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PROPER P-COMPATIBLE
HYPERSUBSTITUTIONS

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Abstract

An identity \( s \approx t \) is called a hyperidentity in a variety \( V \) if by substituting terms of appropriate arity for the operation symbols in \( s \approx t \), one obtains an identity satisfied in \( V \). If every identity in \( V \) is a hyperidentity, the variety \( V \) is called solid. All solid varieties of a given type \( \tau \) form a complete sublattice \( \mathcal{S}(\tau) \) of the lattice \( \mathcal{L}(\tau) \) of all varieties of type \( \tau \). The concept of an \( M \)-solid variety generalizes that of a solid variety. An equation \( s \approx t \) of terms of type \( \tau \) is called \( P \)-compatible where \( P \) is a partition of the set \( F = \{ f_i | i \in I \} \) of operation symbols of type \( \tau \) if it has the form \( x_i \approx x_i \) or \( f_i(t_1, \ldots, t_{n_1}) \approx f_j(t_1, \ldots, t'_{n_2}) \) with \( f_j \in [f_i]_P \), where \( [f_j]_P \) is the block of \( P \) containing \( f_j \). A variety is called \( P \)-compatible if it contains only \( P \)-compatible identities. All \( P \)-compatible varieties of type \( \tau \) form also a sublattice of the lattice of all varieties of type \( \tau \). We ask for the intersection of both lattices, i.e. we want to characterize solid varieties which are \( P \)-compatible or \( M \)-solid varieties which are \( P \)-compatible.

1 Preliminaries

Our informal definition of a hyperidentity shows that we are interested in a map which associates to each \( n_i \)-ary operation symbol \( f_i \) an \( n_i \)-ary term \( \sigma(f_i) \).

Key words and phrases: P-compatible identity, hyperidentity, hypersubstitution.

Any such map is called a hypersubstitution. Let \( W_\tau(X) \) be the set of all terms of type \( \tau \) on an alphabet \( X = \{ x_1, x_2, \ldots, x_n, \ldots \} \). Using a hypersubstitution \( \sigma \) we can define a uniquely determined mapping \( \delta \) defined on terms by

(i) \( \delta[x] := x \),
(ii) \( \delta(f(t_1, \ldots, t_n)) := \sigma(f_i)(\delta[t_1], \ldots, \delta[t_n]). \)

By \( Hyp(\tau) \) we denote the set of all these hypersubstitutions. If we define a multiplication \( \circ \) on the set \( Hyp(\tau) \) by \( \sigma_1 \circ \sigma_2 := \sigma_1 \circ \sigma_2 \) where \( \circ \) is the usual composition of functions, together with \( \sigma_{id}(f_i) := f_i(x_1, \ldots, x_n) \) we obtain a monoid \( Hyp(\tau) = (Hyp(\tau); \circ, \sigma_{id}) \). If \( M \) is a submonoid of the monoid of all hypersubstitutions of type \( \tau \) then an equation \( s \approx t \) of terms of type \( \tau \) is called \( M \)-hyperidentity in the variety \( V \) of groupoids if for all \( \sigma \in M \) the equations \( \delta[s] \approx \delta[t] \) are satisfied as identities of \( V \). Hyperidentities are \( M \)-hyperidentities for \( M = Hyp(\tau) \). A variety \( V \) of type \( \tau \) is called \( M \)-solid if each of its identities is an \( M \)-hyperidentity for \( M = Hyp(\tau) \). All \( M \)-solid varieties of type \( \tau \) form a complete sublattice \( S_M(\tau) \) of the lattice \( L(\tau) \) of all varieties of type \( \tau \) with

\[
M_1 \subseteq M_2 \Rightarrow S_{M_1}(\tau) \supseteq S_{M_2}(\tau).
\]

To test whether an identity \( s \approx t \) of a variety \( V \) is an \( M \)-hyperidentity of \( V \) our definition requires that we check, for each hypersubstitution in \( M \), that \( \delta[s] \approx \delta[t] \) is an identity of \( V \). Indeed, we can restrict our testing to certain "special hypersubstitutions". We recall of two concepts, both introduced by J. Plonka ([4]).

**Definition 1.1** Let \( V \) be a variety of type \( \tau \). A hypersubstitution \( \sigma \) is called \( V \)-proper if for every identity \( s \approx t \) in \( V \), the identity \( \delta[s] \approx \delta[t] \) also holds in \( V \). We use \( P(V) \) for the set of all \( V \)-proper hypersubstitutions.

It is clear that \( (P(V); \circ, \sigma_{id}) \) is a submonoid of \( Hyp(\tau) = (Hyp(\tau); \circ, \sigma_{id}) \) and that a variety \( V \) is \( M \)-solid for \( M = P(V) \) and \( P(V) \) is the largest \( M \) for which \( V \) is \( M \)-solid.

**Definition 1.2** Let \( V \) be a variety of type \( \tau \). Two hypersubstitutions \( \sigma_1, \sigma_2 \) are called \( V \)-equivalent \((\sigma_1 \sim_V \sigma_2)\) if \( \sigma_1(f_i) \approx \sigma_2(f_i) \) are identities in \( V \) for all \( i \in I \).

This relation can be extended to arbitrary terms \( t \), i.e.

\[
\sigma_1 \sim_V \sigma_2 \iff \delta_1[t] \approx \delta_2[t]
\]

is an identity in \( V \). Then one can prove: If \( \sigma_1 \sim_V \sigma_2 \) and \( \delta_1[s] \approx \delta_1[t] \) then \( \delta_2[s] \approx \delta_2[t] \) is an identity in \( V \).
We will use the following denotations:

- \( \text{Id } V \) - the set of all identities satisfied in the variety \( V \),
- \( C_P(\tau) \) - the set of all \( P \)-compatible equations of type \( \tau \),
- \( C_P(V) = C_P(\tau) \cap \text{Id } V \) - the set of all \( P \)-compatible identities of \( V \),
- \( E_X(\tau) \) - the set of all externally compatible equations of type \( \tau \), i.e. \( P \)-compatible for \( P = \{ \{ f_i \} | i \in I \} \),
- \( E_X(V) = E_X(\tau) \cap \text{Id } V \),
- \( N(\tau) \) - the set of all normal identities of type \( \tau \), i.e. \( P \)-compatible for \( P = \{ \{ f_i \} | i \in I \} \),
- \( N(V) = N(\tau) \cap \text{Id } V \).

It is easy to see that \( C_P(\tau) \) and \( C_P(V) \) are equational theories, i.e. closed under the rules of consequences for identities.

2 \( C_P(V) \)-proper hypersubstitutions

**Definition 2.1** A hypersubstitution \( \sigma \in \text{Hyp}(\tau) \) is called \( C_P(V) \)-proper if for all \( s \approx t \in C_P(V) \) we have \( \dot{\sigma}[s] \approx \dot{\sigma}[t] \in C_P(V) \) (i.e. \( \dot{\sigma}[s] \approx \dot{\sigma}[t] \in \text{Id } V \) and \( \dot{\sigma}[s] \approx \dot{\sigma}[t] = x_i \in X \) or \( \text{ex}(\dot{\sigma}[s]) \in [\text{ex}(\dot{\sigma}[t])]_P \) where \( \text{ex}(\dot{\sigma}[t]) \) denotes the first operation symbol occurring in the term \( \dot{\sigma}[t] \).

Let \( M_{C_P}(V) \) be the set of all \( C_P(V) \)-proper hypersubstitutions of type \( \tau \). Then we have

**Lemma 2.2** \( M_{C_P}(V) \) forms a submonoid of \( \text{Hyp}(\tau) \).

**Proof.** If \( s \approx t \in C_P(V) \) then \( \sigma_{id}[s] = s \approx t = \sigma_{id}[t] \in C_P(V) \), thus \( \sigma_{id} \in M_{C_P}(V) \). If \( \sigma_1, \sigma_2 \in M_{C_P}(V) \) then for all \( s \approx t \in C_P(V) \) we have \( \sigma_2[s] \approx \sigma_2[t] \in C_P(V) \) and then \( \sigma_1[\sigma_2[s]] \approx \sigma_1[\sigma_2[t]] \in C_P(V) \), i.e. \( (\sigma_1 \circ h \sigma_2)[s] \approx (\sigma_1 \circ h \sigma_2)[t] \in C_P(V) \). Therefore \( \sigma_1 \circ h \sigma_2 \in M_{C_P}(V) \).

Remark that there are different possibilities to define sets of hypersubstitutions which are connected with \( P \)-compatible identities of the variety \( V \). For instance we could also define a hypersubstitution to be \( C_P(V) \)-generating if for all \( s \approx t \in \text{Id } V \) it follows that \( \dot{\sigma}[s] \approx \dot{\sigma}[t] \in C_P(V) \). If we denote by \( G_{C_P}(V) \) the set of all \( C_P(V) \)-generating hypersubstitutions we have

**Lemma 2.3** \( G_{C_P}(V) \) is a semigroup of hypersubstitutions which in general is not a monoid.
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Proof. If $\sigma_1, \sigma_2 \in G_{C_P}(V)$ then for all $s \approx t \in \text{Id } V$ we get $\delta_2[\sigma_2[s] \approx \delta_2[t]] \in C_P(V)$ and thus $\delta_1[\delta_2[\sigma_2[s]] \approx \delta_1[\delta_2[t]]] \in C_P(V)$. This means $G_{C_P}(V)$ is closed under the product $\circ_h$. But in general $G_{C_P}(V)$ is not a monoid since $s \approx t \in \text{Id } V$, but $s \approx t \not\in C_P(V)$ and then $\delta_{\text{id}}[s] \approx \delta_{\text{id}}[t] \not\in C_P(V)$. □

Remarks:

1. If $V$ is an idempotent variety, (i.e. $(f_i(x,\ldots,x) = x) \in \text{Id } V$) then a hypersubstitution belonging to $G_{C_P}(V)$ has to map each $f_i$ to one of the variables $x_1,\ldots,x_n$.

2. Clearly, $G_{C_P}(V)$ is a subsemigroup of the monoid $P(V)$ of all proper hypersubstitutions of type $\tau$ and $M_{C_P}(V)$ is the monoid of all proper hypersubstitutions of the variety $V_{C_P} := \text{Mod}(C_P(V))$ which is defined by all $P$-compatible identities of the variety $V$.

That means, the variety $V_{C_P}$ is $M$-solid for the monoid $M_{C_P}(V)$ and $M_{C_P}(V)$ is the greatest monoid of hypersubstitutions such that $V_{C_P}$ is $M$-solid.

Theorem 2.4 Let $V$ be a variety of type $\tau$ and let $P$ be a partition of the set $\{f_i|i \in I\}$ of operation symbols. Let $M_{C_P}(V)$ be the monoid of all $C_P(V)$-proper hypersubstitutions. If $M_{C_P}(V) = \text{Hyp}(\tau)$ then $P = \{\{f_i|i \in I\}$ or $P = \{f_i|i \in I\}$.

Proof. Let $s \approx t$ be an arbitrary identity of $C_P(V)$ and assume that $P \neq \{\{f_i|i \in I\}$. Then we can assume that $s = f_i(s_1,\ldots,s_n)$ and $t = f_j(t_1,\ldots,t_n)$ with $f_j \in [f_i]_P, f_i \neq f_j$. (Such an identity exists since $P \neq \{\{f_i|i \in I\}$). Consider now a hypersubstitution which maps $f_i$ to $f_i(x_1,\ldots,x_m)$ and $f_j$ to $h(f_j(x_1,\ldots,x_m),\ldots,f_j(x_1,\ldots,x_m))$, where $h$ is an arbitrary operation symbol of $P = \{f_i|i \in I\}$. We may assume that $h$ is not nullary, otherwise we change the role of $f$ and $g$. Since $\text{Hyp}(\tau) = C_P(V)$ we obtain $\delta[s] \approx \delta[t] \in C_P(V)$ and $h \in [f_i]_P$ and $P = \{f_i|i \in I\}$. □

Note that Theorem 2.4 is a reformulation of [2,Theorem 8] which says that if $\text{Mod}(C_P(V))$ is solid and $\text{Mod}(C_P(V)) \neq \text{Mod}(\text{Ex}(V))$ then $\text{Mod}(C_P(V))$ is normal. The proof is also only a reformulation of the proof of [2,Theorem 8].

We consider some more examples. We will call a hypersubstitution $\sigma$ of type $\tau$ a pre-hypersubstitution if for every $i \in I$ the term $\sigma(f_i)$ is not a variable ([3]). Let $\text{Pre}(\tau)$ be the set of all pre-hypersubstitutions of type $\tau$. Let $T$ be the trivial variety of type $\tau$, i.e. $T = \text{Mod}(x \approx y)$ and let $\text{Id } T$ be the set of all
identities of type $\tau$. Therefore $C_P(T) = C_P(\tau)$ and $M_{C_P}(T) = \text{Pre}(\tau) \cap \{ \sigma | f_i \in [f_j]_P \Rightarrow \text{ex}(\sigma(f_i)) \in [\text{ex}(\sigma(f_j))]_P \}$. If $V = \text{Alg}(\tau)$ is the class of all algebras of type $\tau$, i.e. $V = \text{Mod}\{ x \approx x \}$ then $V$ is solid. The set $\text{Id} V$ consists of all equations where the terms on the left and on the right hand side are the same. Clearly, $C_P(V) = \text{Hyp}(\tau)$ for any partition $P$ of $\{ f_i | i \in I \}$.

Since $M_{C_P}(V)$ is a monoid we can apply the theory of $M$-hypersubstitutions and $M$-solid varieties developed in [1]. We can apply hypersubstitutions $\sigma \in M_{C_P}(V)$ to both, to equations and to algebras. If $s \approx t$ is an equation of terms of type $\tau$ then we can form $\hat{\sigma}[s] \approx \hat{\sigma}[t]$, and define an operator $\mathcal{X}_{C_P}^E$ by

$$\mathcal{X}_{C_P}^E[\Sigma] := \{ \hat{\sigma}[s] \approx \hat{\sigma}[t] | \sigma \in M_{C_P}(V), s \approx t \in \Sigma, \Sigma \subseteq W_\tau(X)^2 \}.$$ 

The application of hypersubstitutions to an algebra $A = (A; (f_i^A)_{i \in I})$ of type $\tau$ is defined by

$$\mathcal{X}_{C_P}^A[K] := \{ \sigma[A] | \sigma \in M_{C_P}(V), A \in K, K \subseteq \text{Alg}(\tau) \},$$

where $\sigma[A] := (A; (\sigma(f_i)A^A)_{i \in I})$.

It is easy to see that both operators have the properties of closure operators which are defined for arbitrary non-empty sets as union of the results which we obtain if we apply them to one-element sets, i.e.

$$\mathcal{X}_{C_P}^E[\Sigma] = \bigcup_{s \approx t \in \Sigma} \mathcal{X}_{C_P}^E(\{ s \approx t \}),$$

$$\mathcal{X}_{C_P}^A[K] = \bigcup_{A \in K} \mathcal{X}_{C_P}^A(\{ A \}).$$

Such operators are called additive. Further they are connected by the property

$$s \approx t \in \text{Id} \Rightarrow \mathcal{X}_{C_P}^A[K] \ni \mathcal{X}_{C_P}^E(\{ s \approx t \}) \in \text{Id} K.$$ 

Because of this property we speak of a conjugate pair of additive closure operators.

Further we use the following denotations:

$H_{M_{C_P}(V)} \text{Id}(V)$ - the set of all $M_{C_P}(V)$-hyperidentities satisfied in the variety $V$ and $H_{M_{C_P}(V)} \text{Mod}(\Sigma)$ - the class of all algebras of type $\tau$, such that every equation of $\Sigma$ is an $M_{C_P}(V)$-hyperidentity of this algebra.

Further, we say that a variety $K$ is $M_{C_P}(V)$-solid if $\mathcal{X}_{M_{C_P}(V)}^A[K] = K$. From the properties of the pair $(\mathcal{X}_{M_{C_P}(V)}^A, \mathcal{X}_{M_{C_P}(V)}^E)$ as a conjugate pair of additive closure operators we obtain the following characterization of $M_{C_P}(V)$-solid varieties ([1]).
Theorem 2.5 For all varieties $K$ of type $\tau$ and for all equational theories $\Sigma$ of type $\tau$ the following conditions (i)-(iv) and the conditions (i')-(iv') are equivalent:

(i) $K = H_{MC_p}(V) \text{Mod} H_{MC_p}(V) \text{Id}(K)$ ($K$ is an $MC_p(V)$-hyperequational class),

(ii) $X_{MC_p}(V)[K] = K$, i.e. $K$ is $MC_p(V)$-solid,

(iii) $X_{MC_p}(V)[\text{Id}(K)] = \text{Id}(K)$,

(iv) $\text{Id}(K) = H_{MC_p}(V) \text{Id}(K)$, and

(i') $\sum = H_{MC_p}(V) \text{Id} H_{MC_p}(V) \text{Mod}(\sum)$,

(ii') $X_{MC_p}(V)[\sum] = \sum$,

(iii') $X_{MC_p}(V)[\text{Mod}(\sum)] = \text{Mod}(\sum)$,

(iv') $\text{Mod}(\sum) = H_{MC_p}(V) \text{Mod}(\sum)$.

We have already mentioned that the variety $V_{CP} = \text{Mod}(C_P(V))$ is $MC_p(V)$-solid. Therefore $V_{CP}$ satisfies the equivalent conditions (i),(ii),(iii),(iv).

From the general theory (see [1]) it follows also that the class of all $MC_p(V)$-solid varieties forms a complete lattice which is a complete sublattice of the lattice of all varieties of type $\tau$.

3 P-compatible relations on hypersubstitutions

In analogy to the relation $\sim_V$ we define the following binary relation on the set $Hyp(\tau)$ and on submonoids of $Hyp(\tau)$.

Definition 3.1 Let $V$ be a variety of type $\tau$ and let $P$ be a partition of the set of operation symbols $\{f_i | f \in I\}$ of $V$. Let $C_P(V)$ be the set of all $P$-compatible identities satisfied in $V$. Then for any two hypersubstitutions $\sigma_1, \sigma_2 \in Hyp(\tau)$ we define

$$\sigma_1 \sim_{C_P(V)} \sigma_2 : \Leftrightarrow \forall i \in I(\sigma_1(f_i) \approx \sigma_2(f_i) \in C_P(V)).$$
We notice that $\sim_{C_P(V)}$ is an equivalence relation on $Hyp(\tau)$. It can be easily shown that for all terms $t \in W_\tau(X)$, if $\sigma_1 \sim_{C_P(V)} \sigma_2$ then $\delta_1[t] \approx \delta_2[t] \in C_P(V)$.

The relation $\sim_{C_P(V)}$ can also be restricted to the monoid $M_{C_P(V)}$.

Theorem 3.2 The monoid $M_{C_P(V)}$ is a union of full equivalence classes with respect to the relation $\sim_{C_P(V)}$.

Proof. We have to show that if $\sigma_1 \in M_{C_P(V)}$ and if $\sigma_1 \sim_{C_P(V)} \sigma_2$ then $\sigma_2 \in M_{C_P(V)}$. Indeed, $\sigma_1 \in M_{C_P(V)}$ means that for each $s \approx t \in C_P(V)$ the identity $\delta_1[s] \approx \delta_1[t]$ belongs also to $C_P(V)$. The relation $\sigma_2 \sim_{C_P(V)} \sigma_1$ implies $\delta_2[t] \approx \delta_1[t] \in C_P(V)$ for all $t \in W_\tau(X)$. But then, by transitivity we get $\delta_2[s] \approx \delta_2[t] \in C_P(V)$, and this means $\sigma_2 \in M_{C_P(V)}$. \qed

Theorem 3.2 shows also that, if we want to check whether an identity is an $M_{C_P(V)}$-hyperidentity, we can restrict our checking to one representative from each equivalence class with respect to $\sim_{C_P(V)}$. We can also show

Corollary 3.3 The restriction of the relation $\sim_{C_P(V)}$ to the submonoid $M_{C_P(V)}$ is a congruence relation on the monoid $M_{C_P(V)}$.

Proof. We show that the restricted relation $\sim_{C_P(V)} | M_{C_P(V)}$ is a right and a left congruence on $M_{C_P(V)}$. Assume that $\sigma_1 \sim_{C_P(V)} | M_{C_P(V)} \sigma_2$ and that $\sigma \in M_{C_P(V)}$. Then for the term $\sigma(f_i)$ we have $\delta_1[\sigma(f_i)] \approx \delta_2[\sigma(f_i)] \in C_P(V)$ and therefore $\sigma_1 \circ_{h} \sigma \sim_{C_P(V)} | M_{C_P(V)} \sigma_2 \circ_{h} \sigma$. From $\sigma_1 \sim_{C_P(V)} | M_{C_P(V)} \sigma_2$ it follows $\sigma_1(f_i) \approx \sigma_2(f_i) \in C_P(V)$ and for every $\sigma \in M_{C_P(V)}$ also $\delta[\sigma_1(f_i)] \approx \delta[\sigma_2(f_i)] \in C_P(V)$, i.e. $\sigma \circ_{h} \sigma_1 \sim_{C_P(V)} | M_{C_P(V)} \sigma \circ_{h} \sigma_1 \in C_P(V)$. \qed

If we consider the class of the identity hypersubstitution, we notice that it forms a submonoid of $M_{C_P(V)}$ since if $\sigma_1 \sim_{C_P(V)} \sigma_{id}$ and $\sigma_2 \sim_{C_P(V)} \sigma_{id}$ then we have $\sigma_1(f_i) \approx \sigma_{id}(f_i) = f_i(x_1, \ldots, x_n) \in C_P(V)$ and $\sigma_2(f_i) \approx \sigma_{id}(f_i) = f_i(x_1, \ldots, x_n) \in C_P(V)$ and then also $\sigma(\sigma_1 \circ_{h} \sigma_2)(f_i) = \delta_1[\sigma_2(f_i)] \approx \delta_1[\sigma_{id}(f_i)] \in C_P(V)$, i.e. $(\sigma_1 \circ_{h} \sigma_2)(f_i) = \sigma_1[\sigma_2(f_i)] \approx \sigma_1(f_i) = \sigma_{id} \in C_P(V)$, that means $\sigma_1 \circ_{h} \sigma_2 \sim_{C_P(V)} \sigma_{id}$.

Finally we want to remark that the previous definitions and theorems can be generalized in the following way:

Let $V$ be a variety of type $\tau$ and let $T(\tau)$ be the set of all identities of type $\tau$ which fulfills a given property. For example the property could also be that
Proper $P$-compatible Hypersubstitutions

the set of variables occurring on both sides of the identity agree, so that $T(\tau)$ is the set of all regular identities of type $\tau$. We will also assume that $T(\tau)$ is an equational theory. We set $T(V) := T(\tau) \cap \text{Id} V$ and if $\Sigma$ is an equational theory we set $T(\Sigma) := T(\tau) \cap \Sigma$.

Then we define a hypersubstitution to be $T(V)$-proper if for all $s \approx t \in T(V)$ we obtain $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in T(V)$.

Let $M_{T(V)}$ be the set of all $T(V)$-proper hypersubstitutions of type $\tau$. Clearly $M_{T(V)}$ is a submonoid of $\text{Hyp}(\tau)$ and we get a theorem similar to Theorem 2.6. The relation $\sim_{C_P(V)}$ can be also generalized and we define

$$\sigma_1 \sim_{T(V)} \sigma_2 :\iff \forall i \in I(\sigma_1(f_i) \approx \sigma_2(f_i) \in T(V)).$$

Using this definition we obtain theorems similar to Theorem 3.2 and to Corollary 3.3.

References

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