot t_2 + \ldots + k_s \cdot t_s = (k_2, \ldots, k_s)$. Putting $x_j = x_1^{p_j}$ for $j \in \{2, \ldots, s\}$ in the identity (1) we get, that $(x_1 \approx x_1 \cdot x_1^{k_2 \cdot p_2 + \ldots + k_s \cdot p_s}) \in S_1$ and thus $(x_1 \cdot x_1^0 \approx x_1 \cdot x_1^0 \cdot x_1^{k_2 \cdot p_2 + \ldots + k_s \cdot p_s}) \in S_1$, so we have $(x_1 \approx x_1^0 \cdot x_1^{d+1}) \in S_1$. Finally, we have $S_2 \subseteq S_1$.

To prove the opposite inclusion let us note, that from the condition $(x_1 \approx x_1^0 \cdot x_1^{d+1}) \in S_2$ it follows that $(x_1^0 \approx x_1^0 \cdot x_1^d) \in S_2$. The immediate consequence of these conditions is $(x_1 \approx x_1^0 \cdot x_1) \in S_2$. The definition of d implying that for each j from the set $\{2, \ldots, s\}$ a number d is a divisor of k_j. Hence there exist elements $p_2, \ldots, p_s$ in the set Z_n such that $k_j = p_j \cdot d$. As a result of the condition $(x_1^0 \approx x_1^0 \cdot x_1^d) \in S_2$ we have that for each $j \in \{2, \ldots, s\}$ the identity $x_j^0 \approx x_j^0 \cdot x_j^{p_j \cdot d}$ belongs to S_2. From the fact that $(x_1^0 \approx x_1^0 \cdot x_2^0 \cdot \ldots \cdot x_s^0) \in S_2$, we obtain $(x_1^0 \approx x_1^0 \cdot x_2^{p_2 \cdot d} \cdot \ldots \cdot x_s^{p_s \cdot d}) \in S_2$. Using earlier notation we get $(x_1^0 \approx x_1^0 \cdot x_2^{k_2} \cdot \ldots \cdot x_s^{k_s}) \in S_2$ and of course

$(x_1 \cdot x_1^0 \approx x_1^0 \cdot x_1 \cdot x_2^{k_2} \cdot \ldots \cdot x_s^{k_s}) \in S_2$. From this and from the condition $(x_1 \approx x_1 \cdot x_1^0) \in S_2$ we get $(x_1 \approx x_1^0 \cdot x_1 \cdot x_2^{k_2} \cdot \ldots \cdot x_s^{k_s}) \in S_2$. It completes the proof.

(d) Let $S_2 = Cn(Ex(\mathcal{G}^n) \cup \{x_1 \approx x_1^0 \cdot x_1^{d+1}\})$, where $d = (k_1 - 1, k_2, \ldots, k_s)$. From the fact that $(x_1 \approx x_1^0 \cdot x_1^{k_1} \cdot \ldots \cdot x_s^{k_s}) \in S_1$ we get that $(x_1^0 \approx x_1^0 \cdot x_1^{k_1 - 1} \cdot \ldots \cdot x_s^{k_s}) \in S_1$. It is obvious that $(x_1^0 \approx x_1^0 \cdot x_1^d) \in S_1$, where $d = (k_1 - 1, k_2, \ldots, k_s)$. From the other hand we have that $(x_1 \approx x_1^0 \cdot x_1^{k_1}) \in S_1$ (we get it putting $x_j = x_1^0$ for $j \in \{2, \ldots, s\}$). From this we obtain $(x_1^0 \approx x_1^0 \cdot x_1^{k_1 - 1}) \in S_1$, and of course we get $(x_1 \approx x_1^0 \cdot x_1) \in S_1$. From this we get $(x_1 \approx x_1^0 \cdot x_1^{d+1}) \in S_1$. So, we have proved that $S_2 \subseteq S_1$.

Now, let we prove the opposite inclusion. Analogously to the proof of (c) we can show that $(x_1 \approx x_1 \cdot x_1^0) \in S_2$. From the fact that $(x_1 \approx x_1^0 \cdot x_1^{d+1}) \in S_2$ we obtain that $(x_1^0 \approx x_1^0 \cdot x_1^d) \in S_2$. The number d is a divisor of $k_1 - 1$ then there exists $p_1 \in Z_n$ such that $d \cdot p_1 = k_1 - 1$. Putting $x_1 = x_1^{p_1}$ in the identity $x_1^0 \approx x_1^0 \cdot x_1^d$ we get $(x_1^0 \approx x_1^0 \cdot x_1^{k_1 - 1}) \in S_2$. From this we have $(x_1 \cdot x_1^0 \approx x_1^0 \cdot x_1^{k_1}) \in S_2$. By this and by the condition $(x_1 \approx x_1 \cdot x_1^0) \in S_2$ we have that $(x_1 \approx x_1^0 \cdot x_1^{k_1}) \in S_2$. Now it is easy to verify that $(x_1 \approx x_1^0 \cdot x_1^{k_1} \cdot x_2^{k_2} \cdot \ldots \cdot x_s^{k_s}) \in S_2$ (similarly as in proof of (c)). So we get the inclusion $S_1 \subset S_2$. It completes the proof of Lemma 2. ■

Now, let us regard the identity (3).

LEMMA 3. (a) *If* $k_j = 0$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(3)\}) = Cn(Ex(\mathcal{G}^n) \cup \{((x_j^0)^{-1})^{-1} \approx x_j\}).$$

(b) *If* $k_j = 1, k_1 = k_2 = \ldots = k_{j-1} = k_{j+1} = \ldots = k_s = 0$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(3)\}) = Cn(Ex(\mathcal{G}^n) \cup \{((x_j^0 \cdot x_j)^{-1})^{-1} \approx x_j\}).$$

(c) *If* $k_j = 1, k_1^2 + \ldots + k_{j-1}^2 + k_{j+1}^2 + \ldots + k_s^2 > 0$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(3)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_j \approx ((x_j^0 \cdot x_j^{d+1})^{-1})^{-1}\}),$$

where $d = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_s)$.

(d) *If* $k_j \geq 2$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(3)\}) = Cn(Ex(\mathcal{G}^n) \cup \{((x_j^0 \cdot x_j^{d+1})^{-1})^{-1} \approx x_j\}),$$

where $d = (k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_s)$.

Proof. The proof of this lemma is analogously to the proof of the Lemma 2. ■

Let we study the identity (4).

LEMMA 4. (a) *If* $l_1 = k_1, \cdots, l_s = k_s$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(4)\}) = Ex(\mathcal{G}^n).$$

(b) *If* $l_j \neq k_j$ *for some* $j \in \{1, \ldots, s\}$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(4)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^d\}),$$

where $d = (l_1 - k_1, \ldots, l_s - k_s)$.

Proof. The proof of (a) is obvious.

To prove (b) let us use some notation. Let $S_1 = Cn(Ex(\mathcal{G}^n) \cup \{(4)\})$ and $S_2 = Cn(Ex(\mathcal{G}^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^d\})$. It is easy to check that $(x_1^0 \approx x_1^0 \cdot x_1^{r_1} \ldots x_s^{r_s}) \in S_1$, where $r_i = l_i - k_i$, if $l_i \geq k_i$ or $r_i = n - (l_i - k_i)$ in opposite case. From this it follows directly that for each i from the set $\{1, \ldots, s\}$ it holds $(x_1^0 \approx x_1^0 \cdot x_1^{r_i}) \in S_1$, and thereby $(x_1 \approx x_1^0 \cdot x_1^{(r_1, \ldots, r_s)}) \in S_1$. We have proved that $S_2 \subseteq S_1$.

To prove the opposite inclusion let us observe that $(r_1, \ldots, r_s) | r_i$ for each $i \in \{1, \ldots, s\}$. Hence, for each $i \in \{1, \ldots, s\}$ there exists $p_i \in \{0, \ldots, n-1\}$ such that $r_i = p_i \cdot (r_1, \ldots, r_s)$. Putting $x_1 = x_1^{p_1} \cdot \ldots \cdot x_s^{p_s}$ in the identity $x_1^0 \approx x_1^0 \cdot x_1^{(r_1, \ldots, r_s)}$ we get $(x_1^0 \approx x_1^0 \cdot x_1^{r_1} \cdot \ldots \cdot x_s^{r_s}) \in S_2$. From the above it follows directly that the identity (4) belongs to the set S_2, thus $S_1 \subseteq S_2$. So, the lemma has been proved. ∎

Now we consider the identity (5).

LEMMA 5. (a) *If* $l_1 = k_1, \cdots, l_s = k_s$, *then* $Cn(Ex(\mathcal{G}^n) \cup \{(5)\}) = Ex(\mathcal{G}^n)$.

(b) *If* $l_j \neq k_j$ *for some* $j \in \{1, \ldots, s\}$, *then* $Cn(Ex(\mathcal{G}^n) \cup \{(5)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^d\})$, *where* $d = (l_1 - k_1, \ldots, l_s - k_s)$.

Proof. The proof of this lemma is analogous to the proof of the last lemma. ∎

Now, let us regard the identity (6).

LEMMA 6. (a) *If* $l_i = k_i = 0$ *for* $i \in \{1, \cdots, s\}$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(6)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_1^0 \approx ((x_1^0)^{-1})^{-1}\}).$$

(b) *If* $k_i = l_i$ *for each* $i \in \{1, \ldots, s\}$ *and* $k_j \neq 0$ *for some* $j \in \{1, \ldots s\}$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(6)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_1^0 \cdot x_1^{(k_1, \ldots, k_s)} \approx ((x_1^{(k_1, \ldots, k_s)})^{-1})^{-1}\}.$$

(c) *If* $k_j \neq l_j$ *for some* $j \in \{1, \ldots, s\}$, *then*

$$Cn(Ex(\mathcal{G}^n) \cup \{(6)\} = Cn(Ex(\mathcal{G}^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^{(k_1 - l_1, \ldots, k_s - l_s)}, x_1^0 \cdot x_1^{(l_1, \ldots, l_s)} \approx ((x_1^{p_1 \cdot k_1 + \ldots + p_s \cdot k_s})^{-1})^{-1}\}),$$

where $p_1, \ldots, p_s$ *satisfy the following condition* $p_1 \cdot l_1 + \ldots + p_s \cdot l_s = (l_1, \ldots l_s)$, $p_1, \ldots, p_s \in Z_n$.

Proof. (a) The proof is obvious.

(b) It is enough to observe that the equation $t_1 \cdot k_1 + \ldots + t_s \cdot k_s = (k_1, \ldots k_s)$ has a solution in the set Z_n.

(c) Let $S_1 = Cn(Ex(\mathcal{G}^n) \cup \{(6)\})$ and $S_2 = Cn(Ex(\mathcal{G}^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^{(k_1-l_1,\ldots,k_s-l_s)}, x_1^0 \cdot x_1^{(l_1,\ldots,l_s)} \approx ((x_1^{p_1\cdot k_1+\ldots+p_s\cdot k_s})^{-1})^{-1}\})$, where $p_1,\ldots,p_s$ are defined above.

In the identity (6) let us put $x_i = x_1^{p_i}$. We get, that $(x_1^0 \cdot x_1^{(l_1,\ldots,l_s)} \approx ((x_1^{p_1\cdot k_1+\ldots+p_s\cdot k_s})^{-1})^{-1}) \in S_1$. It is clear, that from the definition of the set S_1 it follows that for each $i \in \{1,\ldots,s\}$ the identity $x_i^0 \approx x_i^0 \cdot x_i^{(k_i-l_i)}$ belongs to S_1. Analogously, as in the proof of Lemma 2 we get, that $(x_1^0 \approx x_1^0 \cdot x_1^{(k_1-l_1,\ldots,k_s-l_s)}) \in S_1$. We have proved, that $S_2 \subseteq S_1$.

To prove the opposite inclusion in the identity

$$x_1^0 \cdot x_1^{(l_1,\ldots,l_s)} \approx ((x_1^{p_1\cdot k_1+\ldots+p_s\cdot k_s})^{-1})^{-1}$$

we put $x_1 = x_1^{\frac{l_1}{(l_1,\ldots,l_s)}} \cdot \ldots \cdot x_s^{\frac{l_s}{(l_1,\ldots,l_s)}}$. We get, that the identity

$$(*) \quad x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s} \approx ((x_1^{\frac{l_1}{(l_1,\ldots,l_s)}\cdot(p_1\cdot k_1+\ldots+p_s\cdot k_s)} \cdot \ldots \cdot x_s^{\frac{l_s}{(l_1,\ldots,l_s)}\cdot(p_1\cdot k_1+\ldots+p_s\cdot k_s)})^{-1})^{-1}$$

belongs to S_2.

For each $i \in \{1,\ldots,s\}$ let us consider the equation $h_i \cdot (k_1-l_1,\ldots,k_s-l_s) + \frac{l_i}{(l_1,\ldots,l_s)} \cdot (p_1\cdot k_1 + \ldots + p_s \cdot k_s) = k_i$. We show, that $h_i = \frac{k_i-l_i}{(k_1-l_1,\ldots,k_s-l_s)} - \frac{p_1\cdot l_i(k_1-l_1)}{(l_1,\ldots,l_s)\cdot(k_1-l_1,\ldots,k_s-l_s)} - \ldots - \frac{p_s\cdot l_i(k_s-l_s)}{(l_1,\ldots,l_s)\cdot(k_1-l_1,\ldots,k_s-l_s)}$ is a solution of this equation.

Because $(l_1,\ldots,l_s)|l_i$ and $(k_1-l_1,\ldots,k_s-l_s)|(k_r-l_r)$ for each $r \in \{1,\ldots,s\}$ then $h_i \in Z$. Hence $h_i\cdot(k_1-l_1,\ldots,k_s-l_s) = (k_i-l_i) - \frac{l_i}{(l_1,\ldots,l_s)}(p_1 \cdot (k_1-l_1)+\ldots+p_s\cdot(k_s-l_s))$ and $h_i\cdot(k_1-l_1,\ldots,k_s-l_s) = (k_i-l_i) - \frac{l_i}{(l_1,\ldots,l_s)} \cdot (p_1\cdot k_1 + \ldots + p_s\cdot k_s - p_1\cdot l_1 - \ldots - p_s\cdot l_s) = (k_i-l_i) - \frac{l_i}{(l_1,\ldots,l_s)} \cdot (p_1\cdot k_1 + \ldots + p_s\cdot k_s - (l_1,\ldots,l_s)) = k_i - \frac{l_i}{(l_1,\ldots,l_s)} \cdot (p_1\cdot k_1 + \ldots + p_s\cdot k_s)$.

Thus for each $i \in \{1,\ldots,s\}$ the identity $x_i^0 \approx x_1 \cdot x_i^{h_i\cdot(k_1-l_1,\ldots,k_s-l_s)}$ belongs to S_2. Hence, as a result of the fact, that the identity (*) belongs to S_2 we get, that the identity

$$x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s} \approx ((x_1^0 \cdot x_1^{h_1\cdot(k_1-l_1,\ldots,k_s-l_s)+\frac{l_1}{(l_1,\ldots,l_s)}\cdot(p_1\cdot k_1+\ldots+p_s\cdot k_s)} \cdot \ldots \cdot x_s^{h_s\cdot(k_1-l_1,\ldots,k_s-l_s)+\frac{l_s}{(l_1,\ldots,l_s)}\cdot(p_1\cdot k_1+\ldots+p_s\cdot k_s)})^{-1})^{-1}$$

belongs to S_2. So, we get that $(x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s} \approx ((x_1^{k_1} \cdot \ldots \cdot x_s^{k_s})^{-1})^{-1}) \in S_2$. It implies that $S_1 \subseteq S_2$.

Finally, we have proved that $S_1 = S_2$. ■

From Lemmas 1–6 we obtain

THEOREM 1. *If E is a finite set of identities of type τ then there exists a set E_1 of one variable identities such that $Cn(Ex(\mathcal{G}^n) \cup E) = Cn(Ex(\mathcal{G}^n) \cup E_1)$.*

By $\mathcal{G}^n_{Ex}$ we denote the variety defined by the set $Ex(\mathcal{G}^n)$. The consequence of Theorem 1 is the following theorem

THEOREM 2. *Let τ be a type of Abelian groups with the exponent n and let $\mathcal{A}$ be a free algebra in the variety $\mathcal{G}^n_{Ex}$ with a one element set of generators. Then $Id(\mathcal{A}) = Id(\mathcal{G}^n_{Ex})$.*

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