ON SOME SETS OF IDENTITIES SATISFIED IN ABELIAN GROUPS

Abstract. The equational theories were studied in many works (see [4], [5], [6], [7]). Let \( T \) be a type of Abelian groups. In this paper we consider the extensions of the equational theory \( E_x(G^n) \) defined by so-called externally compatible identities of Abelian groups and the identity \( x^n \approx y^n \). The equational base of this theory was found in [3]. We prove that each equational theory \( C_n(E_x(G^n) \cup \{ \phi \approx \psi \}) \), where \( \phi \approx \psi \) is an identity of type \( \tau \), is equal to the extension of the equational theory \( C_n(E_x(G^n) \cup E) \), where \( E \) is a finite set of one variable identities of type \( \tau \).

The notation in this paper are the same as in [1].

1. Preliminaries

Let \( \tau : \{ , -1 \} \rightarrow \mathbb{N} \) be a type of Abelian groups where \( \tau(,) = 2, \tau(-1) = 1 \). By \( G^n \) we denote the class of all Abelian groups satisfying the identity \( x^n \approx y^n, n \geq 2 \).

The identity of type \( \tau \) is externally compatible (see [2]) if it is one of the form \( x \approx x \) or of the form \( \phi_1 \cdot \phi_2 \approx \psi_1 \cdot \psi_2, \phi_1^{-1} \approx \psi_1^{-1} \) for some terms \( \phi_1, \phi_2, \psi_1, \psi_2 \) of type \( \tau \). Let \( E_x(G^n) \) be a set of all externally compatible identities satisfied in \( G^n \). In [2] it was proved that \( E_x(G^n) \) is the equational theory. Let \( Id(\tau) \) be a set of all identities of type \( \tau \). By \( C_n(\Sigma) \), where \( \Sigma \subseteq Id(\tau) \), we denote the deductive closure of \( \Sigma \).

It is well known fact, that the lattice of all equational theories extending \( Id(G^n) \) is dually isomorphic to the lattice of all natural divisors of \( n \) with divisibility relation. It implies that \( C_n(Id(G^n) \cup \{ \phi \approx \psi \}) = C_n(Id(G^n) \cup \{ \phi_1 \approx \psi_1 \}) \), where \( \phi \approx \psi \) and \( \phi_1 \approx \psi_1 \) are identities of type \( \tau \) and the last of them is the one variable identity. Indeed, let \( \phi \approx \psi \) be an identity of type \( \tau \). So, it is equivalent to the identity of the form \( x_1^{k_1} \cdots x_s^{k_s} \approx x_1^{m_1} \cdots \)
$x^i_s$, where $k_1, \ldots, k_s, l_1, \ldots, l_s \in \mathbb{Z}_n$ and $k_i \neq l_i$ for some $i \in \{1, \ldots, s\}$.\footnote{If $k_1 = l_1, \ldots, k_s = l_s$, then it is obvious that $Cn(Id(G^n) \cup \{\phi \approx \psi\}) = Id(G^n)$.}

Then, it is obvious that $Cn(Id(G^n) \cup \{x^k_1 \cdots x^{k_s}_s \approx x^l_1 \cdots x^l_s\}) = Cn(Id(G^n) \cup \{x^k_1 \cdots x^{k_s}_s \approx x^l_1 \cdots x^l_s\})$. Let $d = (k_1 - l_1, \ldots, k_s - l_s)$. Then $Cn(Id(G^n) \cup \{x^{k_1-l_1}_1 \cdots x^{k_s-l_s}_s \approx x^1_1 \cdots x^1_s\}) = Cn(Id(G^n) \cup \{x^{1}_1 \cdots x^{1}_s\})$.

Because $d = (k_1 - l_1, \ldots, k_s - l_s)$ then there exist $p_1, \ldots, p_s \in \mathbb{Z}_n$ such that $(k_1 - l_1) p_1 + \cdots + (k_s - l_s) p_s = d$ and $Cn(Id(G^n) \cup \{x^{k_1}_1 \cdots x^{k_s}_s \approx x^{l_1}_1 \cdots x^{l_s}_s\}) \subseteq Cn(Id(G^n) \cup \{x^{k_1-l_1}_1 \cdots x^{k_s-l_s}_s \approx x^1_1 \cdots x^1_s\})$. So $Cn(Id(G^n) \cup \{x^{1}_1 \cdots x^{1}_s\}) \subseteq Cn(Id(G^n) \cup \{x_1^{p_1(k_1-l_1)} \cdots x_s^{p_s(k_s-l_s)} \approx x^1_1 \cdots x^1_s\})$ and of course $Cn(Id(G^n) \cup \{x^{k_1}_1 \cdots x^{k_s}_s \approx x^1_1 \cdots x^1_s\}) \subseteq Cn(Id(G^n) \cup \{x^1_1 \cdots x^1_s\})$.

For each $i \in \{1, \ldots, s\}$ we have that $d(k_i - l_i)$, so $x^{k_i-l_i}_i \approx x^1 \cdots x^1_i \approx x^1_1 \cdots x^1_s \approx x^1_1 \cdots x^1_s$. Thus $x^{k_1}_1 \cdots x^{k_s}_s \approx x^1_1 \cdots x^1_s$.

The algorithm presented above neglects the structure of identities, and that is why it is useless in the case of extensions of the theory $Ex(G^n)$.

Using the Galois connection between algebras and identities we have that the lattice of all equational theories of type $\tau$ is dually isomorphic to the lattice of all varieties of the same type. So, if we know all theories $Cn(Ex(G^n) \cup \{\phi \approx \psi\})$, where $\phi$ and $\psi$ are terms of type $\tau$, we can describe all subvarieties of the variety defined by all externally compatible identities of the variety $G^n$.

2. The extension of the theory $Ex(G^n)$

In this paper, as in [3], by $x^0$ we denote $x \cdot x^{-1}$. Let us consider the following identities:

- (1) $x_i \approx x_j,$
- (2) $x^0_0 \cdot x^k_1 \cdots x^{k_s}_s \approx x_j,$
- (3) $((x^k_1 \cdots x^{k_s}_s)^{-1})^{-1} \approx x_j,$
- (4) $x^0_0 \cdot x^k_1 \cdots x^k_s \approx x^0_1 \cdot x^k_1 \cdots x^k_s,$
- (5) $((x^0_0 \cdot x^k_1 \cdots x^{k_s}_s)^{-1})^{-1} \approx ((x^0_1 \cdot x^k_1 \cdots x^{k_s}_s)^{-1})^{-1},$
- (6) $x^0_0 \cdot x^k_1 \cdots x^{k_s}_s \approx ((x^0_1 \cdot x^k_1 \cdots x^{k_s}_s)^{-1})^{-1},$

where $s \geq 2$, $i, j \in \{1, \ldots, s\}$, $l_1, \ldots, l_s, k_1, \ldots, k_s \in \{0, \ldots, n-1\}$. It is possible to prove that every term of type $\tau$ of variables $x_1, \ldots, x_s$ ($s \geq 2$) has one of the following canonical forms in the variety defined by the set $Ex(G^n)$: $x_j$, $x^0_0 \cdot x^k_1 \cdots x^{k_s}_s$, $((x^k_1 \cdots x^{k_s}_s)^{-1})^{-1}$, where $j \in \{1, \ldots, s\}$, $k_1, \ldots, k_s \in \{0, \ldots, n-1\}$. It implies that each identity of type $\tau$ is equivalent one of the identities (1)–(6).
Let us consider the identity (1). The following lemma is obvious.

**Lemma 1.** (a) If \( i = j \), then \( Cn(Ex(G^n) \cup \{1\}) = Ex(G^n) \).

(b) If \( i \neq j \), then \( Cn(Ex(G^n) \cup \{1\}) = Cn(Ex(G^n) \cup \{x_i \approx x_j\}) \). \( \blacksquare \)

Now, we study the identity (2).

**Lemma 2.** (a) If \( k_j = 0 \), then
\[ Cn(Ex(G^n) \cup \{(2)\}) = Cn(Ex(G^n) \cup \{x_j^0 \approx x_j\}) \]

(b) If \( k_j = 1, k_1 = k_2 = \ldots = k_{j-1} = k_{j+1} = \ldots = k_s = 0 \), then
\[ Cn(Ex(G^n) \cup \{(2)\}) = Cn(Ex(G^n) \cup \{x_j^0 \cdot x_j \approx x_j\}) \]

(c) If \( k_j = 1, k_1^2 + \ldots + k_{j-1}^2 + k_{j+1}^2 + \ldots + k_s^2 > 0 \), then
\[ Cn(Ex(G^n) \cup \{(2)\}) = Cn(Ex(G^n) \cup \{x_j^0 \cdot x_j^{d+1}\}) \]

where \( d = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_s) \).

(d) If \( k_j \geq 2 \), then
\[ Cn(Ex(G^n) \cup \{(2)\}) = Cn(Ex(G^n) \cup \{x_j^0 \cdot x_j^{d+1} \approx x_j\}) \]

where \( d = (k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_s) \).

**Proof.** Without losing generality we can assume that \( j = 1 \). Let \( S_1 = Cn(Ex(G^n) \cup \{(2)\}) \).

(a) Let \( S_2 = Cn(Ex(G^n) \cup \{x_j^0 \approx x_1\}) \). If we put \( x_j = x_1^0, j = 2, \ldots, s \) we get \( S_2 \subseteq S_1 \). From the fact that \((x_1^0 \approx y_1^0) \in Ex(G^n)\) we get \((x \approx y) \in S_2 \).

From this we obtain immediately \( S_1 \subseteq S_2 \).

(b) Let \( S_2 = Cn(Ex(G^n) \cup \{x_1 \cdot x_j^0 \approx x_1\}) \). Because \( k_1 - 1 = k_2 = \ldots = k_s = 0 \) then \( S_1 = S_2 \) is obvious.

(c) Let \( S_2 = Cn(Ex(G^n) \cup \{x_1 \approx x_1^0 \cdot x_1^{d+1}\}) \). Putting \( x_j = x_1^0 \) for \( j \geq 2 \) in the identity (1) we get \((x_1 \approx x_1 \cdot x_1^0) \in S_1 \). Let the sequence \( p_2, \ldots, p_s \) of integers be a solution of the equation \( k_2 \cdot t_2 + \ldots + k_s \cdot t_s = (k_2, \ldots, k_s) \).

Putting \( x_j = x_1^0 \) for \( j \in \{2, \ldots, s\} \) in the identity (1) we get, that \((x_1 \approx x_1^0 \cdot x_1 \cdot x_1^0 \cdot x_1 \cdot \ldots \cdot x_1^0 \cdot x_1) \in S_1 \) and thus \((x_1 \cdot x_1^0 \approx x_1 \cdot x_1^0 \cdot x_1 \cdot x_1^0 \cdot x_1 \cdot \ldots \cdot x_1^0 \cdot x_1) \in S_1 \), so we have \((x_1 \approx x_1^0 \cdot x_1^{d+1}) \in S_1 \). Finally, we have \( S_2 \subseteq S_1 \).

To prove the opposite inclusion let us note, that from the condition \((x_1 \approx x_1 \cdot x_1^{d+1}) \in S_2 \) it follows that \((x_1^0 \approx x_1^0 \cdot x_1^0) \in S_2 \). The immediate consequence of these conditions is \((x_1 \approx x_1 \cdot x_1^0) \in S_2 \). The definition of \( d \) implying that for each \( j \) from the set \( \{2, \ldots, s\} \) a number \( d \) is a divisor of \( k_j \). Hence there exist elements \( p_2, \ldots, p_s \) in the set \( Z_n \) such that \( k_j = p_j \cdot d \).

As a result of the condition \((x_1^0 \approx x_1^0 \cdot x_1^0) \in S_2 \) we have that for each \( j \in \{2, \ldots, s\} \) the identity \( x_j^0 \approx x_j^0 \cdot x_j^0 \cdot p_j^d \) belongs to \( S_2 \). From the fact that \((x_1 \approx x_1^0 \cdot x_2^0 \cdot \ldots \cdot x_s^0) \in S_2 \), we obtain \((x_1^0 \approx x_1^0 \cdot x_2^{d^2} \cdot \ldots \cdot x_s^{d^s}) \in S_2 \). Using earlier notation we get \((x_1^0 \approx x_1^0 \cdot x_2^0 \cdot \ldots \cdot x_s^0) \in S_2 \) and of course
(x_1 \approx x_1 \cdot x_1^k \cdot \ldots \cdot x_s^k) \in S_2. \text{ From this and from the condition } (x_1 \approx x_1 \cdot x_1) \in S_2 \text{ we get } (x_1 \approx x_1 \cdot x_1^k \cdot \ldots \cdot x_s^k) \in S_2. \text{ It completes the proof.}

(d) Let S_2 = Cn(Ex(\mathcal{G}^n) \cup \{x_1 \approx x_1^0 \cdot x_1^{d+1}\}), \text{ where } d = (k_1 - 1, k_2, \ldots, k_s).

From the fact that \((x_1 \approx x_1^0 \cdot x_1^{k_1} \cdot \ldots \cdot x_s^k) \in S_1\) we get that \((x_1^0 \approx x_1^1 \cdot x_1^{k_1-1} \cdot \ldots \cdot x_s^k) \in S_1). \text{ It is obvious that } (x_1 \approx x_1^0 \cdot x_1^d) \in S_1, \text{ where } d = (k_1 - 1, k_2, \ldots, k_s).

From the other hand we have that \((x_1 \approx x_1^0 \cdot x_1^{k_1}) \in S_1) \text{ (we get it putting } x_j = x_j^0 \text{ for } j \in \{2, \ldots, s\}). \text{ From this we obtain } (x_1^0 \approx x_1^0 \cdot x_1^{k_1-1}) \in S_1, \text{ and of course we get } (x_1 \approx x_1^0 \cdot x_1) \in S_1. \text{ From this we get } (x_1 \approx x_1^0 \cdot x_1^{d+1}) \in S_1.

So, we have proved that \(S_2 \subseteq S_1\).

Now, let we prove the opposite inclusion. Analogously to the proof of (c) we can show that \((x_1 \approx x_1 \cdot x_1^d) \in S_2). \text{ From the fact that } (x_1 \approx x_1^0 \cdot x_1^{d+1}) \in S_2 \text{ we obtain that } (x_1 \approx x_1^0 \cdot x_1^d) \in S_2). \text{ The number } d \text{ is a divisor of } k_1 - 1 \text{ then there exists } p_1 \in Z_n \text{ such that } d \cdot p_1 = k_1 - 1. \text{ Putting } x_1 = x_1^p_1 \text{ in the identity } x_1 \approx x_1^0 \cdot x_1^d \text{ we get } (x_1^0 \approx x_1^0 \cdot x_1^{k_1-1}) \in S_2. \text{ From this we have } (x_1 \approx x_1^0 \cdot x_1^{k_1}) \in S_2. \text{ By this and by the condition } (x_1 \approx x_1 \cdot x_1^d) \in S_2 \text{ we have that } (x_1 \approx x_1^0 \cdot x_1^{k_1}) \in S_2. \text{ Now it is easy to verify that } (x_1 \approx x_1^0 \cdot x_1^{k_1} \cdot x_s^k) \in S_2) \text{ (similarly as in proof of (c))}. \text{ So we get the inclusion } S_1 \subseteq S_2. \text{ It completes the proof of Lemma 2.}

Now, let us regard the identity (3).

Lemma 3. (a) If \(k_j = 0\), then
\[ Cn(Ex(\mathcal{G}^n) \cup \{(3)\}) = Cn(Ex(\mathcal{G}^n) \cup \{(x_j)^{d-1} \approx x_j\}). \]

(b) If \(k_j = 1, k_1 = k_2 = \ldots = k_{j-1} = k_{j+1} = \ldots = k_s = 0\), then
\[ Cn(Ex(\mathcal{G}^n) \cup \{(3)\}) = Cn(Ex(\mathcal{G}^n) \cup \{(x_j)^{d-1} \approx x_j\}). \]

(c) If \(k_j = 1, k_1^2 + \ldots + k_{j-1}^2 + k_{j+1}^2 + \ldots + k_s^2 > 0\), then
\[ Cn(Ex(\mathcal{G}^n) \cup \{(3)\}) = Cn(Ex(\mathcal{G}^n) \cup \{x_j \approx ((x_j)^d)^{d-1}\})), \]
where \(d = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_s)\).

(d) If \(k_j \geq 2\), then
\[ Cn(Ex(\mathcal{G}^n) \cup \{(3)\}) = Cn(Ex(\mathcal{G}^n) \cup \{(x_j)^{d+1})^{d-1} \approx x_j\}), \]
where \(d = (k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_s)\).

Proof. The proof of this lemma is analogously to the proof of the Lemma 2.

Let us regard the identity (4).

Lemma 4. (a) If \(l_1 = k_1, \ldots, l_s = k_s\), then
\[ Cn(Ex(\mathcal{G}^n) \cup \{(4)\}) = Ex(\mathcal{G}^n). \]
(b) If \( l_j \neq k_j \) for some \( j \in \{1, \ldots, s\} \), then
\[
C_n(Ex(G^n) \cup \{(4)\}) = C_n(Ex(G^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^d\}),
\]
where \( d = (l_1 - k_1, \ldots, l_s - k_s) \).

Proof. The proof of (a) is obvious.

To prove (b) let us use some notation. Let \( S_1 = C_n(Ex(G^n) \cup \{(4)\}) \) and \( S_2 = C_n(Ex(G^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^d\}) \). It is easy to check that \( (x_1^0 \approx x_1^0 \cdot x_1^d) \) is satisfied in \( S_1 \), where \( r_i = l_i - k_i \), if \( l_i \geq k_i \) or \( r_i = n - (l_i - k_i) \) in opposite case. From this it follows directly that for each \( i \) from the set \( \{1, \ldots, s\} \) it holds \( (x_1^0 \approx x_1^0 \cdot x_1^{r_1}) \in S_1 \), and thereby \( (x_1^0 \approx x_1^0 \cdot x_1^{r_1}) \in S_1 \). We have proved that \( S_2 \subseteq S_1 \).

To prove the opposite inclusion let us observe that \( (r_1, \ldots, r_s) | r_1 \) for each \( i \in \{1, \ldots, s\} \). Hence, for each \( i \in \{1, \ldots, s\} \) there exists \( p_i \in \{0, \ldots, n-1\} \) such that \( r_i = p_i \cdot (r_1, \ldots, r_s) \). Putting \( x_1 = x_1^{p_1} \cdot \ldots \cdot x_1^{p_s} \) in the identity \( x_1^0 \approx x_1^0 \cdot x_1^{r_1} \cdot x_1^{r_2} \ldots x_s^{r_s} \) we get \( (x_1^0 \approx x_1^0 \cdot x_1^{r_1}) \in S_2 \). From the above it follows directly that the identity (4) belongs to the set \( S_2 \), thus \( S_1 \subseteq S_2 \). So, the lemma has been proved.

Now we consider the identity (5).

**Lemma 5.** (a) If \( l_1 = k_1, \ldots, l_s = k_s, \) then \( C_n(Ex(G^n) \cup \{(5)\}) = Ex(G^n) \).

(b) If \( l_j \neq k_j \) for some \( j \in \{1, \ldots, s\} \), then \( C_n(Ex(G^n) \cup \{(5)\}) = C_n(Ex(G^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^d\}) \), where \( d = (l_1 - k_1, \ldots, l_s - k_s) \).

Proof. The proof of this lemma is analogous to the proof of the last lemma.

Now, let us regard the identity (6).

**Lemma 6.** (a) If \( l_i = k_i = 0 \) for \( i \in \{1, \ldots, s\} \), then
\[
C_n(Ex(G^n) \cup \{(6)\}) = C_n(Ex(G^n) \cup \{x_1^0 \approx (x_1^{k_1} \ldots k_s)^{-1}\}).
\]

(b) If \( k_i = l_i \) for each \( i \in \{1, \ldots, s\} \) and \( k_j \neq 0 \) for some \( j \in \{1, \ldots, s\} \), then
\[
C_n(Ex(G^n) \cup \{(6)\}) = C_n(Ex(G^n) \cup \{x_1^0 \cdot x_1^{(k_1 \ldots k_s)} \approx (x_1^{(k_1 \ldots k_s)})^{-1}\}).
\]

(c) If \( k_j \neq l_j \) for some \( j \in \{1, \ldots, s\} \), then
\[
C_n(Ex(G^n) \cup \{(6)\}) = C_n(Ex(G^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^{(k_1 \ldots k_s)} \cdot x_1^{(l_1, \ldots, l_s)} \approx ((x_1^{p_1 \ldots k_s} + \ldots + p_s \cdot k_s)^{-1})^{-1}\}),
\]
where \( p_1, \ldots, p_s \) satisfy the following condition \( p_1 \cdot l_1 + \ldots + p_s \cdot l_s = (l_1, \ldots, l_s) \), \( p_1, \ldots, p_s \in Z_n \).

Proof. (a) The proof is obvious.

(b) It is enough to observe that the equation \( t_1 \cdot k_1 + \ldots + t_s \cdot k_s = (k_1, \ldots, k_s) \) has a solution in the set \( Z_n \).
(c) Let $S_1 = Cn(Ex(G^n) \cup \{(6)\})$ and $S_2 = Cn(Ex(G^n) \cup \{x_1^0 \approx x_1^0 \cdot x_1^{(k_1-l_1, \ldots, k_s-l_s)}, x_1^0 \cdot x_1^{(l_1, \ldots, l_s)} \approx ((x_1^{p_1 \cdot k_1 + \ldots + p_s \cdot k_s})^{-1})^{-1}\})$, where $p_1, \ldots, p_s$ are defined above.

In the identity (6) let us put $x_i = x_i^{P_i}$. We get, that $(x_1^0 \cdot x_1^{(l_1, \ldots, l_s)} \approx ((x_1^{p_1 \cdot k_1 + \ldots + p_s \cdot k_s})^{-1})^{-1}) \in S_1$. It is clear, that from the definition of the set $S_1$ it follows that for each $i \in \{1, \ldots, s\}$ the identity $x_i^0 \approx x_i^0 \cdot x_i^{(k_i-l_i)}$ belongs to $S_1$. Analogously, as in the proof of Lemma 2 we get, that $(x_i^0 \approx x_i^0 \cdot x_i^{(k_1-l_1, \ldots, k_s-l_s)}) \in S_1$. We have proved, that $S_2 \subseteq S_1$.

To prove the opposite inclusion in the identity

$$x_1^0 \cdot x_1^{(l_1, \ldots, l_s)} \approx ((x_1^{p_1 \cdot k_1 + \ldots + p_s \cdot k_s})^{-1})^{-1}$$

we put $x_1 = x_1^{(l_1, \ldots, l_s)}$. We get, that the identity

$$(*) \quad x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s} \approx \frac{(p_1 \cdot k_1 + \ldots + p_s \cdot k_s)}{(l_1, \ldots, l_s)-(l_1, \ldots, l_s)} \cdot x_1 \cdot x_1^{l_1} \cdot \ldots \cdot x_s \cdot x_i \cdot (x_i^{p_1 \cdot k_1 + \ldots + p_s \cdot k_s})^{-1}$$

belongs to $S_2$.

For each $i \in \{1, \ldots, s\}$ let us consider the equation $h_i \cdot (k_1 - l_1 - \ldots, k_s - l_s) + \frac{\langle l_i \rangle}{(l_1, \ldots, l_s)} \cdot (p_1 \cdot k_1 + \ldots + p_s \cdot k_s) = k_i$. We show, that $h_i = \frac{\langle k_i - l_i \rangle}{(l_1, \ldots, l_s)-(k_1-l_1, \ldots, k_s-l_s)} - \frac{\langle p_i \cdot k_i \rangle}{(l_1, \ldots, l_s)} = (k_i - l_i) - (k_i - l_i) \cdot (p_1 \cdot k_1 + \ldots + p_s \cdot k_s)$.

Because $(l_1, \ldots, l_s)\langle l_1 \rangle$ and $(k_1 - l_1 - \ldots, k_s - l_s)\langle k_i - l_i \rangle$ for each $r \in \{1, \ldots, s\}$ then $h_i \in Z$. Hence $h_i \cdot (k_1 - l_1, \ldots, k_s - l_s) = (k_i - l_i) - \frac{\langle l_i \rangle}{(l_1, \ldots, l_s)} \cdot (p_1 \cdot k_1 + \ldots + p_s \cdot k_s)$.

Thus for each $i \in \{1, \ldots, s\}$ the identity $x_i^0 \approx x_1 \cdot x_i^{(k_1-l_1, \ldots, k_s-l_s)}$ belongs to $S_2$. Hence, as a result of the fact, that the identity (*) belongs to $S_2$ we get, that the identity

$$x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s} \approx ((x_1^0 \cdot x_1^{h_1}) \cdot (x_1^{p_1 \cdot k_1 + \ldots + p_s \cdot k_s})^{-1})^{-1}$$

belongs to $S_2$. So, we get that $(x_1^0 \cdot x_1^{l_1} \cdot \ldots \cdot x_s^{l_s}) \approx ((x_1^{p_1 \cdot k_1 + \ldots + p_s \cdot k_s})^{-1})^{-1}) \in S_2$.

It implies that $S_1 \subseteq S_2$.

Finally, we have proved that $S_1 = S_2$. ■

From Lemmas 1–6 we obtain
THEOREM 1. If $E$ is a finite set of identities of type $\tau$ then there exists a set $E_1$ of one variable identities such that $Cn(Ex(G^n) \cup E) = Cn(Ex(G^n) \cup E_1)$.

By $G^n_{Ex}$ we denote the variety defined by the set $Ex(G^n)$. The consequence of Theorem 1 is the following theorem

THEOREM 2. Let $\tau$ be a type of Abelian groups with the exponent $n$ and let $A$ be a free algebra in the variety $G^n_{Ex}$ with a one element set of generators. Then $Id(A) = Id(G^n_{Ex})$.

References