Cycle-finite module categories

Piotr Malicki, José Antonio de la Peña and Andrzej Skowroński

Abstract We describe the structure of module categories of finite dimensional algebras over an algebraically closed field for which the cycles of nonzero nonisomorphisms between indecomposable finite dimensional modules are finite (do not belong to the infinite Jacobson radical of the module category). Moreover, geometric and homological properties of these module categories are exhibited.

1 Introduction

Throughout the article K denotes a fixed algebraically closed field.

By an algebra we mean an associative finite dimensional K-algebra with an identity which we shall assume (without loss of generality) to be basic and connected. For an algebra A, by an A-module we mean a finite dimensional right A-module. We shall denote by $\operatorname{mod} A$ the category of A-modules, by $\operatorname{ind} A$ its full subcategory formed by the indecomposable modules, by Γ_A the Auslander-Reiten quiver of A, and by τ_A the Auslander-Reiten translation $D\operatorname{Tr}$ in Γ_A. We shall identify an indecomposable A-module with the vertex of Γ_A corresponding to it. From Drozd's Tame and Wild Theorem [30] (see also [26]) the class of algebras may be divided into two classes. One class consists of the wild algebras whose representation theory comprises the representation theories of all algebras over K (see [68, Chapter

Piotr Malicki

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland, e-mail: `pmalicki@mat.uni.torun.pl`

José A. de la Peña

Instituto de Matemáticas UNAM, Ciudad Universitaria 04510 México, D.F. México, e-mail: `jap@matem.unam.mx`

Andrzej Skowroński

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland, e-mail: `skowron@mat.uni.torun.pl`

XIX]). The second class consists of the tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. Hence, a classification of the finite dimensional modules is only feasible for tame algebras. It has been shown by Crawley-Boevey [26] that, if A is a tame algebra, then, for any dimension $d \geq 1$, all but finitely many isomorphism classes of indecomposable A-modules of dimension d are invariant on the action of τ_A, and hence, by a result due to Hoshino [35], lie in stable tubes of rank one in Γ_A. The indecomposable modules over tame algebras which do not lie in stable tubes of rank one are called discrete. A distinguished class of tame algebras is formed by the algebras of finite representation type, having only finitely many isomorphism classes of indecomposable modules, for which the representation theory is presently rather well understood (see [13], [20], [21], [22], [24]). On the other hand, the representation theory of arbitrary tame algebras is still only emerging. At present the most accessible seem to be the (tame) algebras of polynomial growth, for which there exists an integer m such that the number of one-parameter families of indecomposable modules is bounded, in each dimension d, by d^m. This class of algebras has been subject of intensive research over the last 30 years.

A prominent role in the representation theory of algebras is played by cycles of modules, or more generally cycles of complexes of modules. Recall that a cycle in a module category $\mathrm{mod}A$ is a sequence

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_r} X_r = X_0$$

of nonzero nonisomorphisms in $\mathrm{ind}A$, and the cycle is said to be finite if the homomorphisms $f_1, \ldots, f_r$ do not belong to the infinite Jacobson radical of $\mathrm{mod}A$. Following Ringel [65] a module in $\mathrm{ind}A$ which does not lie on cycle in $\mathrm{ind}A$ is called directing. It has been proved independently by Peng and Xio [53] and the third named author [74] that the Auslander-Reiten quiver Γ_A of an arbitrary algebra contains at most finitely many τ_A-orbits containing directing modules. Hence, in order to obtain information on nondirecting indecomposable modules of a module category, we may study properties of cycles in $\mathrm{mod}A$ containing these modules. We also note that, by a result of Ringel [65] the support algebras of directing modules are tilted algebras. Following [5] an algebra A is said to be cycle-finite if all cycles in $\mathrm{mod}A$ are finite. It has been proved by the third named author in [76] that every cycle-finite algebra A is of polynomial growth and the support algebras of the one-parametric families of indecomposable A-modules are tame concealed algebras (preprojective tilts of the path algebras of Euclidean quivers) and Ringel's tubular algebras, which are distinguished classes of cycle-finite algebras. The class of cycle-finite algebras is wide and contains the algebras of finite representation type, the tame tilted algebras [37], the tame double tilted algebras [62], the tame generalized double tilted algebras [63], the tubular algebras [65], the iterated tubular algebras [61], the tame quasi-tilted algebras [78], [40], the tame coil and multicoil algebras [5], [6], [7], the tame generalized multicoil algebras [51], and the strongly simply connected algebras of polynomial growth [77]. It has been also proved in [2], [3], [4] that the class of algebras A for which the derived category $D^b(\mathrm{mod}A)$ of

bounded complexes of A-modules is cycle-finite coincides with the class of piece-wise hereditary algebras of Dynkin, Euclidean, and tubular type, and consequently these algebras are also cycle-finite. Moreover, frequently an algebra A admits a Galois covering $R \to R/G = A$ where R is a cycle-finite locally bounded category and G is an admissible group of automorphisms of R, which allows to reduce the representation theory of A to the representation theory of cycle-finite algebras being finite convex subcategories of R. For example, every selfinjective algebra A of polynomial growth admits a canonical standard form $\overline{A}$ (geometric socle deformation of A) such that $\overline{A}$ has a Galois covering $R \to R/G = \overline{A}$, where R is a cycle-finite selfinjective locally bounded category and G is an admissible infinite cyclic group of automorphisms of R, the Auslander-Reiten quiver $\Gamma_{\overline{A}}$ of $\overline{A}$ is the orbit quiver Γ_R/G of Γ_R, and the stable Auslander-Reiten quivers of A and $\overline{A}$ are isomorphic (see [69], [80] for details). We also mention that by the main result of [59], every algebra A which admits a cycle-finite Galois covering $R \to R/G = A$ with G torsion-free is tame.

One of the objectives of this article is to describe the structure of the category $\mathrm{ind}\,A$ of an arbitrary cycle-finite algebra A, by showing that it can be covered by the categories of indecomposable modules of tame generalized multicoil algebras and tame generalized double tilted algebras. Here, a crucial role will be played by description of support algebras of cyclic components of the Auslander-Reiten quivers of cycle-finite algebras. The second objective of the article is to exhibit geometric and homological properties of indecomposable modules over cycle-finite algebras. We are interested in the class of coherent cycle-finite algebras for which all cyclic components of the Auslander-Reiten quivers are coherent (see Section 2 for definition). Every coherent cycle-finite algebra A is triangular, and hence the (geometric) Tits quadratic form q_A and the (homological) Euler form χ_A of A are well defined. For a vector $\mathbf{d}$ in the Grothendieck group $K_0(A)$ of A with nonnegative coordinates, we denote by $\mathrm{mod}_A(\mathbf{d})$ the affine variety of A-modules of dimension vector $\mathbf{d}$ and by $G(\mathbf{d})$ the corresponding product of general linear groups acting on $\mathrm{mod}_A(\mathbf{d})$ in such a way that the $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$ correspond to the isomorphism classes of A-modules of dimension vector $\mathbf{d}$. The third main aim of the article is to establish a common bound on the numbers of discrete indecomposable modules in each dimension vector over cycle-finite algebras, generalizing results proved in [84] for strongly simply connected algebras of polynomial growth.

For basic background from the representation theory of algebras we refer to the books [1], [11], [65], [67], [68], [82].

2 Preliminaries

In this section we recall some concepts and results from the representation theory of algebras important for further considerations.

Let A be an algebra (which by our assumption is basic and connected). Then there is an isomorphism $A \cong KQ/I$ of K-algebras, where KQ is the path algebra of the *Gabriel quiver* $Q = Q_A$ of A and I is an admissible ideal of KQ. Equivalently,

$A = KQ/I$ may be considered as a K-category whose class of objects is the set Q_0 of vertices of Q, and the set of morphisms $A(x,y)$ from x to y is the quotient of K-space $KQ(x,y)$, formed by the K-linear combinations of paths in Q from x to y, by the subspace $I(x,y) = KQ(x,y) \cap I$. We shall identify an algebra A with its K-category. Moreover, the module category $\mathrm{mod}\,A$ may be identified with the category $\mathrm{rep}_K(Q,I)$ of finite dimensional K-linear representations of the bound quiver (Q,I). An algebra A with Q_A acyclic (without oriented cycles) is said to be *triangular*. A full subcategory C of A is said to be *convex* if any path in Q_A with source and target in Q_C lies entirely in Q_C. Recall also that the *Jacobson radical* $\mathrm{rad}(\mathrm{mod}\,A)$ of the module category $\mathrm{mod}\,A$ is the ideal of $\mathrm{mod}\,A$ generated by all noninvertible morphisms in $\mathrm{ind}\,A$. Then the *infinite radical* $\mathrm{rad}^\infty(\mathrm{mod}\,A)$ of $\mathrm{mod}\,A$ is the intersection of all powers $\mathrm{rad}^i(\mathrm{mod}\,A)$, $i \geq 1$, of $\mathrm{rad}(\mathrm{mod}\,A)$. A *path of length $t \geq 1$* in $\mathrm{mod}\,A$ is a sequence of nonzero nonisomorphisms

$$M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{t-1} \xrightarrow{f_t} M_t$$

and modules $M_0, M_1, \ldots, M_t$ in $\mathrm{ind}\,A$. Such a path is said to be *finite* if $f_1, \ldots, f_t$ do not belong to $\mathrm{rad}^\infty(\mathrm{mod}\,A)$, and otherwise *infinite*. Moreover, if $M_0 \cong M_t$ then the path is called a *cycle* of length t. A module M from $\mathrm{ind}\,A$ is called *directing* if it does not lie on a cycle in $\mathrm{mod}\,A$. For a module M in $\mathrm{mod}\,A$, we denote by $\mathbf{dim}\,M$ its *dimension vector* $(\dim_K M(i))_{i \in Q_0}$. The *support* $\mathrm{supp}\,M$ of a module M in $\mathrm{mod}\,A$ is the full subcategory of A given by all vertices i of Q_A such that $M(i) \neq 0$. A module M in $\mathrm{mod}\,A$ with $\mathrm{supp}\,M = A$ is said to be *sincere*. Recall also that the Grothendieck group $K_0(A) = K_0(\mathrm{mod}\,A)$ is isomorphic to $\mathbb{Z}^{Q_0}$.

Let A be an algebra and $K[x]$ the polynomial algebra in one variable x. Following [30] A is said to be *tame* if, for any dimension d, there exists a finite number of $K[x] - A$-bimodules M_i, $1 \leq i \leq n_d$, which are finitely generated and free as left $K[x]$-modules, and all but a finite number of isoclasses of indecomposable A-modules of dimension d are of the form $K[x]/(x - \lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some $i \in \{1, \ldots, n_d\}$. Let $\mu_A(d)$ be the least number of $K[x] - A$-bimodules M_i satisfying the above condition for d. Then A is said to be of *polynomial growth* (respectively, *domestic*) if there exists a positive integer m such that $\mu_A(d) \leq d^m$ (respectively, $\mu_A(d) \leq m$) for any $d \geq 1$ (see [27], [70]). Recall that from the validity of the second Brauer-Thrall conjecture we know that A is representation-finite if and only if $\mu_A(d) = 0$ for any $d \geq 1$.

The *Tits form* of a triangular algebra $A = KQ/I$ is the integral quadratic form $q_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$, defined, for $\mathbf{x} = (x_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$, by

$$q_A(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{(i \to j) \in Q_1} x_i x_j + \sum_{i,j \in Q_0} r(i,j) x_i x_j,$$

where Q_1 is the set of arrows in Q and $r(i,j)$ is the cardinality of $L \cap I(i,j)$, for a minimal set of generators $L \subset \bigcup_{i,j \in Q_0} I(i,j)$ of the ideal I (see [18]). Moreover, the *Euler form* of A is the integral quadratic form $\chi_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ defined in [65, 2.4] such that for any module M in $\mathrm{mod}\,A$, we have

$$\chi_A(\mathbf{dim}\,M) = \sum_{i=0}^{\infty} (-1)^i \dim_K \mathrm{Ext}_A^i(M,M).$$

Observe that A is of finite global dimension, because A is triangular. It is also known that if $\mathrm{gl\,dim}\,A \leq 2$ then $q_A = \chi_A$ (see [18]). Finally, it is known (see [54]) that, if A is tame, then q_A is *weakly nonnegative*, that is, $q_A(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{N}^{Q_0}$. Unfortunately, the reverse implication is not true in general. However, it has been proved recently in [25] that a strongly simply connected algebra A is tame if and only if the Tits form q_A is weakly nonnegative. Recall also that a triangular algebra A is called *strongly simply connected* [71] if the first Hochschild cohomology $H^1(C,C)$ of every convex subcategory C of A vanishes.

We need also special types of components of the Auslander-Reiten quivers of algebras.

Recall from [28], [65] that a translation quiver Γ is called a *tube* if it contains a cyclical path and if its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$, where S^1 is the unit circle and $\mathbb{R}^+$ is the nonnegative real line. A tube has only two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. Tubes containing neither projective vertices nor injective vertices are called *stable*, and are as follows. For the infinite quiver

$$\mathbb{A}_\infty : 0 \to 1 \to 2 \to \cdots$$

the translation quiver $\mathbb{Z}\mathbb{A}_\infty$ is of the form
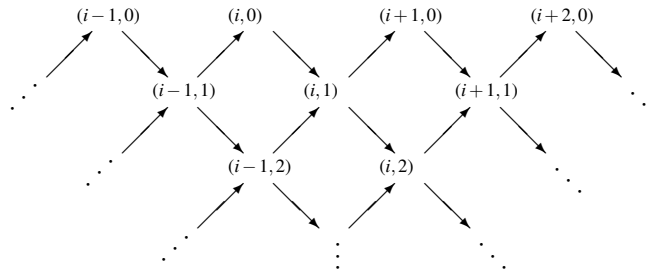
with the translation τ given by $\tau(i,j) = (i-1,j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$. For each $r \geq 1$, denote by $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ the translation quiver Γ obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex (i,j) of $\mathbb{Z}\mathbb{A}_\infty$ with the vertex $\tau^r(i,j)$ and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r x \to \tau^r y$. The translation quiver $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is called the *stable tube of rank r*. The τ-orbit of a stable tube Γ formed by all vertices having exactly one immediate predecessor (equivalently, successor) is called the *mouth* of Γ. A tube Γ without injective vertices (respectively, without projective vertices) is called a *ray tube* (respectively, *coray tube*).

Let A be an algebra. A component $\mathscr{C}$ of Γ_A is called *regular* if $\mathscr{C}$ contains neither a projective module nor an injective module, and *semiregular* if $\mathscr{C}$ does not contain both a projective and an injective module. It has been shown in [41] and [87] that a regular component $\mathscr{C}$ of Γ_A contains an oriented cycle if and only if $\mathscr{C}$ is a stable tube. Moreover, Liu proved in [42] that a semiregular component $\mathscr{C}$ of Γ_A contains

an oriented cycle if and only if $\mathscr{C}$ is a ray or coray tube. A component $\mathscr{P}$ of Γ_A is called *postprojective* if $\mathscr{P}$ is acyclic and every module in $\mathscr{P}$ lies in the τ_A-orbit of a projective module. Dually, a component $\mathscr{Q}$ of Γ_A is called *preinjective* if $\mathscr{Q}$ is acyclic and every module in $\mathscr{Q}$ lies in the τ_A-orbit of an injective module. A component Γ of Γ_A is said to be *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$ (that is, $X_i \neq \tau_A X_{i+2}$ for any $i \geq 1$) in Γ.

(C2) For each injective module I in Γ there is an infinite sectional path $\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ (that is, $Y_{j+2} \neq \tau_A Y_j$ for any $j \geq 1$) in Γ.

Further, a component Γ of Γ_A is said to be *almost cyclic* if all but finitely many modules of Γ lie on oriented cycles in Γ_A, so contained entirely in Γ. We note that the stable tubes, ray tubes and coray tubes of Γ_A are special types of almost cyclic coherent components. In general, it has been proved in [50] that a component Γ of Γ_A is almost cyclic and coherent if and only if Γ is a *generalized multicoil*, which can be obtained from a family of stable tubes by a sequence of admissible operations (see Section 4 for algebras having such components). A component Γ of Γ_A is said to be *almost acyclic* if all but finitely many modules of Γ are acyclic (do not lie on oriented cycles in Γ_A, hence in Γ), and *acyclic* if all modules of Γ are acyclic. Finally, following [73] a component $\mathscr{C}$ of Γ_A is said to be *generalized standard* if $\mathrm{rad}_A^\infty(X,Y) = 0$ for all modules X and Y from $\mathscr{C}$. It has been proved in [73] that every generalized standard component $\mathscr{C}$ of Γ_A is *almost periodic*, that is, all but finitely many τ_A-orbits in $\mathscr{C}$ are periodic. Clearly, the postprojective and preinjective components are acyclic, and the Auslander-Reiten quivers of representation-finite algebras are almost acyclic. Moreover, these components are generalized standard (see [74]). General results on almost acyclic components and related algebras have been proved by Reiten and third named author in [62], [63] (see Section 5). For a component $\mathscr{C}$ of Γ_A, we denote by $\mathrm{ann}_A(\mathscr{C})$ the annihilator of $\mathscr{C}$ in A, that is, the intersection of the annihilators $\mathrm{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all modules X in $\mathscr{C}$. We note that $\mathscr{C}$ is a component of $\Gamma_{A/\mathrm{ann}_A(\mathscr{C})}$. Moreover, if $\mathrm{ann}_A(\mathscr{C}) = 0$, $\mathscr{C}$ is said to be a *faithful component* of Γ_A. By the *support* of a subquiver Γ of Γ_A we mean the full subcategory $\mathrm{supp}\,\Gamma$ of A given by the supports $\mathrm{supp}\,M$ of all modules M in Γ, and, if $\mathrm{supp}\,\Gamma = A$ then Γ is said to be *sincere*. We note that a faithful component $\mathscr{C}$ of Γ_A is sincere.

3 Semiregular components of cycle-finite algebras

In this section we recall the shapes of the Auslander-Reiten quivers of representation-infinite tilted algebras of Euclidean type and tubular algebras, as well as results from [76] on semiregular components of the Auslander-Reiten quivers of cycle-finite algebras, important for further considerations.

By a *tame concealed algebra* we mean a tilted algebra $C = \mathrm{End}_H(T)$, where H is the path algebra $K\Delta$ of a quiver Δ of Euclidean type (the underlying graph $\overline{\Delta}$ of

Δ of type $\widetilde{\mathbb{A}}_m$ ($m \geq 1$), $\widetilde{\mathbb{D}}_n$ ($n \geq 4$), or $\widetilde{\mathbb{E}}_p$ ($6 \leq p \leq 8$)) and T is a (multiplicity-free) postprojective tilting H-modules. The tame concealed algebras have been described by quivers and relations by Bongartz [19] and Happel-Vossieck [34]. Recall also that the Auslander-Reiten quiver Γ_C of a tame concealed algebra C is of the form

$$\Gamma_C = \mathscr{P}^C \cup \mathscr{T}^C \cup \mathscr{Q}^C,$$

where $\mathscr{P}^C$ is a postprojective component containing all indecomposable projective C-modules, $\mathscr{Q}^C$ is a preinjective component containing all indecomposable injective C-modules, and $\mathscr{T}^C$ is a $\mathbb{P}_1(K)$-family $\mathscr{T}_\lambda^C$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized standard stable tubes, all but a finite number of them of rank one. The ordering from the left to right indicates that there are nonzero homomorphisms only from any of these classes to itself and to the classes to its right. We refer to [65, Chapter 4] and [67] for more details on the module categories of tame concealed algebras.

Recall also that, if B is a representation-infinite tilted algebra of Euclidean type Δ, then one of the following holds:

(a) B is a domestic tubular extension of a tame concealed algebra C and

$$\Gamma_B = \mathscr{P}^B \cup \mathscr{T}^B \cup \mathscr{Q}^B,$$

where $\mathscr{P}^B = \mathscr{P}^C$ is the postprojective component of Γ_C, $\mathscr{T}^B$ is a $\mathbb{P}_1(K)$-family $\mathscr{T}_\lambda^B$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized standard ray tubes, obtained from the $\mathbb{P}_1(K)$-family $\mathscr{T}^C$ of stable tubes of Γ_C by ray insertions, and $\mathscr{Q}^B$ is a preinjective component containing all indecomposable injective B-modules and a section of type Δ;

(b) B is a domestic tubular coextension of a tame concealed algebra C and

$$\Gamma_B = \mathscr{P}^B \cup \mathscr{T}^B \cup \mathscr{Q}^B,$$

where $\mathscr{P}^B$ is the postprojective component containing all indecomposable projective B-modules and a section of type Δ, $\mathscr{T}^B$ is a $\mathbb{P}_1(K)$-family $\mathscr{T}_\lambda^B$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized standard coray tubes, obtained from the $\mathbb{P}_1(K)$-family $\mathscr{T}^C$ of stable tubes of Γ_C by coray insertions, and $\mathscr{Q}^B = \mathscr{Q}^C$ is the preinjective component of Γ_C. We refer to [65, Chapter 4] and [68, Chapters XV-XVII] for more details on the module categories of representation-infinite tilted algebras of Euclidean type.

By a *tubular algebra* we mean a tubular extension (equivalently tubular coextension) of a tame concealed algebra of tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$, as defined by Ringel in [65, Chapter 5]. Recall that a tubular algebra B admits two different tame concealed convex subcategories C_0 and C_∞ such that B is a tubular extension of C_0, and a tubular coextension of C_∞, and the Auslander-Reiten quiver Γ_B of B is of the form

$$\Gamma_B = \mathscr{P}_0^B \cup \mathscr{T}_0^B \cup \left(\bigcup_{q \in \mathbb{Q}^+} \mathscr{T}_q^B \right) \cup \mathscr{T}_\infty^B \cup \mathscr{Q}_\infty^B,$$

where $\mathscr{P}_0^B = \mathscr{P}^{C_0}$ is the postprojective component of Γ_{C_0}, $\mathscr{T}_0^B$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal generalized standard ray tubes, obtained from the $\mathbb{P}_1(K)$-family $\mathscr{T}^{C_0}$ of stable tubes of Γ_{C_0} by ray insertions, $\mathscr{Q}_\infty^B = \mathscr{Q}^{C_\infty}$ is the preinjective component of Γ_{C_∞}, $\mathscr{T}_\infty^B$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal generalized standard coray tubes, obtained from the $\mathbb{P}_1(K)$-family $\mathscr{T}^{C_\infty}$ of stable tubes of Γ_{C_∞} by coray insertions, and, for each $q \in \mathbb{Q}^+$ (the set of positive rational numbers) $\mathscr{T}_q^B$ is a $\mathbb{P}_1(K)$-family of pairwise orthogonal generalized standard stable tubes. We refer to [65, Chapter 5] for more details on the module categories of tubular algebras.

The following characterization of tame concealed and tubular algebras has been established in [76, Theorem 4.1].

Theorem 3.1. *Let A be an algebra. The following statements are equivalent:*

(i) *A is cycle-finite and Γ_A admits a sincere stable tube.*
(ii) *A is either tame concealed or tubular.*

Moreover, we have also the following consequence [76, Theorem 4.3] of the above theorem, the tameness of cycle-finite algebras [5, Proposition 1.4], and a result of Crawley-Boevey [26, Corollary E].

Theorem 3.2. *Let A be a cycle-finite algebra. Then A is of polynomial growth.*

An algebra A is said to be *minimal representation-infinite* if A is representation-infinite but every proper convex subcategory of A is representation-finite. Then we have the following characterization of minimal representation-infinite cycle-finite algebras established in [76, Corollary 4.4].

Theorem 3.3. *Let A be an algebra. The following statements are equivalent:*

(i) *A is minimal representation-infinite and cycle-finite.*
(ii) *A is a tame concealed algebra.*

We also have the following characterization of domestic cycle-finite algebras established in [76, Theorem 5.1].

Theorem 3.4. *Let A be a cycle-finite algebra. The following statements are equivalent:*

(i) *A is domestic.*
(ii) *A does not contain a tubular algebra as a convex subcategory.*
(iii) *All but finitely many components of Γ_A are stable tubes of rank one.*

In general, we have the following information on the Auslander-Reiten quiver of a cycle-finite algebra, which is a consequence of results proved in [74], [76], and results on the shapes of regular components from [41], [87].

Theorem 3.5. *Let A be a cycle-finite algebra. Then every regular component of Γ_A is a generalized standard stable tube. In particular, all but finitely many components of Γ_A are stable tubes.*

Finally, we give a complete description of semiregular components of the Auslander-Reiten quivers of cycle-finite algebras, established in [76, Proposition 3.3].

Theorem 3.6. *Let A be a cycle-finite algebra and $\mathscr{C}$ be a semiregular component of Γ_A containing a projective module. Then $B = \operatorname{supp}\mathscr{C}$ is a convex subcategory of A and one of the following holds:*

(i) *B is a domestic tubular coextension of a tame concealed algebra and $\mathscr{C}$ is the postprojective component of Γ_B.*
(ii) *B is either a domestic tubular extension of a tame concealed algebra or a tubular algebra, and $\mathscr{C}$ is a generalized standard ray tube of Γ_B.*

Theorem 3.7. *Let A be a cycle-finite algebra and $\mathscr{C}$ be a semiregular component of Γ_A containing an injective module. Then $B = \operatorname{supp}\mathscr{C}$ is a convex subcategory of A and one of the following holds:*

(i) *B is a domestic tubular extension of a tame concealed algebra and $\mathscr{C}$ is the preinjective component of Γ_B.*
(ii) *B is either a domestic tubular coextension of a tame concealed algebra or a tubular algebra, and $\mathscr{C}$ is a generalized standard coray tube of Γ_B.*

4 Tame generalized multicoil algebras

In this section we introduce and exhibit basic properties of the class of tame generalized multicoil algebras, plying a prominent role in the description of infinite cyclic components of the Auslander-Reiten quivers of cycle-finite algebras. This is the class of tame algebras among the class of all algebras having a separating family of almost cyclic coherent components investigated in [51], [52]. Recall that a family $\mathscr{C} = (\mathscr{C}_i)_{i\in I}$ of components of the Auslander-Reiten quiver Γ_A of an algebra A is called *separating* in $\operatorname{mod}A$ if the modules in $\operatorname{ind}A$ split into three disjoint classes $\mathscr{P}^A$, $\mathscr{C}^A = \mathscr{C}$ and $\mathscr{Q}^A$ such that:

(S1) $\mathscr{C}^A$ is a sincere generalized standard family of components;
(S2) $\operatorname{Hom}_A(\mathscr{Q}^A, \mathscr{P}^A) = 0$, $\operatorname{Hom}_A(\mathscr{Q}^A, \mathscr{C}^A) = 0$, $\operatorname{Hom}_A(\mathscr{C}^A, \mathscr{P}^A) = 0$;
(S3) any morphism from $\mathscr{P}^A$ to $\mathscr{Q}^A$ factors through the additive category add$\mathscr{C}^A$ of $\mathscr{C}^A$.

We then say that $\mathscr{C}^A$ *separates* $\mathscr{P}^A$ from $\mathscr{Q}^A$ and write $\Gamma_A = \mathscr{P}^A \cup \mathscr{C}^A \cup \mathscr{Q}^A$. We note that then $\mathscr{P}^A$ and $\mathscr{Q}^A$ are uniquely determined by $\mathscr{C}^A$ (see [8, (2.1)] or [65, (3.1)]).

We also recall a characterization of generalized standard stable tubes of an Auslander-Reiten quiver established in [73, Corollary 5.3] (see also [75, Lemma 3.1].

Proposition 4.1. *Let A be an algebra and Γ be a stable tube of Γ_A. The following statements are equivalent:*

(i) Γ *is generalized standard.*

(ii) *The mouth of* Γ *consists of pairwise orthogonal bricks.*

(iii) $\mathrm{rad}_A^\infty(X,X) = 0$ *for any module X in* Γ.

Recall that a module X in $\mathrm{mod}\,A$ is called a *brick* if $\mathrm{End}_A(X) \cong K$.

It has been proved in [50, Theorem A] that a connected component Γ of an Auslander-Reiten quiver Γ_A is almost cyclic and coherent if and only if Γ is a generalized multicoil, obtained from a family of stable tubes by a sequence of operations called admissible. We recall the letter and simultaneously define the corresponding enlargements of algebras.

We start with the one-point extensions and one-point coextensions of algebras. Let A be an algebra and M be a module in $\mathrm{mod}\,A$. Then the *one-point extension* of A by M is the matrix algebra

$$A[M] = \begin{bmatrix} A & 0 \\ M & K \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ m & \lambda \end{bmatrix} ; \ \lambda \in K, \ a \in A, \ m \in M \right\}$$

with the usual addition and multiplication. The quiver $Q_{A[M]}$ of $A[M]$ contains the quiver Q_A of A as a convex subquiver, and there is an additional (extension) vertex which is a source. The $A[M]$-modules are usually identified with the triples (V, X, φ), where V is a K-vector space, X an A-module and $\varphi : V \to \mathrm{Hom}_A(M, X)$ is a K-linear map. An $A[M]$-linear map $(V, X, \varphi) \to (W, Y, \psi)$ is then identified with a pair (f, g), where $f : V \to W$ is K-linear, $g : X \to Y$ is A-linear and $\psi f = \mathrm{Hom}_A(M, g)\varphi$. Dually, one defines also the *one-point coextension* of A by M as the matrix algebra

$$[M]A = \begin{bmatrix} K & 0 \\ D(M) & A \end{bmatrix}.$$

For K and $r \geq 1$, we denote by $T_r(K)$ the $r \times r$-lower triangular matrix algebra

$$\begin{bmatrix} K & 0 & 0 & \dots & 0 & 0 \\ K & K & 0 & \dots & 0 & 0 \\ K & K & K & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K & K & K & \dots & K & 0 \\ K & K & K & \dots & K & K \end{bmatrix}$$

Given a generalized standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathscr{S}(X)$ of the functor $\mathrm{Hom}_A(X, -)|_\Gamma$ is the R-linear category defined as follows [7]. Let $\mathscr{H}_X$ denote the full subcategory of Γ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X, M) \neq 0$, and $\mathscr{I}_X$ denote the ideal of $\mathscr{H}_X$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathscr{H}_X$) such that $\mathrm{Hom}_A(X, f) = 0$. We define $\mathscr{S}(X)$ to be the quotient category $\mathscr{H}_X / \mathscr{I}_X$. Following the above convention, we usually identify the R-linear category $\mathscr{S}(X)$ with its quiver.

From now on let A be an algebra and Γ be a family of generalized standard infinite components of Γ_A. For an indecomposable brick X in Γ, called the *pivot*, one defines five admissible operations (ad 1)-(ad 5) and their dual (ad 1*)-(ad 5*) modifying the translation quiver $\Gamma = (\Gamma, \tau)$ to a new translation quiver (Γ', τ') and the algebra A to a new algebra A', depending on the shape of the support $\mathscr{S}(X)$ (see [50, Section 2] for the figures illustrating the modified translation quivers Γ').

(ad 1) Assume $\mathscr{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

In this case, we let $t \geq 1$ be a positive integer, $D = T_t(K)$ and $Y_1, Y_2, \ldots, Y_t$ denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective D-module. We define the *modified algebra A'* of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the *modified translation quiver Γ'* of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(K, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau' X_0' = Y_t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1} X_i) = X_i'$ provided X_i is not an injective A-module, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_D, respectively.

If $t = 0$ we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified translation quiver Γ' to be the translation quiver obtained from Γ by inserting only the sectional path consisting of the vertices X_i', $i \geq 0$.

The non-negative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 4.2. Γ' *is a generalized standard family of components of $\Gamma_{A'}$.*

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation. Actually, the above operation is, in this case, the tube insertion as considered in [28].

(ad 2) Suppose that $\mathscr{S}(X)$ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

where $t \geq 1$. In particular, X is necessarily injective. We define the *modified algebra A'* of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver Γ'* of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(K, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $i \geq 1$, $1 \leq j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 1$. The

translation τ' of Γ' is defined as follows: X_0' is projective-injective, $\tau'Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau'Z_{i1} = X_{i-1}$ if $i \geq 1, \tau'Z_{1j} = Y_{j-1}$ if $j \geq 2, \tau'X_i' = Z_{i-1,t}$ if $i \geq 2, \tau'X_1' = Y_t, \tau'(\tau^{-1}X_i) = X_i'$ provided X_i is not an injective A-module, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

The integer $t \geq 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 4.3. Γ' *is a generalized standard family of components of $\Gamma_{A'}$.*

(ad 3) Assume $\mathscr{S}(X)$ is the mesh-category of two parallel sectional paths:

$$
\begin{array}{ccccccc}
Y_1 & \to & Y_2 \to \cdots \to & & Y_t & & \\
\uparrow & & \uparrow & & \uparrow & & \\
X = X_0 & \to & X_1 \to \cdots \to & X_{t-1} & \to & X_t \to \cdots &
\end{array}
$$

where $t \geq 2$. In particular, X_{t-1} is necessarily injective. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1}Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1}Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(K, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $i \geq 1$, $1 \leq j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: X_0' is projective, $\tau'Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $2 \leq j \leq t$, $\tau'Z_{i1} = X_{i-1}$ if $i \geq 1, \tau'X_i' = Y_i$ if $1 \leq i \leq t$, $\tau'X_i' = Z_{i-1,t}$ if $i \geq t+1$, $\tau'Y_j = X_{j-2}'$ if $2 \leq j \leq t$, $\tau'(\tau^{-1}X_i) = X_i'$, if $i \geq t$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ^*. We note that X_{t-1}' is injective.

The integer $t \geq 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 4.4. Γ' *is a generalized standard family of components of $\Gamma_{A'}$.*

(ad 4) Suppose that $\mathscr{S}(X)$ consists an infinite sectional path, starting at X

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

and

$$Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

with $t \geq 1$, be a finite sectional path in Γ_A. Let r be a positive integer. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1}Y_{i-1}$.

We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1}Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. For $r = 0$ we define the *modified algebra* A' of A to be the one-point extension $A' = A[X \oplus Y]$ and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(K, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: $\tau'Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau'Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau'Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau'X_0' = Y_t$, $\tau'X_i' = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1}X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*.

For $r \geq 1$, let $G = T_r(K)$, $U_{1,t+1}$, $U_{2,t+1}$, ..., $U_{r,t+1}$ denote the indecomposable projective G-modules, $U_{r,t+1}$, $U_{r,t+2}$, ..., $U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective-injective G-module. We define the *modified algebra* A' of A to be the triangular matrix algebra of the form:

$$A' = \begin{bmatrix} A & 0 & 0 & \ldots & 0 & 0 \\ Y & K & 0 & \ldots & 0 & 0 \\ Y & K & K & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & K & K & \ldots & K & 0 \\ X \oplus Y & K & K & \ldots & K & K \end{bmatrix}$$

with $r + 2$ columns and rows and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangles consisting of the modules $U_{kl} = Y_l \oplus U_{k,t+k}$ for $1 \leq k \leq r$, $1 \leq l \leq t$, and $Z_{ij} = \left(K, X_i \oplus U_{rj}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $i \geq 0$, $1 \leq j \leq t + r$, and $X_i' = (K, X_i, 1)$ for $i \geq 0$. The translation τ' of Γ' is defined as follows: $\tau'Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau'Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau'Z_{0j} = U_{r,j-1}$ if $2 \leq j \leq t+r$, $Z_{01}, U_{k1}, 1 \leq k \leq r$ are projective, $\tau'U_{kl} = U_{k-1,l-1}$ if $2 \leq k \leq r$, $2 \leq l \leq t+r$, $\tau'U_{1l} = Y_{l-1}$ if $2 \leq l \leq t+1$, $\tau'X_0' = U_{r,t+r}$, $\tau'X_i' = Z_{i-1,t+r}$ if $i \geq 1$, $\tau'(\tau^{-1}X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*, or Γ_G, respectively.

We note that the quiver $Q_{A'}$ of A' is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r+1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

The integers $t \geq 1$ and $r \geq 0$ are such that the number of infinite sectional paths parallel to $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ in the inserted rectangles equals $t + r + 1$. We call $t + r$ the *parameter* of the operation.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 4.5. Γ' *is a generalized standard family of components of* $\Gamma_{A'}$.

(fad 1) Assume $\mathscr{S}(X)$ consists of a finite sectional path starting at X:

$$X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_s$$

where $s \geq 0$ and X_s is injective. Let $t \geq 1$ be a positive integer, $D = T_t(K)$ and Y_1, Y_2, …, Y_t denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective D-module. We define the *modified algebra A'* of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(K, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $0 \leq i \leq s$, $1 \leq j \leq t$, $X'_i = (K, X_i, 1)$ for $0 \leq i \leq s$, $Y'_j = (K, Y_j, 1)$ for $1 \leq j \leq t$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2, Z_{01}$ is projective, $\tau' X'_0 = Y_t, \tau' X'_i = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ', $\tau' Y'_1 = X_s, \tau' Y'_j = Z_{s,j-1}$ if $2 \leq j \leq t, \tau' W = Z_{st}$. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_D, respectively. If $t = 0$ we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified translation quiver Γ' to be the component obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $0 \leq i \leq s$, and W.

Observe that for $s = 0 = t$ the new translation quiver Γ' is obtained from Γ by adding the projective-injective vertex X'_0 and the vertex W with $\tau' W = X_0$.

(fad 2) Suppose that $\mathscr{S}(X)$ admits two finite sectional paths starting at X, each of them with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_s$$

where $t \geq 1$ and $s \geq 1$. In particular, X and X_s are necessarily injective. We define the *modified algebra A'* of A to be the one-point extension $A' = A[X]$ and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(K, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $1 \leq i \leq s$, $1 \leq j \leq t$, $X'_i = (K, X_i, 1)$ for $1 \leq i \leq s$, $Y'_j = (K, Y_j, 1)$ for $1 \leq j \leq t$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: X'_0 is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{1j} = Y_{j-1}$ if $j \geq 2, \tau' X'_i = Z_{i-1,t}$ if $i \geq 2, \tau' X'_1 = Y_t, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ', $\tau' Y'_1 = X_s, \tau' Y'_j = Z_{s,j-1}$ if $2 \leq j \leq t, \tau' W = Z_{st}$. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

(fad 3) Assume $\mathscr{S}(X)$ is the mesh-category of two finite parallel sectional paths:

$$
\begin{array}{ccccccc}
Y_1 & \rightarrow & Y_2 & \rightarrow \cdots \rightarrow & Y_t & & \\
\uparrow & & \uparrow & & \uparrow & & \\
X = X_0 & \rightarrow & X_1 & \rightarrow \cdots \rightarrow & X_{t-1} & \rightarrow X_t \rightarrow \cdots \rightarrow X_s
\end{array}
$$

where $s \geq t-1, t \geq 2$. In particular, X_{t-1} and X_s are necessarily injective. We define the *modified algebra A'* of A to be the one-point extension $A' = A[X]$ and the *modified*

translation quiver Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(K, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $1 \le i \le s$, $1 \le j \le t$, $X_i' = (K, X_i, 1)$ for $1 \le i \le s$, $Y_j' = (K, Y_j, 1)$ for $1 \le j \le t$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: X_0' is projective, $\tau'Z_{ij} = Z_{i-1,j-1}$ if $i \ge 2, 2 \le j \le t$, $\tau'Z_{i1} = X_{i-1}$ if $i \ge 1$, $\tau'X_i' = Y_i$ if $1 \le i \le t$, $\tau'X_i' = Z_{i-1,t}$ if $i \ge t+1$, $\tau'Y_j = X_{j-2}'$ if $2 \le j \le t$, $\tau'(\tau^{-1}X_i) = X_i'$, if $i \ge t$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. In both cases, X_{t-1}' is injective, $\tau'Y_1' = X_s$, $\tau'Y_j' = Z_{s,j-1}$ if $2 \le j \le t$, $\tau'W = Z_{st}$. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ. Observe that for $s = t-1$ we have $Z_{tt} = Y_t'$ and $X_t' = W$.

(fad 4) Suppose that $\mathscr{S}(X)$ consists of a finite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_s$$

with $s \ge 1$ and X_s injective, and

$$Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

$t \ge 1$, be a finite sectional path in Γ_A. Let r be a positive integer. For $r = 0$ we define the *modified algebra* A' of A to be the one-point extension $A' = A[X \oplus Y]$ and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(K, X_i \oplus Y_j, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $0 \le i \le s$, $1 \le j \le t$, $X_i' = (K, X_i, 1)$ for $0 \le i \le s$, $Y_j' = (K, Y_j, 1)$ for $1 \le j \le t$, and $W = S_p$, where p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: $\tau'Z_{ij} = Z_{i-1,j-1}$ if $i \ge 1, j \ge 2$, $\tau'Z_{i1} = X_{i-1}$ if $i \ge 1$, $\tau'Z_{0j} = Y_{j-1}$ if $j \ge 2$, Z_{01} is projective, $\tau'X_0' = Y_t$, $\tau'X_i' = Z_{i-1,t}$ if $i \ge 1$, $\tau'(\tau^{-1}X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ', $\tau'Y_1' = X_s$, $\tau'Y_j' = Z_{s,j-1}$ if $2 \le j \le t$, $\tau'W = Z_{st}$. For the remaining vertices of Γ', τ' coincides with the translation of Γ.

For $r \ge 1$, let $G = T_r(K)$, $U_{1,t+1}$, $U_{2,t+1}$, …, $U_{r,t+1}$ denote the indecomposable projective G-modules, $U_{r,t+1}$, $U_{r,t+2}$, …, $U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective-injective G-module. We define the *modified algebra* A' of A to be the triangular matrix algebra of the form:

$$A' = \begin{bmatrix} A & 0 & 0 & \ldots & 0 & 0 \\ Y & K & 0 & \ldots & 0 & 0 \\ Y & K & K & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & K & K & \ldots & K & 0 \\ X \oplus Y & K & K & \ldots & K & K \end{bmatrix}$$

with $r + 2$ columns and rows and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangles consisting of the modules $U_{kl} = Y_l \oplus U_{k,t+k}$ for $1 \le k \le r$, $1 \le l \le t$, $Z_{ij} = \left(K, X_i \oplus U_{rj}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ for $0 \le i \le s$, $1 \le j \le t+r$, $X_i' = (K, X_i, 1)$ for $0 \le i \le s$, $Y_j' = (K, U_{rj}, 1)$ for $1 \le j \le t+r$, and $W = S_p$, where

p is the extension vertex of $A[X]$. The translation τ' of Γ' is defined as follows: $\tau'Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2$, $\tau'Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau'Z_{0j} = U_{r,j-1}$ if $2 \leq j \leq t+r$, $Z_{01}, U_{k1}, 1 \leq k \leq r$ are projective, $\tau'U_{kl} = U_{k-1,l-1}$ if $2 \leq k \leq r$, $2 \leq l \leq t+r$, $\tau'U_{1l} = Y_{l-1}$ if $2 \leq l \leq t+1$, $\tau'X_0' = U_{r,t+r}$, $\tau'X_i' = Z_{i-1,t+r}$ if $i \geq 1$, $\tau'(\tau^{-1}X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ', $\tau'Y_1' = X_s$, $\tau'Y_j' = Z_{s,j-1}$ if $2 \leq j \leq t+r$, $\tau'W = Z_{s,t+r}$. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_G, respectively.

(ad 5) We define the *modified algebra A'* of A to be the iteration of the extensions described in the definitions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), and their finite versions corresponding to the operations (fad 1), (fad 2), (fad 3) and (fad 4). The *modified translation quiver Γ'* of Γ is obtained in the following three steps: first we are doing on Γ one of the operations (fad 1), (fad 2) or (fad 3), next a finite number (possibly empty) of the operation (fad 4) and finally the operation (ad 4), and in such a way that the sectional paths starting from all the new projective vertices have a common cofinite (infinite) sectional subpath.

Since Γ is a generalized standard family of components of Γ_A, we then have

Lemma 4.6. *Γ' is a generalized standard family of components of $\Gamma_{A'}$.*

Finally, together with each of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4) and (ad 5), we consider its dual, denoted by (ad 1*), (ad 2*), (ad 3*), (ad 4*) and (ad 5*). These ten operations are called the *admissible operations*. Following [50] a connected translation quiver Γ is said to be a *generalized multicoil* if Γ can be obtained from a finite family $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s$ of stable tubes by an iterated application of admissible operations (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3), (ad 3*), (ad 4), (ad 4*), (ad 5) or (ad 5*). If $s = 1$, such a translation quiver Γ is said to be a *generalized coil*. The admissible operations of types (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*) and (ad 3*) have been introduced in [5], [7], [8], and the admissible operations (ad 4) and (ad 4*) for $r = 0$ in [45].

Observe that any stable tube is trivially a generalized coil. A *tube* (in the sense of [28]) is a generalized coil having the property that each admissible operation in the sequence defining it is of the form (ad 1) or (ad 1*). If we apply only operations of type (ad 1) (respectively, of type (ad 1*)) then such a generalized coil is called a *ray tube* (respectively, a *coray tube*). Observe that a generalized coil without injective (respectively, projective) vertices is a ray tube (respectively, a coray tube). A *quasi-tube* (in the sense of [70]) is a generalized coil having the property that each of the admissible operations in the sequence defining it is of type (ad 1), (ad 1*), (ad 2) or (ad 2*). Finally, following [7] a coil is a generalized coil having the property that each of the admissible operations in the sequence defining it is one of the forms (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*). We note that any generalized multicoil Γ is a coherent translation quiver with trivial valuations and its cyclic part $_c\Gamma$ (the translation subquiver of Γ obtained by removing from Γ all acyclic vertices and the arrows attached to them) is infinite, connected and cofinite in Γ, and so Γ is almost cyclic.

Let C be the product $C_1 \times \ldots \times C_m$ of a family $C_1, \ldots, C_m$ of tame concealed algebras and $\mathcal{T}^C$ the disjoin union $\mathcal{T}^{C_1} \cup \ldots \cup \mathcal{T}^{C_m}$ of $\mathbb{P}_1(K)$-families $\mathcal{T}^{C_1}, \ldots, \mathcal{T}^{C_m}$ of

pairwise orthogonal generalized standard stable tubes of $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$, respectively. Following [51], we say that an algebra A is a *generalized multicoil enlargement* of $C_1, \ldots, C_m$ if A is obtained from C by an iteration of admissible operations of types (ad 1)-(ad 5) and (ad 1^*)-(ad 5^*) performed either on stable tubes of $\mathcal{T}^C$ or on generalized multicoils obtained from stable tubes of $\mathcal{T}^C$ by means of operations done so far. It follows from [51, Corollary B] that then A is a triangular algebra, and hence the Tits and Euler forms q_A and χ_A of A are defined. In fact, in [51] generalized multicoil enlargements of finite families of arbitrary concealed canonical algebras have been introduced and investigated. But in the tame case we may restrict to the generalized multicoil enlargements of tame concealed algebras. Namely, we have the following consequence of [51, Theorems A and F].

Theorem 4.7. *Let A be an algebra. The following statements are equivalent:*

(i) *A is tame and Γ_A admits a separating family of almost cyclic coherent components.*

(ii) *A is a tame generalized multicoil enlargement of a finite family of tame concealed algebras.*

(iii) *A is a generalized multicoil enlargement of a finite family of tame concealed algebras and the Tits form q_A is weakly nonnegative.*

From now on, by a *tame generalized multicoil algebra* we mean a connected tame generalized multicoil enlargement of a finite family of tame concealed algebras. The following consequence of [51, Theorems C and F] describes the structure of the Auslander-Reiten quivers of tame generalized multicoil algebras.

Theorem 4.8. *Let A be a tame generalized multicoil algebra obtained from a family $C_1, \ldots, C_m$ of tame concealed algebras. There are convex subcategories $A^{(l)} = A_1^{(l)} \times \ldots \times A_m^{(l)}$ and $A^{(r)} = A_1^{(r)} \times \ldots \times A_m^{(r)}$ of A such that the following statement hold:*

(i) *For each $i \in \{1, \ldots, m\}$, $A_i^{(l)}$ is a domestic tubular or tubular coextension of the tame concealed algebra C_i.*

(ii) *For each $i \in \{1, \ldots, m\}$, $A_i^{(r)}$ is a domestic tubular or tubular extension of the tame concealed algebra C_i.*

(iii) *The Auslander-Reiten quiver Γ_A of A is of the form*

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A,$$

where $\mathcal{C}^A$ is a family of generalized multicoils separating $\mathcal{P}^A$ from $\mathcal{Q}^A$ such that:

(1) *$\mathcal{C}^A$ is obtained from the $\mathbb{P}_1(K)$-families $\mathcal{T}^{C_1}, \ldots, \mathcal{T}^{C_m}$ of stable tubes of $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$ by admissible operations corresponding to the admissible operations leading from $C_1, \ldots, C_m$ to A;*

(2) *$\mathcal{P}^A$ is the disjoint union $\mathcal{P}^{A_1^{(l)}} \cup \ldots \cup \mathcal{P}^{A_m^{(l)}}$, where, for each $i \in \{1, \ldots, m\}$, $\mathcal{P}^{A_i^{(l)}}$ is either the postprojective component of $\Gamma_{A_i^{(l)}}$, if $A_i^{(l)}$ is tilted of Euclidean type, or $\mathcal{P}^{A_i^{(l)}} = \mathcal{P}_0^{A_i^{(l)}} \cup \mathcal{T}_0^{A_i^{(l)}} \cup \left(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{A_i^{(l)}} \right)$, if $A_i^{(l)}$ is a tubular algebra;*

(3) $\mathscr{Q}^A$ is the disjoint union $\mathscr{Q}^{A_1^{(r)}} \cup \ldots \cup \mathscr{Q}^{A_m^{(r)}}$, where, for each $i \in \{1, \ldots, m\}$, $\mathscr{Q}^{A_i^{(r)}}$ is either the preinjective component of $\Gamma_{A_i^{(r)}}$, if $A_i^{(r)}$ is tilted of Euclidean type, or

$$\mathscr{Q}^{A_i^{(r)}} = \left(\bigcup_{q \in \mathbb{Q}^+} \mathscr{T}_q^{A_i^{(r)}} \right) \cup \mathscr{T}_\infty^{A_i^{(r)}} \cup \mathscr{Q}_\infty^{A_i^{(r)}}, \text{ if } A_i^{(r)} \text{ is a tubular algebra.}$$

In particular, we have the following consequence of Theorems 3.1 and 4.8.

Corollary 4.9. *Let A be a tame generalized multicoil algebra. Then A is cycle-finite.*

Further, as a consequence of Theorems 3.4 and 4.8, we obtain the following fact.

Corollary 4.10. *Let A be a tame generalized multicoil algebra and $\Gamma_A = \mathscr{P}^A \cup \mathscr{C}^A \cup \mathscr{Q}^A$ the canonical decomposition of Γ_A. The following statements are equivalent:*

(i) *A is domestic.*
(ii) *$\mathscr{P}^A$ is a disjoint union of postprojective components of Euclidean type and $\mathscr{Q}^A$ is a disjoint union of preinjective components of Euclidean type.*

Moreover, the following consequence of [51, Theorem E] describes the homological properties of modules over tame generalized multicoil algebras.

Theorem 4.11. *Let A be a tame generalized multicoil algebra and $\Gamma_A = \mathscr{P}^A \cup \mathscr{C}^A \cup \mathscr{Q}^A$ the canonical decomposition of Γ_A described above. Then the following statements hold:*

(i) $\mathrm{pd}_A X \leq 1$ *for any module X in $\mathscr{P}^A$.*
(ii) $\mathrm{id}_A X \leq 1$ *for any module X in $\mathscr{Q}^A$.*
(iii) $\mathrm{pd}_A X \leq 2$ *and $\mathrm{id}_A X \leq 2$ for any module X in $\mathscr{C}^A$.*
(iv) $\mathrm{gl\,dim} A \leq 3$.

5 Tame generalized double tilted algebras

In this section we introduce and describe basic properties of the class of tame generalized double tilted algebras, which is the class of tame algebras in the class of generalized double tilted algebras investigated in [62], [63], [79].

Let H be a hereditary algebra, T a (multiplicity-free) tilting module in $\mathrm{mod}\,H$ and $B = \mathrm{End}_H(T)$ the associated tilted algebra. Then T induces the torsion pair $(\mathscr{T}(T), \mathscr{F}(T))$ in $\mathrm{mod}\,H$, with the torsion class $\mathscr{T}(T) = \{M \in \mathrm{mod}\,H; \mathrm{Ext}_A^1(T, M) = 0\}$ and the torsion-free class $\mathscr{T}(F) = \{N \in \mathrm{mod}\,H; \mathrm{Hom}_H(T, N) = 0\}$, and the torsion pair $(\mathscr{X}(T), \mathscr{Y}(T))$ in $\mathrm{mod}\,B$, with the torsion class $\mathscr{X}(T) = \{X \in \mathrm{mod}\,B; X \otimes_B T = 0\}$ and the torsion-free class $\mathscr{Y}(T) = \{Y \in \mathrm{mod}\,H; \mathrm{Tor}_1^B(Y, T) = 0\}$. Then, by the Brenner-Butler theorem, the functors $\mathrm{Hom}_A(T, -)$ and $- \otimes_B T$ induce quasi-inverse equivalence between $\mathscr{T}(T)$ and $\mathscr{Y}(T)$, and the functors $\mathrm{Ext}_A^1(T, -)$ and $\mathrm{Tor}_1^B(-, T)$ induce quasi-inverse equivalence between $\mathscr{F}(T)$ and $\mathscr{X}(T)$ (see [23], [33]). Moreover, $(\mathscr{X}(T), \mathscr{Y}(T))$ is a splitting torsion pair of $\mathrm{mod}\,B$, that is, every indecomposable module in $\mathrm{mod}\,B$ belongs either to $\mathscr{X}(T)$ or $\mathscr{Y}(T)$. Further, the

images $\mathrm{Hom}_H(T,I)$ of the indecomposable injective H-modules I via the functor $\mathrm{Hom}_H(T,I)$ form a section Σ of an acyclic component $\mathscr{C}_T$ of Γ_B such that Σ is isomorphic to the opposite quiver Q_H^{op} of the quiver Q_H of H, any predecessor of Σ in $\mathscr{C}_T$ lies in $\mathscr{Y}(T)$, and any proper successor of Σ in $\mathscr{C}_T$ lies in $\mathscr{X}(T)$. Therefore, the component $\mathscr{C}_T$ of Γ_B connects the torsion-free part $\mathscr{Y}(T)$ with the torsion part $\mathscr{X}(T)$ along the section Σ, and hence $\mathscr{C}_T$ is called the *connecting component* of Γ_B determined by T.

The following theorem proved independently by Liu [43] and Skowroński [72] gives a handy criterion for an algebra to be a tilted algebra.

Theorem 5.1. *An algebra B is a tilted algebra if and only if Γ_B contains a component $\mathscr{C}$ with a faithful section Σ such that $\mathrm{Hom}_B(U,\tau_B V) = 0$ for all modules U, V from Σ. Moreover, in this case, the direct sum T of all modules on Σ is a tilting B-module, $H = \mathrm{End}_B(T)$ is a hereditary algebra, $T^* = D({}_H T)$ is a tilting H-module with $B \cong \mathrm{End}_H(T^*)$, and $\mathscr{C}$ is the connecting component $\mathscr{C}_{T^*}$ of Γ_B determined by T^*.*

The general shape of the Auslander-Reiten quiver of a tilted algebra has been described by Kerner in [37]. We will describe only the Auslander-Reiten quivers of tame tilted algebras, which are exactly the cycle-finite tilted algebras.

Theorem 5.2. *Let $H = K\Delta$ be a hereditary algebra, T a tilting H-module, and assume that the associated tilted algebra $B = \mathrm{End}_H(T)$ is tame. Then the connecting component $\mathscr{C}_T$ of Γ_B determined by T admits a finite (possibly empty) family of pairwise disjoint translation subquivers $\mathscr{D}_1^{(l)}, \ldots, \mathscr{D}_m^{(l)}, \mathscr{D}_1^{(r)}, \ldots, \mathscr{D}_n^{(r)}$ such that the following statements hold.*

(i) *For each $i \in \{1, \ldots, m\}$, there exists an isomorphism of translation quivers $\mathscr{D}_i^{(l)} \cong \mathbb{N}\Delta_i^{(l)}$, where $\Delta_i^{(l)}$ is a connected convex subquiver of Δ of Euclidean type and $\mathscr{D}_i^{(l)}$ is closed under predecessors in $\mathscr{C}_T$.*

(ii) *For each $j \in \{1, \ldots, n\}$, there exists an isomorphism of translation quivers $\mathscr{D}_j^{(r)} \cong (-\mathbb{N})\Delta_j^{(r)}$, where $\Delta_j^{(r)}$ is a connected convex subquiver of Δ of Euclidean type and $\mathscr{D}_j^{(r)}$ is closed under successors in $\mathscr{C}_T$.*

(iii) *All but finitely many modules of $\mathscr{C}_T$ lie in $\mathscr{D}_1^{(l)} \cup \cdots \cup \mathscr{D}_m^{(l)} \cup \mathscr{D}_1^{(r)} \cup \cdots \cup \mathscr{D}_n^{(r)}$.*

(iv) *For each $i \in \{1, \ldots, m\}$, there exists a tilted algebra $B_i^{(l)} = \mathrm{End}_{H_i^{(l)}}(T_i^{(l)})$, where $H_i^{(l)}$ is the path algebra $K\Delta_i^{(l)}$, $T_i^{(l)}$ is a tilting $H_i^{(l)}$-module without nonzero preinjective direct summands, $B_i^{(l)}$ is a quotient algebra of B, and $\mathscr{D}_i^{(l)}$ coincides with the torsion-free part $\mathscr{Y}(T_i^{(l)}) \cap \mathscr{C}_{T_i^{(l)}}$ of the connecting component $\mathscr{C}_{T_i^{(l)}}$ of $\Gamma_{B_i^{(l)}}$ determined by $T_i^{(l)}$.*

(v) *For each $j \in \{1, \ldots, n\}$, there exists a tilted algebra $B_j^{(r)} = \mathrm{End}_{H_j^{(r)}}(T_j^{(r)})$, where $H_j^{(r)}$ is the path algebra $K\Delta_j^{(r)}$, $T_j^{(r)}$ is a tilting $H_j^{(r)}$-module without nonzero postprojective direct summands, $B_j^{(r)}$ is a quotient algebra of B, and $\mathscr{D}_j^{(r)}$ coincides*

with the torsion part $\mathscr{X}(T_j^{(r)}) \cap \mathscr{C}_{T_j^{(r)}}$ of the connecting component $\mathscr{C}_{T_j^{(r)}}$ of $\Gamma_{B_j^{(r)}}$ determined by $T_j^{(r)}$.

(vi) $\mathscr{Y}(T) = \mathrm{add}(\mathscr{Y}(T_1^{(l)}) \cup \cdots \cup \mathscr{Y}(T_m^{(l)}) \cup (\mathscr{Y}(T) \cap \mathscr{C}_T))$.

(vii) $\mathscr{X}(T) = \mathrm{add}((\mathscr{X}(T) \cap \mathscr{C}_T) \cup \mathscr{X}(T_1^{(r)}) \cup \cdots \cup \mathscr{X}(T_n^{(r)}))$.

(viii) *The Auslander-Reiten quiver Γ_B of B has the disjoint union decomposition*

$$\Gamma_B = \left(\bigcup_{i=1}^{m} \mathscr{Y}\Gamma_{B_i^{(l)}} \right) \cup \mathscr{C}_T \cup \left(\bigcup_{j=1}^{n} \mathscr{X}\Gamma_{B_j^{(r)}} \right),$$

where

(a) *For each $i \in \{1, \ldots, m\}$, $\mathscr{Y}\Gamma_{B_i^{(l)}}$ is the union of all components of $\Gamma_{B_i^{(l)}}$ contained entirely in $\mathscr{Y}(T_i^{(l)})$, and hence consists of a unique postprojective component $\mathscr{P}^{B_i^{(l)}}$ and a $\mathbb{P}_1(K)$-family $\mathscr{T}^{B_i^{(l)}} = (\mathscr{T}_\lambda^{B_i^{(l)}})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard ray tubes;*

(b) *For each $j \in \{1, \ldots, n\}$, $\mathscr{X}\Gamma_{B_j^{(r)}}$ is the union of all components of $\Gamma_{B_j^{(r)}}$ contained entirely in $\mathscr{X}(T_j^{(r)})$, and hence consists of a unique preinjective component $\mathscr{Q}^{B_j^{(r)}}$ and a $\mathbb{P}_1(K)$-family $\mathscr{T}^{B_j^{(r)}} = (\mathscr{T}_\lambda^{B_j^{(r)}})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard coray tubes.*

The following theorem follows from [18](part (i)) and [65, p.376](parts (ii) and (iii)).

Theorem 5.3. *Let A be a cycle-finite algebra, X a directing module in $\mathrm{mod}A$, and $B = \mathrm{supp}X$. Then the following statements hold:*

(i) *B is a convex subcategory of A.*
(ii) *B is a tame tilted algebra.*
(iii) *X belongs to a connecting component of Γ_B.*

We refer to [17] and [29] (respectively, [55] and [56]) for a classification of representation-finite (respectively, tame representation-infinite) tame tilted algebras with sincere directing modules.

The class of tilted algebras was extended in [62] to the class of double tilted algebras, and next in [63] to the class of generalized double tilted algebras, containing the class of all algebras of finite type, by extending the concept of a section to the concept of a multisection.

Following [63], a full connected subquiver Δ of a component $\mathscr{C}$ of the Auslander-Reiten quiver Γ_A of an algebra A is said to be a *multisection* if the following conditions are satisfied:

(i) Δ is almost acyclic.
(ii) Δ is convex in $\mathscr{C}$.

(iii) For each τ_A-orbit $\mathcal{O}$ in $\mathcal{C}$, we have $1 \leq |\Delta \cap \mathcal{O}| < \infty$.
(iv) $|\Delta \cap \mathcal{O}| = 1$ for all but finitely many τ_A-orbits $\mathcal{O}$ in $\mathcal{C}$.
 (v) No proper full convex subquiver of Δ satisfies (i)–(iv).

It has been proved in [63, Theorem 2.5] that a component $\mathcal{C}$ of Γ_A is almost acyclic if and only if $\mathcal{C}$ admits a multisection Δ. Moreover, for a multisection Δ of a component $\mathcal{C}$ of Γ_A, the following full subquivers of $\mathcal{C}$ were defined in [63]:

 (i) $\Delta'_l = \{X \in \Delta;$ there is a nonsectional path in $\mathcal{C}$ from X to a projective module $P\}$,
 (ii) $\Delta'_r = \{X \in \Delta;$ there is a nonsectional path in $\mathcal{C}$ from an injective module I to $X\}$,
(iii) $\Delta''_l = \{X \in \Delta'_l; \tau_A^{-1} X \notin \Delta'_l\}$, $\Delta''_r = \{X \in \Delta'_r; \tau_A X \notin \Delta'_r\}$,
(iv) $\Delta_l = (\Delta \setminus \Delta'_r) \cup \tau_A \Delta''_r$, $\Delta_c = \Delta'_l \cap \Delta'_r$, $\Delta_r = (\Delta \setminus \Delta'_l) \cup \tau_A^{-1} \Delta''_l$.

Then Δ_l is called the *left part* of Δ, Δ_r the *right part* of Δ, and Δ_c the *core* of Δ.

Lemma 5.4. *Let A be an algebra, $\mathcal{C}$ a component of Γ_A and Δ a multisection of $\mathcal{C}$. The following statements hold*

 (i) *Every cycle of $\mathcal{C}$ lies in Δ_c.*
 (ii) *Δ_c is finite.*
(iii) *Every indecomposable module X in $\mathcal{C}$ is in Δ_c, or a predecessor of Δ_l or a successor of Δ_r in $\mathcal{C}$.*
(iv) *Δ is faithful if and only if $\mathcal{C}$ is faithful.*

Moreover, in [63] a numerical invariant $w(\Delta) \in \mathbb{N} \cup \{\infty\}$ of a multisection Δ of $\mathcal{C}$, called the *width* of Δ, was introduced such that $\mathcal{C}$ is acyclic if and only if $w(\Delta) < \infty$, and $w(\Delta) = 1$ if and only if Δ is a section.

The following facts proved in [63, Proposition 2.11] show that the core and the width of a multisection of an almost cyclic component $\mathcal{C}$ are uniquely determined by $\mathcal{C}$.

Proposition 5.5. *Let A be an algebra, $\mathcal{C}$ a component of Γ_A and Δ, Σ multisections of $\mathcal{C}$. Then $\Delta_c = \Sigma_c$ and $w(\Delta) = w(\Sigma)$.*

Following [63], an algebra B is said to be a *generalized double tilted algebra* if the following conditions are satisfied:

(1) Γ_B admits a component $\mathcal{C}$ with a faithful multisection Δ.
(2) There exists a tilted quotient algebra $B^{(l)}$ of B (not necessarily connected) such that Δ_l is a disjoint union of sections of the connecting components of the connected parts of $B^{(l)}$ and the category of all predecessors of Δ_l in ind B coincides with the category of all predecessors of Δ_l in ind $B^{(l)}$.
(3) There exists a tilted quotient algebra $B^{(r)}$ of B (not necessarily connected) such that Δ_r is a disjoint union of sections of the connecting components of the connected parts of $B^{(r)}$, and the category of all successors of Δ_r in ind B coincides with the category of all successors of Δ_r in ind $B^{(r)}$.

Then $B^{(l)}$ is called a *left tilted part* of B and $B^{(r)}$ a *right tilted part* of B.

The following generalization of Theorem 5.1, proved in [63, Theorem 3.1], gives a handy criterion for an algebra to be a generalized double tilted algebra.

Theorem 5.6. *Let B be an algebra. The following conditions are equivalent:*

(i) *B is a generalized double tilted algebra.*
(ii) *The quiver Γ_B admits a component $\mathscr{C}$ with a faithful multisection Δ such that $\mathrm{Hom}_B(U, \tau_B V) = 0$, for all modules $U \in \Delta_r$ and $V \in \Delta_l$.*
(iii) *The quiver Γ_B admits a faithful generalized standard almost acyclic component $\mathscr{C}$.*

In particular, we obtain the following characterization of tame generalized double tilted algebras.

Theorem 5.7. *Let B be a generalized double tilted algebra, $\mathscr{C}$ a faithful generalized standard almost cyclic component of Γ_B, and Δ a multisection of $\mathscr{C}$. The following conditions are equivalent:*

(i) *B is tame.*
(ii) *B is cycle-finite.*
(iii) *Δ_l and Δ_r are disjoint unions of Euclidean quivers.*
(iv) *The tilted algebras $B^{(l)}$ and $B^{(r)}$ are tame.*
(v) *The Auslander-Reiten quiver Γ_B of B has disjoint union decomposition*

$$\Gamma_B = \mathscr{Y}\Gamma_{B^{(l)}} \cup \mathscr{C} \cup \mathscr{X}\Gamma_{B^{(r)}},$$

where

(a) *$\mathscr{Y}\Gamma_{B^{(l)}}$ is the union of all connected components of $\Gamma_{B^{(l)}}$ contained entirely in the torsion-free part $\mathscr{Y}(B^{(l)})$, and $\mathscr{Y}\Gamma_{B^{(l)}}$ is a disjoint union of postprojective components of Euclidean type and $\mathbb{P}_1(K)$-families of pairwise orthogonal generalized standard ray tubes.*
(b) *$\mathscr{X}\Gamma_{B^{(r)}}$ is the union of all connected components of $\Gamma_{B^{(r)}}$ contained entirely in the torsion part $\mathscr{X}(B^{(r)})$, and $\mathscr{X}\Gamma_{B^{(r)}}$ is a disjoint union of preinjective components of Euclidean type and $\mathbb{P}_1(K)$-families of pairwise orthogonal generalized standard coray tubes.*

We end this section with an example of a tame generalized double tilted algebra, illustrating the above considerations.

Example 5.8. Let $B = KQ/I$, where Q is the quiver
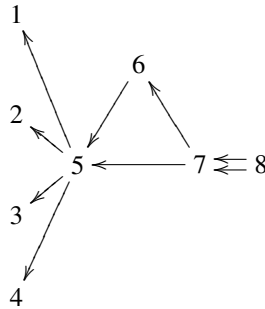
and I is the ideal of KQ generated by all paths of Q of length 2 (see [63, Example 4.3]). Then B is a tame generalized double tilted algebra of global dimension 4 and Γ_B admits a generalized standard component $\mathscr{C}$ of the form
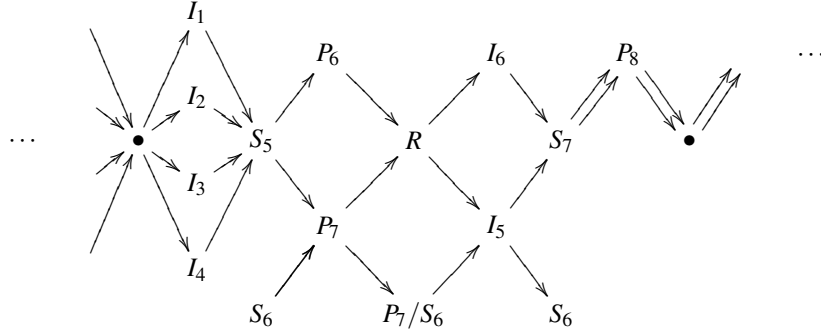
with a faithful multisection Δ formed by the indecomposable injective modules $I_1, I_2, I_3, I_4, I_5, I_6$ (at the vertices 1, 2, 3, 4, 5, 6), the indecomposable projective modules P_6, P_7, P_8 (at the vertices 6, 7, 8), the simple modules S_5, S_6, S_7 (at the vertices 5, 6, 7) and the modules P_7/S_6, $R = \tau_B^{-1} S_5 = \tau_B S_7$. Then the left part Δ_l of Δ consists of the modules $I_1, I_2, I_3, I_4, S_5, P_6$ and is a section of the preinjective connecting component of the tame tilted algebra $B^{(l)}$ being the convex subcategory of B given by the vertices 1, 2, 3, 4, 5, 6. The right part Δ_r of Δ consists of the modules I_6, S_7, P_8 and is a section of the postprojective connecting component of the tame tilted algebra $B^{(r)}$ being the convex subcategory of B given by the vertices 6, 7, 8. Moreover, the core Δ_c of Δ consists of the modules $S_6, P_7, P_7/S_6, R, I_5$ and is the cyclic part of the Auslander-Reiten quiver $\Gamma_{B^{(c)}}$ of the representation-finite convex subcategory $B^{(c)} = \operatorname{supp} \Delta_c$ of B given by the vertices 5, 6, 7. We also note that every module in $\operatorname{ind} B$ belongs to one of its full subcategories $\operatorname{ind} B^{(l)}$, $\operatorname{ind} B^{(c)}$, or $\operatorname{ind} B^{(r)}$.

6 Cyclic components of cycle-finite algebras

Let A be an algebra. We denote by $_c\Gamma_A$ the translation subquiver of Γ_A, called the *cyclic part* of Γ_A, obtained by removing from Γ_A all acyclic modules and the arrows attached to them. The connected components of $_c\Gamma_A$ are said to be *cyclic components* of Γ_A (see [50]). The following result from [50, Proposition 5.1] will be very useful.

Proposition 6.1. *Let A be an algebra and X, Y be two cyclic modules of Γ_A. Then X and Y belong to the same cyclic component of Γ_A if and only if there is an oriented cycle in Γ_A passing through X and Y.*

Moreover, we have the following property of the support algebras of cyclic components of the Auslander-Reiten quivers of cycle-finite algebras (see [49]).

Proposition 6.2. *Let A be a cycle-finite algebra, Γ a cyclic component of Γ_A and $B = \operatorname{supp} \Gamma$. Then B is a convex subcategory of A.*

Let A be an algebra and $\mathscr{C}$ be a component of Γ_A. We denote by $_l\mathscr{C}$ the *left stable part* of $\mathscr{C}$, obtained by removing from $\mathscr{C}$ the τ_A-orbits containing projective modules, and by $_r\mathscr{C}$ the *right stable part* of $\mathscr{C}$, obtained by removing from $\mathscr{C}$ the τ_A-orbits containing injective modules. We note that if $\mathscr{C}$ is an infinite component of Γ_A then $_l\mathscr{C}$ or $_r\mathscr{C}$ is not empty.

The following theorem from [49, Theorem 1] describes the supports of infinite cyclic components of the Auslander-Reiten quivers of cycle-finite algebras.

Theorem 6.3. *Let A be a cycle-finite algebra and Γ an infinite cyclic component of Γ_A. Then there exist infinite full translation subquivers $\Gamma_1, \ldots, \Gamma_r$ of Γ such that the following statements hold.*

(i) *For each $i \in \{1, \ldots, r\}$, Γ_i is a cyclic coherent full translation subquiver of Γ_A.*
(ii) *For each $i \in \{1, \ldots, r\}$, $B^{(i)} = \operatorname{supp}\Gamma_i$ is a tame generalized multicoil algebra and a quotient algebra of A.*
(iii) *$\Gamma_1, \ldots, \Gamma_r$ are pairwise disjoint full translation subquivers of Γ and $\Gamma^{cc} = \Gamma_1 \cup \ldots \cup \Gamma_r$ is a maximal cyclic coherent and cofinite full translation subquiver of Γ.*
(iv) *$B(\Gamma \setminus \Gamma^{cc}) = A/\operatorname{ann}_A(\Gamma \setminus \Gamma^{cc})$ is of finite representation type.*

The following example illustrates the above theorem.

Example 6.4. Let $A = KQ/I$, where Q is the quiver
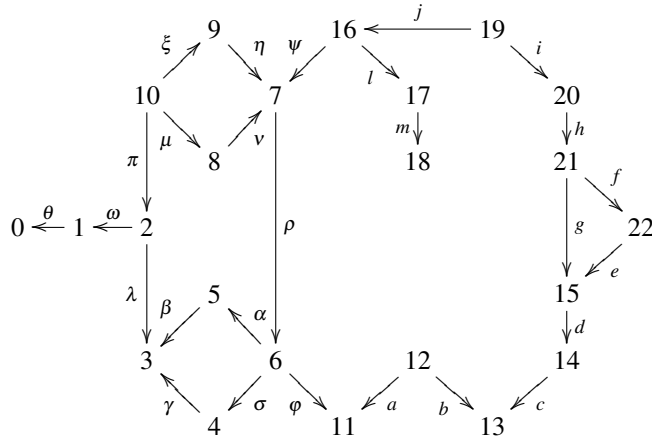
and I is the ideal in the path algebra KQ of Q generated by the elements $\alpha\beta - \sigma\gamma$, $\xi\eta - \mu\nu$, $\pi\lambda - \xi\eta\rho\alpha\beta$, $\rho\varphi$, $\psi\rho$, jl, dc, ed, gd, hg, hf, ih. Then A is a cycle-finite and Γ_A admits a component $\mathscr{C}$ of the form
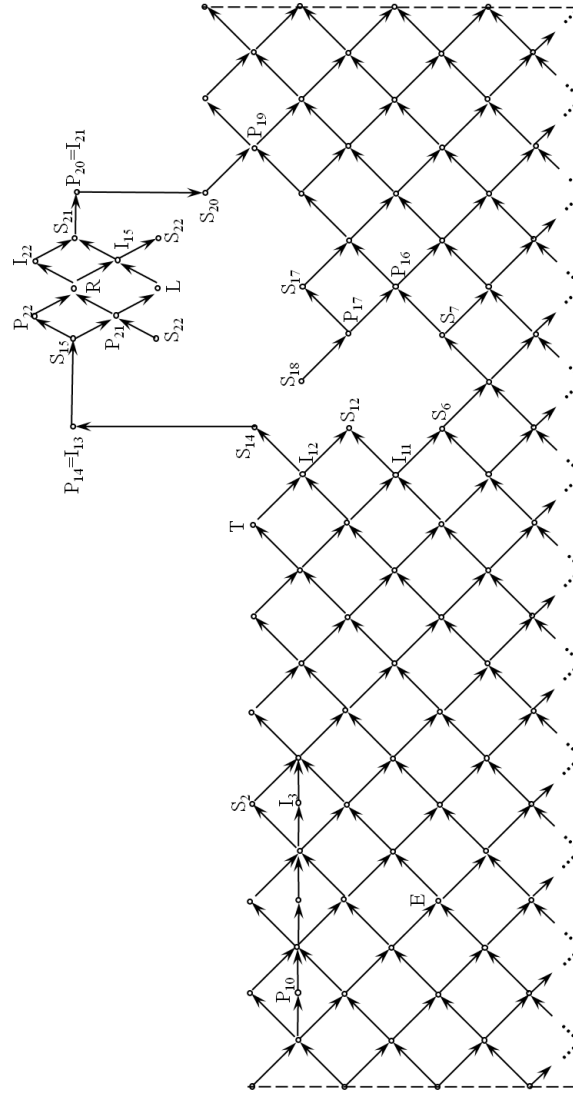
The cyclic part Γ of $\mathscr{C}$ is obtain by removing from $\mathscr{C}$ the (directing) modules $S_{12}, S_{17}, S_{18}, P_{17}$ and the arrows attached to them. The maximal cyclic coherent part Γ^{cc} of Γ is the full translation subquiver of $\mathscr{C}$ obtained by removing the modules $S_{12}, I_{12}, T, S_{14}, P_{14} = I_{13}, S_{15}, P_{21}, S_{22}, L, P_{22}, R, I_{15}, I_{22}, S_{21}, P_{20} = I_{21}, S_{20}, S_{17}, P_{17}, S_{18}$ and the arrows attached to them. Further, Γ^{cc} is the cyclic part of the maximal almost cyclic coherent full translation subquiver Γ^* of $\mathscr{C}$ obtained by removing the modules $P_{14} = I_{13}, S_{15}, P_{21}, S_{22}, L, P_{22}, R, I_{15}, I_{22}, S_{21}, P_{20} = I_{21}$ and the arrows attached to them.

Let $B = A/\operatorname{ann}\Gamma$. Then $B = A/\operatorname{ann}\Gamma^*$, because $\operatorname{ann}\Gamma = \operatorname{ann}\Gamma^*$. Observe that $B = KQ_B/I_B$, where Q_B is the full subquiver of Q given by all vertices of Q except

15, 21, 22, and $I_B = I \cap KQ_B$. We claim that B is a tame generalized multicoil algebra. Consider the path algebra $C = K\Delta$ of the full subquiver of Q given by the vertices 4, 5, 6, 7, 8, 9. Then C is a hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_5$, and hence a tame concealed algebra. It is know that Γ_C admits $\mathbb{P}_1(K)$-family $\mathscr{T}_\lambda^C$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized standard stable tubes, having a unique stable tube $\mathscr{T}_\lambda$ of rank 3 with the mouth formed by the modules $S_6 = \tau_C S_7$, $S_7 = \tau_C E$, $E = \tau_C S_6$, where E is the unique indecomposable C-module with the dimension

vector $\mathbf{dim}\, E = \begin{smallmatrix} & 1 & \\ & 1 & \\ 1 & & \\ & 1 & \\ & 1 & \\ & 1 & \end{smallmatrix}$, (see [67, Theorem XIII 2.9]).

Then B is the generalized multicoil enlargement of C, obtained by applications

- two admissible operations of types (ad 1*) with the pivots S_6 and S_{12}, creating the vertices 11, 12, 13 and the arrows φ, a, b, c;
- two admissible operations of types (ad 1*) with the pivots E and S_2, creating the vertices 3, 2, 1, 0 and the arrows β, γ, λ, ω, θ;
- two admissible operations of types (ad 1) with the pivots S_7 and S_{16}, creating the vertices 16, 17, 18, 19, 20 and the arrows ψ, l, m, j, i;
- one admissible operation of type (ad 3) with the pivot the radical of P_{10}, creating the vertex 10 and the arrows ξ, μ, π.

Then the left part $B^{(l)}$ of B is the convex subcategory of B (and of A) given by the vertices 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, and is a tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_{13}$ with the connecting postprojective component $\mathscr{P}^{B^{(l)}}$ containing all indecomposable projective $B^{(l)}$-modules. The right part $B^{(r)}$ of B is the convex subcategory of B (and of A) given by the vertices 4, 5, 6, 7, 8, 9, 16, 17, 18, 19, 20, and is a tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_{10}$ with the connecting preinjective component $\mathscr{Q}^{B^{(r)}}$ containing all indecomposable injective $B^{(r)}$-modules. We also note that the algebra $B(\Gamma \setminus \Gamma^{cc}) = A / \mathrm{ann}_A(\Gamma \setminus \Gamma^{cc})$ is the representation-finite convex subcategory of A given by the vertices 12, 13, 14, 15, 20, 21, 22. It follows from Theorem 4.8 that the Auslander-Reiten quiver Γ_B of the generalized multicoil enlargement B of C is of the form

$$\Gamma_B = \mathscr{P}^B \cup \mathscr{C}^B \cup \mathscr{Q}^B,$$

where $\mathscr{P}^B = \mathscr{P}^{B^{(l)}}$, $\mathscr{Q}^B = \mathscr{Q}^{B^{(r)}}$, and $\mathscr{C}^B$ is the $\mathbb{P}_1(K)$-family $\mathscr{C}_\lambda^B$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized multicoils such that $\mathscr{C}_1^B = \Gamma^*$ and $\mathscr{C}_\lambda^B = \mathscr{T}_\lambda^C$ for all $\lambda \in \mathbb{P}_1(K) \setminus \{1\}$. Hence Γ_A is of the form

$$\Gamma_A = \mathscr{P}^A \cup \mathscr{C}^A \cup \mathscr{Q}^A,$$

where $\mathscr{P}^A = \mathscr{P}^{B^{(l)}}$, $\mathscr{Q}^A = \mathscr{Q}^{B^{(r)}}$, and $\mathscr{C}^A$ is the $\mathbb{P}_1(K)$-family $\mathscr{C}_\lambda^A$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal generalized standard components such that $\mathscr{C}_1^A = \mathscr{C}$, $\mathscr{C}_\lambda^A = \mathscr{T}_\lambda^C$ for all $\lambda \in \mathbb{P}_1(K) \setminus \{1\}$. Moreover, we have

$$\mathrm{Hom}_A(\mathscr{C}^A, \mathscr{P}^A) = 0, \mathrm{Hom}_A(\mathscr{Q}^A, \mathscr{C}^A) = 0, \mathrm{Hom}_A(\mathscr{Q}^A, \mathscr{P}^A) = 0.$$

In particular, A is a cycle-finite algebra with $(\mathrm{rad}^\infty(\mathrm{mod}\,A))^3 = 0$.

The following theorem from [49, Theorem 2] describes the supports of finite cyclic components of the Auslander-Reiten quivers of cycle-finite algebras.

Theorem 6.5. *Let A be a cycle-finite algebra and Γ a finite cyclic component of Γ_A. Then the following statements hold.*

(i) $B = \mathrm{supp}\,\Gamma$ *is a tame generalized double tilted algebra.*
(ii) Γ *is the core of the connecting component $\mathscr{C}_B$ of Γ_B.*

We note that if the core of an almost cyclic component of an Auslander-Reiten quiver Γ_A is not empty that it contains a projective module and an injective module. Then we obtain the following consequence of Theorem 6.5.

Corollary 6.6. *Let A be a cycle-finite algebra. Then the number of finite cyclic components of Γ_A is bounded by the rank of $K_0(A)$.*

Observe also that for a cycle-finite algebra A of infinite representation type there are infinitely many (infinite) cyclic components of Γ_A, since Γ_A contains infinitely many stable tubes (see Theorems 3.1 and 3.3).

7 The structure theorems

Let A be a cycle-finite algebra and X a module in $\mathrm{ind}\,A$. Then X is a directing module if and only if X is an acyclic vertex of Γ_A. Hence X is nondirecting if and only if X belongs to a cyclic component of Γ_A. Then the following structure theorem is a direct consequence of Theorems 5.3, 6.3 and Propositions 6.1, 6.2.

Theorem 7.1. *Let A be a cycle-finite algebra. Then there exist quotient algebras $B_1, \ldots, B_p$ of A such that the following statements hold.*

(i) *For each $i \in \{1, \ldots, p\}$, B_i is either a tame generalized multicoil algebra or a tame generalized double tilted algebra.*
(ii) $\mathrm{ind}\,A = \bigcup_{i=1}^p \mathrm{ind}\,B_i$.

It follows from Theorem 5.2 that for a tame tilted algebra B, all but finitely many modules in $\mathrm{ind}\,B$ are indecomposable modules over the left tilted algebras $B_1^{(l)}, \ldots, B_m^{(l)}$ of Euclidean types or over the right tilted algebras $B_1^{(r)}, \ldots, B_n^{(r)}$ of Euclidean types. Moreover, all representation-infinite tilted algebras of Euclidean types and all tubular algebras are tame generalized multicoil algebras. Therefore, applying Theorem 4.8, we obtain the following completion to Theorem 7.1.

Theorem 7.2. *Let A be a cycle-finite algebra. Then there exist tame generalized multicoil quotient algebras $B_1, \ldots, B_q$ of A whose indecomposable finite dimensional modules exhaust all but finitely many isoclasses of modules in $\mathrm{ind}\,A$.*

Moreover, we have the following information on nondirecting indecomposable modules over cycle-finite algebras.

Theorem 7.3. *Let A be a cycle-finite algebra. Then there exist tame generalized multicoil quotient algebras $B_1, \ldots, B_q$ of A such that all but finitely many isomorphism classes of nondirecting modules in ind$A$ belong to generalized multicoils of the Auslander-Reiten quivers $\Gamma_{B_1}, \ldots, \Gamma_{B_q}$ of $B_1, \ldots, B_q$.*

We exhibit an example of a cycle-finite algebra having a nongeneralized standard Auslander-Reiten component.

Example 7.4. Let Q be the quiver
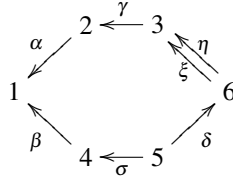
I the ideal in the path algebra KQ of Q generated by the elements $\gamma\alpha$, $\sigma\beta$, $\eta\gamma$, $\xi\gamma$, $\delta\xi$, $\delta\eta$, and $A = KQ/I$ the associated bound quiver algebra. Then A is a cycle-finite algebra whose Auslander-Reiten quiver Γ_A has the disjoint union form

$$\Gamma_A = \mathscr{C} \cup \Big(\bigcup_{\lambda \in \mathbb{P}_1(K)} \mathscr{T}_\lambda \Big),$$

where $\mathscr{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$, is the family of stable tubes of rank 1 over the Kronecker path algebra $H = K\Delta$ given by the subquiver Δ of Q formed by the arrows ξ and η, and $\mathscr{C}$ is the following glueing of the preprojective component $\mathscr{P}(H)$ and the preinjective component $\mathscr{Q}(H)$ of Γ_A into a component of Γ_A
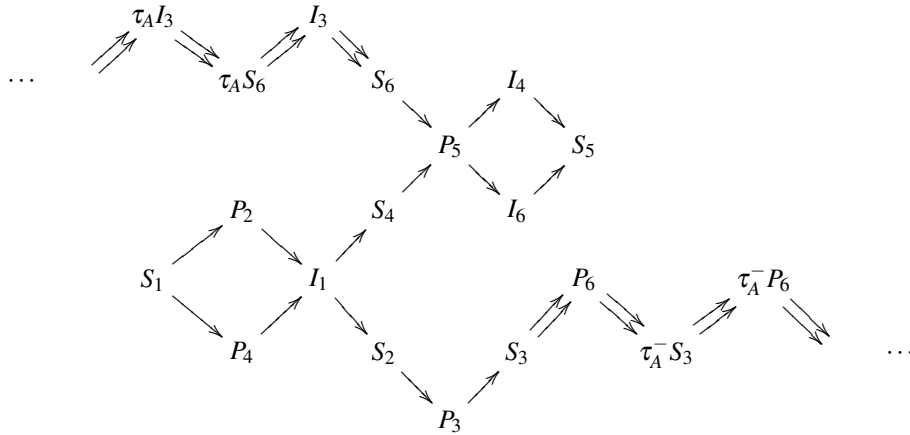

Observe that the simple module S_3 is the socle of the injective module I_3 and the canonical monomorphism $S_3 \to I_3$ belongs to rad$^\infty$(mod$A$), because it factors through any stable tube $\mathscr{T}_\lambda$. Hence $\mathscr{C}$ is not generalized standard. On the other hand, every cycle in ind$A$ consists of modules of a stable tube $\mathscr{T}_\lambda$ and hence is finite, be-

cause the stable tubes $\mathscr{T}_\lambda$ are generalized standard. Therefore, A is a cycle-finite algebra.

We end this section with an example of an algebra A with partially ordered Auslander-Reiten components of Γ_A which is not cycle-finite.

Example 7.5. Let Q be the quiver
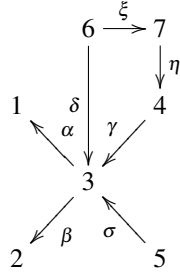
I the ideal in the path algebra KQ of Q generated by the elements $\delta\alpha$, $\delta\beta$, $\xi\eta$ and $\eta\gamma$, and $A = KQ/I$ the associated bound quiver algebra. Then A is a tame algebra whose Auslander-Reiten quiver Γ_A has the disjoint union form

$$\Gamma_A = \mathscr{P}(A) \cup \mathscr{T}^A \cup \mathscr{C},$$

where $\mathscr{P}(A)$ is a preprojective component, $\mathscr{T}^A = (\mathscr{T}^A_\lambda)_{\lambda \in \mathbb{P}_1(K) \setminus \{1\}}$ is a family of pairwise orthogonal generalized standard stable tubes, and $\mathscr{C}$ is a component of the form below
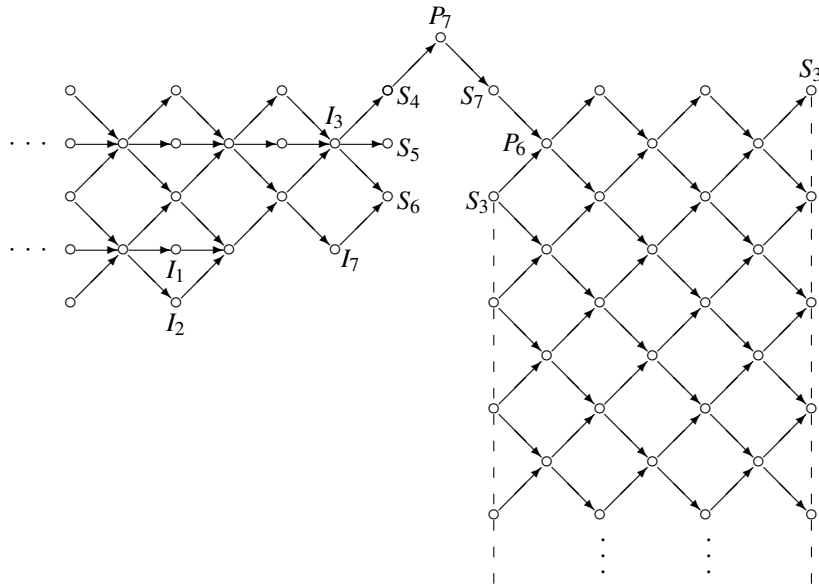
(see [36, Example 2.6]). Then we obtain that the components of Γ_A are partially ordered in $\mathrm{ind}\,A$. In particular, we conclude that for every cycle $X_0 \to X_1 \to \cdots \to X_{r-1} \to X_r = X_0$ in $\mathrm{ind}\,A$, all modules $X_0, X_1, \ldots, X_{r-1}$ belong to the same component of Γ_A. On the other hand, we have in $\mathrm{ind}\,A$ a cycle

$$ I_3 \xrightarrow{\ f_1\ } P_7 \xrightarrow{\ f_2\ } P_6 \xrightarrow{\ f_3\ } I_3 $$

consisting of modules from $\mathscr{C}$ and with f_3 in $\mathrm{rad}^\infty(\mathrm{mod}\,A)$, because there is no path in $\mathscr{C}$ from P_6 to I_3. Therefore, A is not a cycle-finite algebra.

8 Discrete indecomposable modules

The aim of this section is to establish a common bound on the number of discrete indecomposable modules of a fixed dimension vector over a cycle-finite algebra.

Lemma 8.1. *Let A be an algebra, Γ a generalized standard generalized multicoil of Γ_A, n the rank of $K_0(A)$, and $\mathbf{d}$ a nonnegative vector of $K_0(A)$. Then the number of indecomposable modules X in Γ with $\mathbf{dim}\,X = \mathbf{d}$ is bounded by n. Moreover, if Γ consists of modules which do not lie on infinite cycles then the number of indecomposable modules X in Γ with $\mathbf{dim}\,X = \mathbf{d}$ is bounded by $n-1$.*

Proof. Without loss of generality we may assume that A is the support algebra of Γ. Let Γ be an arbitrary generalized multicoil of Γ_A which is generalized standard. We shall prove our claim by induction on the number m of admissible operations which we have to do on a finite family $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_s$ of generalized standard stable tubes in order to obtain the generalized multicoil Γ. If $m = 1$, then we can only do the admissible operation (ad 1) or (ad 1*), so $s = 1$. In this case our statement follows from [46, Proposition 4.1].

Let $m > 1$. If the mth admissible operation is of type (ad 1), then by definition of (ad 1) we have $\mathbf{dim}\,V = \mathbf{dim}\,W$ for any modules $V, W \in \{Z_{ij}, X_i'\}$, $i \geq 0$, $1 \leq j \leq t$ such that $V \neq W$. Therefore, the number of indecomposable modules with the same dimension vector does not change. If it is of type (ad 1*) then the situation is the same. If the mth admissible operation is of type (ad 2), then in the sequence of earlier $m - 1$ admissible operations, there is an operation of type (ad 1*) or (ad 5*) which contains an operation (fad 1*) which gives rise to the pivot X of (ad 2), and the operations done between these two must not affect the support of $\mathrm{Hom}_A(X, -)$ restricted to the generalized multicoil containing X. Let t be the parameter of such operation (ad 1*) or like in definition of (fad 1*). Note that in general, in the sequence of earlier $m - 1$ admissible operations can be an operation of type (ad 5) which contains an operation (fad 4) which gives rise to the pivot X of (ad 2) but from Lemma [51, Lemma 3.10] this case can be reduced to (ad 5*) which contains an operation (fad 1*). Moreover, from [46, Lemma 3.3] we know that, for a fixed $\mathbf{e} \in K_0(A)$, each ray and coray in Γ contains at most one module Z with $\mathbf{dim}\,Z = \mathbf{e}$. Therefore, we get that the number of new indecomposable modules with the same

dimension vector is at most $t+1$ while after applying operations (ad 1*) and (ad 2) or (fad 1*) and (ad 2) the number of new vertices in ordinary quiver of A increases by $t+2$. If the mth admissible operation is of type (ad 2*), then the proof is dual. If the mth admissible operation is of type (ad 3), then in the sequence of earlier $m-1$ admissible operations, there must be at least one operation of type (ad 1*) or (ad 5*) which contains the operation (fad 1*) which gives rise to the pivot X of (ad 3) and to the modules $Y_1, Y_2, \ldots, Y_t$ in the support of $\mathrm{Hom}_A(X, -)$ restricted to the generalized multicoil containing X. The operations done after must not affect this support. Again, in general, in the sequence of earlier $m-1$ admissible operations can be an operation of type (ad 5) which contains an operation (fad 4) which gives rise to the pivot X of (ad 3) but from [51, Lemma 3.10] this case can be reduced to (ad 5*) which contains an operation (fad 1*). Suppose that we had r such consecutive operations of types (ad 1*) or (fad 1*), the first of which had X_t as a pivot, and these admissible operations built up a branch L in A with points $a, a_1, a_2, \ldots, a_t$ in Q_A, so that X_{t-1} and Y_t are the indecomposable injective A-modules corresponding respectively to a and a_1, and both Y_1 and $\tau_A^{-1} Y_1$ are coray modules in the generalized multicoil containing the (ad 3)-pivot X. Again, from [46, Lemma 3.3] we know that, for a fixed $\mathbf{e} \in K_0(A)$, each ray and coray in Γ contains at most one module Z with $\mathbf{dim}\, Z = \mathbf{e}$. Therefore, we get that the number of new indecomposable modules with the same dimension vector is at most $t+1$ while after applying r consecutive operations of types (ad 1*) and (ad 3) or r consecutive operations of types (fad 1*) and (ad 3) the number of new vertices in ordinary quiver of A increases by $t+2$. If the mth admissible operation is of type (ad 3*), then the proof is dual. If the mth admissible operation is of type (ad 4), then A is the algebra obtained from another one, say A', by applying this admissible operation with pivot X and the begin Y_1 of a finite sectional path $Y_1 \to Y_2 \to \cdots \to Y_t$. Note that this finite sectional path is the linearly oriented quiver of type $\mathbb{A}_t$ and its support algebra Λ (given by the vertices corresponding to the simple composition factors of the modules $Y_1, Y_2, \ldots, Y_t$) is a tilted algebra of the path algebra D of the linearly oriented quiver of type $\mathbb{A}_t$. From [65, (4.4)(2)] we know that Λ is a bound quiver algebra given by a truncated branch in x, where x corresponds to the unique projective-injective D-module. Moreover, the modules $Y_1, Y_2, \ldots, Y_t$ are directing in $\Gamma_{A'}$. Again, from [46, Lemma 3.3] we know that, for a fixed $\mathbf{e} \in K_0(A)$, each ray and coray in Γ contains at most one module Z with $\mathbf{dim}\, Z = \mathbf{e}$. Therefore, we get that the number of new indecomposable modules with the same dimension vector is at most $t+r+1$ while after applying operations which give rise to the finite sectional path $Y_1 \to Y_2 \to \cdots \to Y_t$ and (ad 4) the number of new vertices in ordinary quiver of A increases by $t+r+2$. If the mth admissible operation is of type (ad 4*), then the proof is dual. There remains to consider the case where the mth admissible operation is of type (ad 5). Since in the definition of admissible operation (ad 5) we use the finite versions (fad 1), (fad 2), (fad 3), (fad 4) of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4) and the admissible operation (ad 4), we conclude that the lemma follows from the above considerations. If it is of type (ad 5*), then the proof is dual and this finishes the proof of the lemma. □

The following proposition is essential for the proof of Theorem 8.3.

Proposition 8.2. *Let A be a tame generalized multicoil algebra, n the rank of $K_0(A)$, and* **d** *a nonnegative vector of $K_0(A)$. Then*

(i) *The number of isomorphism classes of discrete indecomposable A-modules X with* $\mathbf{dim}\, X = \mathbf{d}$ *is bounded by* $n + 2$.

(ii) *The number of isomorphism classes of indecomposable A-modules X with* $\mathbf{dim}\, X = \mathbf{d}$ *and $q_A(\mathbf{dim}\, X) \neq 0$ is bounded by $n - 1$.*

(iii) *The number of isomorphism classes of indecomposable A-modules X with* $\mathbf{dim}\, X = \mathbf{d}$ *and $\chi_A(\mathbf{dim}\, X) \neq 0$ is bounded by $n - 1$.*

Proof. It is known that an indecomposable A-module M in Γ_A which does not lie on oriented cycle in Γ_A is uniquely determined by $[M]$. Moreover, if $\mathscr{T}$ is a stable tube in Γ_A then the support of $\mathscr{T}$ is a tame concealed or tubular convex subcategory of A. Hence, for any indecomposable A-module X lying in a stable tubes of rank one, we have $q_A(\mathbf{dim}\, X) = \chi_A(\mathbf{dim}\, X) = 0$. Let **d** be a nonnegative vector in $K_0(A)$ such that there exists a nondirecting, discrete indecomposable A-module X with $\mathbf{dim}\, X = \mathbf{d}$. Then X belongs to a generalized multicoil Γ of Γ_A. Assume first that $\mathrm{Hom}_A(P, X) \neq 0$ for some indecomposable projective module in Γ. Then it follows from the proof of [46, Proposition 3.5] that any indecomposable A-module Y with $\mathbf{dim}\, Y = \mathbf{d}$ also lies in Γ. Applying now Lemma 8.1 we conclude that the number of isomorphism classes of indecomposable A-modules Z with $\mathbf{dim}\, Z = \mathbf{dim}\, X = \mathbf{d}$ is bounded by $n - 1$. We get the same statement in the case when $\mathrm{Hom}_A(X, I) \neq 0$ for an indecomposable injective module I in Γ. Note that different tame concealed algebras and different tubular algebras give modules in A with different dimension vectors. Hence, it remains to consider the case when the support of X is contained in a convex subcategory, say C, which is tame concealed or tubular. Then X belongs to a $\mathbb{P}_1(K)$-family $\mathscr{T} = (\mathscr{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of standard stable tubes of Γ_C. Moreover, if Z is a indecomposable A-module with $\mathbf{dim}\, Z = \mathbf{dim}\, X = \mathbf{d}$ then Z is a C-module and lies in one of the tubes $\mathscr{T}_\lambda$ (see [65] or [75]). Denote by m the rank of $K_0(C)$, and by r_λ the rank of the tube $\mathscr{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$. Then the following equality holds

$$\sum_{\lambda \in \mathbb{P}_1(K)} (r_\lambda - 1) = m - 2$$

(see [75]). Further, if $Y \in \mathscr{T}_\lambda$ and $Z \in \mathscr{T}_\mu$ are two nonisomorphic modules in $\mathscr{T}$ with $\mathbf{dim}\, Y = \mathbf{dim}\, Z$ then the quasi-length of Y is divisible by r_λ and the quasi-length of Z is divisible by r_μ. We note that then $q_A(\mathbf{dim}\, Y) = q_C(\mathbf{dim}\, Y) = \chi_A(\mathbf{dim}\, Y) = 0$ and $q_A(\mathbf{dim}\, Z) = q_C(\mathbf{dim}\, Z) = \chi_A(\mathbf{dim}\, Z) = 0$, since gl dim $C \leq 2$. Now a simple inspection of tubular types of tame concealed and tubular algebras shows that, if $\lambda_1, \dots, \lambda_t$ are all indices $\lambda \in \mathbb{P}_1(K)$ with $r_\lambda \neq 1$, then $r_{\lambda_1} + \dots + r_{\lambda_t} \leq m + 2 \leq n + 2$. Therefore, the number of isomorphism classes of indecomposable A-modules Z with $\mathbf{dim}\, Z = \mathbf{dim}\, X = \mathbf{d}$ is bounded by $n + 2$. $\qquad\square$

Theorem 8.3. *Let A be a cycle-finite algebra. Then there is a positive integer m such that, for each nonnegative vector $\mathbf{d} \in K_0(A)$, the number of isomorphism classes of discrete indecomposable A-modules of dimension vector* **d** *is bounded by m.*

Moreover, if A is coherent, n is the rank of $K_0(A)$, then the following statements hold:

(i) *The number of isomorphism classes of discrete indecomposable A-modules X with $\mathbf{dim}\,X = \mathbf{d}$ is bounded by $n + 2$.*

(ii) *The number of isomorphism classes of indecomposable A-modules X with $\mathbf{dim}\,X = \mathbf{d}$ and $q_A(\mathbf{dim}\,X) \neq 0$ is bounded by $n - 1$.*

(iii) *The number of isomorphism classes of indecomposable A-modules X with $\mathbf{dim}\,X = \mathbf{d}$ and $\chi_A(\mathbf{dim}\,X) \neq 0$ is bounded by $n - 1$.*

Proof. It follows from a result due to Ringel [65, (2.4)(8)] that, if X, Y are modules in ind$A$ with $\mathbf{dim}\,X = \mathbf{dim}\,Y$ and X is directing, then X and Y are isomorphic. Further, by Theorem 7.3, there exist tame generalized multicoil quotient algebras $B_1, \ldots, B_q$ of A such that all but finitely many isomorphism classes of nondirecting modules in ind$A$ belong to generalized multicoils of the Auslander-Reiten quivers $\Gamma_{B_1}, \ldots, \Gamma_{B_q}$ of $B_1, \ldots, B_q$. Moreover, it follows from the proof of Theorem 6.3 that, if X and Y are nondirecting discrete modules in ind$A$ with $\mathbf{dim}\,X = \mathbf{dim}\,Y$ lying in generalized multicoils of $\Gamma_{B_1}, \ldots, \Gamma_{B_q}$, then X and Y belong to the same generalized multicoil of Γ_{B_p}, for a fixed $p \in \{1, \ldots, q\}$. Then there is a positive integer m such that, for each nonnegative vector $\mathbf{d} \in K_0(A)$, the number of isomorphism classes of discrete indecomposable A-modules of dimension $\mathbf{d}$ is bounded by m. Finally, if A is coherent, then all nondirecting modules in ind$A$ lie in generalized multicoils of $\Gamma_{B_1}, \ldots, \Gamma_{B_q}$, and the statements (i), (ii), (iii) are consequences of Proposition 8.2.

<div align="right">□</div>

9 Homological properties of indecomposable modules

Let A be an algebra and X be a nonprojective module in ind$A$. Then we have an almost split sequence

$$0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$$

in mod$A$ with $Y_1, \ldots, Y_{s(X)}$ indecomposable modules. We note that $\tau_A = D\mathrm{Tr}$, where $D = \mathrm{Hom}_K(-, K)$ is the standard duality on mod$A$ and the transpose $\mathrm{Tr}\,X$ of a module X in ind$A$ is the cokernel of the homomorphism $\mathrm{Hom}_A(f, A)$ in modA^{op} associated to a minimal projective presentation $P_1 \xrightarrow{f} P_0 \to X \to 0$ of X in mod$A$. Hence τ_A is a homological operator, and $s(X)$ is a numerical homological invariant of a nonprojective module X in ind$A$.

We also recall that for an indecomposable nonsimple projective-injective module P in mod$A$ there is a canonical (up to isomorphism) almost split sequence having P as a middle term, namely

$$0 \to \mathrm{rad}\,P \to P \oplus \mathrm{rad}\,P/\mathrm{soc}\,P \to P/\mathrm{soc}\,P \to 0$$

(see [1, Proposition IV.3.11]).

The following theorem has been proved by Bautista and Brenner in [12] (see also [44]).

Theorem 9.1. *Let A be an algebra of finite representation type and X a nonprojective module in* ind*A. Then*

(i) $s(X) \leq 4$.
(ii) *If $s(X) = 4$, then one of the indecomposable middle terms Y_i of an almost split sequence for X is projective-injective.*

It has been conjectured by Brenner that, for any tame algebra A and every non-projective module X in ind$A$, we have $s(X) \leq 5$. It is still an open problem. The following theorem proved in [60, Theorem 3] by the second named author and Takane confirms the Brenners's conjecture, and extends Theorem 9.1, to cycle-finite algebras.

Theorem 9.2. *Let A be a cycle-finite algebra and X a nonprojective module in* ind*A. Then*

(i) $s(X) \leq 5$.
(ii) *If $s(X) = 5$, then one of the indecomposable middle terms Y_i of an almost split sequence for X is projective-injective.*

It has been proved by Ringel [65, (2.4)(8)] that, for any directing indecomposable module X over an algebra A, we have $\mathrm{End}_A(X) \cong K$ and $\mathrm{Ext}_A^r(X,X) = 0$ for $r \geq 0$. On the other hand, by [52, Theorem B], for a tame generalized multicoil algebra A and an arbitrary module X in ind$A$, we have $\dim_K \mathrm{Ext}_A^1(X,X) \leq \dim_K \mathrm{End}_A(X)$ and $\mathrm{Ext}_A^r(X,X) = 0$ for $r \geq 2$. Hence, applying Theorems 6.3 and 6.5 (and Corollary 6.6) we obtain the following theorem.

Theorem 9.3. *Let A be a cycle-finite algebra. Then for all but finitely many isomorphism classes of module X in* ind*A we have* $\dim_K \mathrm{Ext}_A^1(X,X) \leq \dim_K \mathrm{End}_A(X)$ *and* $\mathrm{Ext}_A^r(X,X) = 0$ *for $r \geq 2$.*

We conclude from the above theorem that, for a cycle-finite algebra A and all but finitely many isomorphism classes X in ind$A$, the Euler form

$$\chi_A([X]) = \sum_{r=0}^{\infty} (-1)^r \dim_K \mathrm{Ext}_A^r(X,X)$$

of X is defined and is nonnegative. For A a coherent cycle-finite algebra, it is the case for all modules X in ind$A$.

It is known from [32, Theorem 2.3] and [63, Theorem 3.4] that if A is a quasitilted algebra or a generalized double tilted algebra then, for all but finitely many isomorphism classes of modules X in ind$A$, we have $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$. Moreover, it has been conjectured in [79] that the converse implication also holds.

We end this section by the following recent result by the third named author and Skowyrski [81] which confirms this conjecture for cycle-finite algebras.

Theorem 9.4. *Let A be a cycle-finite algebra such that* $\mathrm{pd}_A X \leq 1$ *or* $\mathrm{id}_A X \leq 1$ *for all but finitely many isomorphism classes of modules in* $\mathrm{ind}A$*. Then A is a tame quasitilted algebra or a tame generalized double tilted algebra.*

10 Geometric properties of indecomposable modules

The aim of this section is to present some results describing geometric properties of indecomposable modules over cycle-finite algebras.

Let A be an algebra and $A \cong KQ/I$ its bound presentation. Then I is an admissible ideal in the path algebra KQ of Q generated by a finite system of forms $\sum_{1 \leq j \leq t} \lambda_j \alpha_{m_j,j} \dots \alpha_{1,j}$ (called K-linear relations), where $\lambda_1, \dots, \lambda_t$ are elements of K and $\alpha_{m_j,j}, \dots, \alpha_{1,j}$, $1 \leq j \leq t$, are paths of length ≥ 2 in Q having a common source and a common end. Denote by Q_0 the set of vertices of Q, by Q_1 the set of arrows of Q, and by $s, e : Q_1 \to Q_0$ the maps which assign to each arrow α_1 its source $s(\alpha)$ and its end $e(\alpha)$. The category $\mathrm{mod}A$ of finite dimensional right A-modules is equivalent to the category $\mathrm{rep}_K(Q,I)$ of all finite dimensional representations $V = (V_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q, where V_i, $i \in Q_0$, are finite dimensional K-vector spaces and $\varphi_\alpha : V_{s(\alpha)} \to V_{e(\alpha)}$, $\alpha \in Q_1$, are K-linear maps satisfying the equations $\sum_{1 \leq j \leq t} \lambda_j \varphi_{\alpha_{m_j,j}} \dots \varphi_{\alpha_{1,j}} = 0$ for all K-linear relations $\sum_{1 \leq j \leq t} \lambda_j \alpha_{m_j,j} \dots \alpha_{1,j} \in I$ (see [1]). Fix now a vector $\mathbf{d} = (d_i)_{i \in Q_0} \in K_0(A) = \mathbb{Z}^{Q_0}$ with nonnegative coordinates. Denote by $\mathrm{mod}_A(\mathbf{d})$ the set of all representations $V = (V_i, \varphi_\alpha)$ in $\mathrm{rep}_K(Q,I)$ with $V_i = K^{d_i}$ for all $i \in Q_0$. A representation V in $\mathrm{mod}_A(\mathbf{d})$ is given by $d_{e(\alpha)} \times d_{s(\alpha)}$-matrices $V(\alpha)$ determining the maps $\varphi_\alpha : K^{s(\alpha)} \to K^{e(\alpha)}$, $\alpha \in Q_1$, in the canonical bases of K^{d_i}, $i \in Q_0$. Moreover, the matrices $V(\alpha)$, $\alpha \in Q_1$, satisfy the relations

$$\sum_{1 \leq j \leq t} \lambda_j V(\alpha_{m_j,j}) \dots V(\alpha_{1,j}) = 0$$

for all K-linear relations $\sum_{1 \leq j \leq t} \lambda_j \alpha_{m_j,j} \dots \alpha_{1,j} \in I$. Therefore, $\mathrm{mod}_A(\mathbf{d})$ is a closed subset of $\mathbb{A}(\mathbf{d}) = \prod_{\alpha \in Q_1} K^{d_{e(\alpha)} \times d_{s(\alpha)}}$ in the Zariski topology, and so $\mathrm{mod}_A(\mathbf{d})$ is an affine variety. We note that $\mathrm{mod}_A(\mathbf{d})$ is not necessarily irreducible. The affine (reductive) algebraic group $G(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(K)$ acts on the variety $\mathrm{mod}_A(\mathbf{d})$ by conjugation

$$(gV)(\alpha) = g_{e(\alpha)} V(\alpha) g_{s(\alpha)}^{-1}$$

for $g = (g_i) \in G(\mathbf{d})$, $V \in \mathrm{mod}_A(\mathbf{d})$, $\alpha \in Q_1$. We shall identify an A-module V of dimension vector $\mathbf{d}$ with the corresponding point of the variety $\mathrm{mod}_A(\mathbf{d})$. The $G(\mathbf{d})$-orbit $G(\mathbf{d})M$ of a module M in $\mathrm{mod}_A(\mathbf{d})$ will be denoted by $\mathscr{O}(M)$. Observe that two modules M and N in $\mathrm{mod}_A(\mathbf{d})$ are isomorphic if an only if $\mathscr{O}(M) = \mathscr{O}(N)$. For $M, N \in \underline{\mathrm{mod}_A}(\mathbf{d})$, we say that N is a *degeneration* of M if N belongs to the Zariski closure $\overline{\mathscr{O}(M)}$ of $\mathscr{O}(M)$ in $\mathrm{mod}_A(\mathbf{d})$, and we denote this fact by $M \leq_{\mathrm{deg}} N$. We note that $\leq_{\mathrm{deg}}$ is a partial order in $\mathrm{mod}_A(\mathbf{d})$. If $N \in \overline{\mathscr{O}(M)}$ implies $\mathscr{O}(N) = \mathscr{O}(M)$, then the orbit $\mathscr{O}(N)$ is said to be *maximal*. Clearly, an orbit in $\mathrm{mod}_A(\mathbf{d})$ of maximal

dimension is maximal, but the converse is not true in general. It is known that the union of all $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$ of maximal dimension is an open subset of $\mathrm{mod}_A(\mathbf{d})$, called an *open sheet* (see [38], [39]). We note also that for a module M in $\mathrm{mod}_A(\mathbf{d})$, we have $\dim \mathcal{O}(M) = \dim G(\mathbf{d}) - \dim_K \mathrm{End}_A(M)$ (see [38]). Given a module $M \in \mathrm{mod}_A(\mathbf{d})$ we denote by $T_M(\mathrm{mod}_A(\mathbf{d}))$ the tangent space of $\mathrm{mod}_A(\mathbf{d})$ at M and by $T_M(\mathcal{O}(M))$ the tangent space to $\mathcal{O}(M)$ at M. Then there is a canonical monomorphism of K-vector spaces

$$T_M(\mathrm{mod}_A(\mathbf{d}))/T_M(\mathcal{O}(M)) \hookrightarrow \mathrm{Ext}^1_A(M,M)$$

(see [39, (2.7)]). In particular, if $\mathrm{Ext}^1_A(M,M) = 0$ then $\overline{\mathcal{O}(M)}$ is an irreducible component of $\mathrm{mod}_A(\mathbf{d})$ and $\mathcal{O}(M)$ is an open sheet of $\mathrm{mod}_A(\mathbf{d})$. The *local dimension* $\dim_M \mathrm{mod}_A(\mathbf{d})$ of $\mathrm{mod}_A(\mathbf{d})$ is the maximal dimension of the irreducible components of $\mathrm{mod}_A(\mathbf{d})$ containing M. We have $\dim T_M(\mathrm{mod}_A(\mathbf{d})) \geq \dim_M \mathrm{mod}_A(\mathbf{d})$. Further, $M \in \mathrm{mod}_A(\mathbf{d})$ is said to be a *nonsingular point* of $\mathrm{mod}_A(\mathbf{d})$ if $\dim_M \mathrm{mod}_A(\mathbf{d}) = \dim T_M(\mathrm{mod}_A(\mathbf{d}))$. If M is a nonsingular point of $\mathrm{mod}_A(\mathbf{d})$ then M belongs to exactly one irreducible component of $\mathrm{mod}_A(\mathbf{d})$ [66, (II.2.6)]. The nonsingular points of $\mathrm{mod}_A(\mathbf{d})$ form an open nonempty subset. It is known that a module M in $\mathrm{mod}_A(\mathbf{d})$ is nonsingular provided $\mathrm{Ext}^2_A(M,M) = 0$. A module variety $\mathrm{mod}_A(\mathbf{d})$ is said to be a *complete intersection* provided the vanishing ideal of $\mathrm{mod}_A(\mathbf{d})$ in the coordinate ring $K[\mathbb{A}(\mathbf{d})]$ of the affine space $\mathbb{A}(\mathbf{d}) = \prod_{\alpha \in Q_1} K^{d_{e(\alpha)} \times d_{s(\alpha)}}$ is generated by $\dim \mathbb{A}(\mathbf{d}) - \dim \mathrm{mod}_A(\mathbf{d})$ polynomials. Finally, a module variety $\mathrm{mod}_A(\mathbf{d})$ is said to be *normal* if the local ring $\mathcal{O}_M$ of any module M in $\mathrm{mod}_A(\mathbf{d})$ is integrally closed in its total quotient ring. It is known that if $\mathrm{mod}_A(\mathbf{d})$ is normal then it is nonsingular in codimension one, that is, the set of singular points in $\mathrm{mod}_A(\mathbf{d})$ is of codimension at most two (see [31, Chapter 11]). If $\mathrm{mod}_A(\mathbf{d})$ is a complete intersection, then $\mathrm{mod}_A(\mathbf{d})$ is normal if and only if $\mathrm{mod}_A(\mathbf{d})$ is nonsingular in codimension one (consequence of Serre's normality criterion). In the study of the degeneration order on a module variety $\mathrm{mod}_A(\mathbf{d})$ an important role is played by the following related partial orders. Let M and N be modules in a module variety. We define:

- $M \leq_{\mathrm{ext}} N$: $\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\mathrm{mod} A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.
- $M \leq_R N$: $\Leftrightarrow$ there exists in $\mathrm{mod} A$ an exact sequence of the form $0 \to N \to M \oplus Z \to Z \to 0$.
- $M \leq N$: $\Leftrightarrow$ $\dim_K \mathrm{Hom}_A(M,X) \leq \dim_K \mathrm{Hom}_A(N,X)$ for all modules X in $\mathrm{mod} A$.

It follows from the result due to Auslander [9] that $\leq$ is a partial order on the isomorphism classes of modules with the same dimension vector. Further, for modules M and N in $\mathrm{mod}_A(\mathbf{d})$, we have $M \leq N$ if and only if $\dim_K \mathrm{Hom}_A(X,M) \leq \dim_K \mathrm{Hom}_A(X,N)$ for all modules X in $\mathrm{mod} A$ by a result of Auslander and Reiten [10]. Moreover, by a result of Zwara [88], we have $M \leq_R N$ if and only if there exists in $\mathrm{mod} A$ a short exact sequence of the form $0 \to Z' \to Z' \oplus M \to N \to 0$.

The following fundamental result of Zwara from [90] (see also [64] for the sufficiency part) gives an algebraic characterization of degenerations of modules.

Theorem 10.1. *Let A be an algebra, $\mathbf{d}$ a vector in $K_0(A)$ with nonnegative coordinates, and M, N modules in $\mathrm{mod}_A(\mathbf{d})$. Then $M \leq_{\mathrm{deg}} N$ if and only if $M \leq_R N$.*

In general, we have the following relations between the introduced orders. For modules M and N in the module variety $\mathrm{mod}_A(\mathbf{d})$ the following implications hold

$$M \leq_{\mathrm{ext}} N \Longrightarrow M \leq_{\mathrm{deg}} N \Longrightarrow M \leq N.$$

Unfortunately, the reverse implications are not true in general, and it would be interesting to find out when there are true.

The following result of Zwara from [89] gives a combinatorial description of degenerations for modules over algebras of finite representation type.

Theorem 10.2. *Let A be an algebra of finite representation type, $\mathbf{d}$ a vector in $K_0(A)$ with nonnegative coordinates, and M, N modules in $\mathrm{mod}_A(\mathbf{d})$. Then $M \leq_{\mathrm{deg}} N$ if and only if $M \leq N$.*

We also exhibit the following results from [85] and [86] on degenerations of modules from the additive categories of generalized standard Auslander-Reiten components.

Theorem 10.3. *Let A be an algebra, $\mathscr{C}$ a generalized standard quasi-tube of Γ_A, and M, N modules in $\mathrm{add}(\mathscr{C})$. Then $M \leq_{\mathrm{deg}} N$ if and only if $M \leq_{\mathrm{ext}} N$.*

Theorem 10.4. *Let A be an algebra, $\mathscr{C}$ a generalized standard component of Γ_A, N a module in $\mathrm{add}(\mathscr{C})$, and M a module in $\mathrm{mod} A$. If $M \leq_{\mathrm{deg}} N$ then M belongs to $\mathrm{add}(\mathscr{C})$.*

Theorem 10.5. *Let A be an algebra, $\mathscr{C}$ a generalized standard component of Γ_A, M,N modules in $\mathrm{add}(\mathscr{C})$ with $\mathbf{dim} M = \mathbf{dim} N$. The following conditions are equivalent.*

(i) $M \leq_{\mathrm{deg}} N$.
(ii) *There exists an exact sequence $0 \to N \to M \oplus Z \to Z \to 0$ in $\mathrm{mod} A$ with Z from $\mathrm{add}(\mathscr{C})$.*
(iii) *There exists an exact sequence $0 \to Z' \to Z' \oplus M \to N \to 0$ in $\mathrm{mod} A$ with Z' from $\mathrm{add}(\mathscr{C})$.*
(iv) $\dim_K \mathrm{Hom}_A(M,X) \leq \dim_K \mathrm{Hom}_A(N,X)$ *for all modules X in $\mathscr{C}$.*
(v) $\dim_K \mathrm{Hom}_A(X,M) \leq \dim_K \mathrm{Hom}_A(X,N)$ *for all modules X in $\mathscr{C}$.*

Let A be an algebra and M,N be nonisomorphic modules in $\mathrm{ind} A$ with $\mathbf{dim} M = \mathbf{dim} N$. Then $M \leq N$ forces the inequalities $\dim_K \mathrm{Hom}_A(M,M) \leq \dim_K \mathrm{Hom}_A(N,M)$ and $\dim_K \mathrm{Hom}_A(M,M) \leq \dim_K \mathrm{Hom}_A(M,N)$, and consequently we have a cycle $M \to N \to M$. Since $M \leq_{\mathrm{deg}} N$ implies $M \leq N$, we conclude that the directing modules in $\mathrm{ind} A$ are never involved in proper degenerations of indecomposable modules. Observe also that, if A is a cycle-finite algebra and $M <_{\mathrm{deg}} N$, then M and N belong to the same cyclic component of Γ_A. The degenerations of modules in the additive categories of generalized multicoils of Auslander-Reiten quivers of algebras were investigated in [47], [48], [83], [85]. Then, using Theorems 6.3 and 6.5, we obtain the following results.

Theorem 10.6. *Let A be a cycle-finite algebra. Then there exists a positive integer t such that for any sequence*

$$M_r <_{\deg} M_{r-1} <_{\deg} \ldots <_{\deg} M_2 <_{\deg} M_1$$

with $M_1, \ldots, M_r$ modules in ind A, *the inequality $r \leq t$ holds.*

Theorem 10.7. *Let A be a coherent cycle-finite algebra and M, M', N be modules in* mod A *such that $M <_{\deg} N$, $M' <_{\deg} N$ and N is indecomposable. Then $M \cong M'$ and is indecomposable.*

The geometry of directing modules over tame algebras has been described in [14]. In particular, we have the following consequence of [14, Theorems 1 and 2].

Theorem 10.8. *Let A be a tame algebra, M a directing module in* ind A, *and* $\mathbf{d} = \mathbf{dim}\,M$. *Then the following statements hold.*

(i) mod$_A(\mathbf{d})$ *is a complete intersection and has at most two irreducible components.*
(ii) *The maximal $G(\mathbf{d})$-orbits in* mod$_A(\mathbf{d})$ *consist of nonsingular modules.*
(iii) $\mathscr{O}(M)$ *is an open sheet of* mod$_A(\mathbf{d})$.
(iv) *All but finite number of $G(\mathbf{d})$-orbits in* mod$_A(\mathbf{d})$ *have codimension at least two.*
(v) *All $G(\mathbf{d})$-orbits of codimension one are contained in $\overline{\mathscr{O}(M)}$.*
(vi) *If M is not projective-injective over* supp M *then* mod$_A(\mathbf{d}) = \mathscr{O}(M)$, *is normal and a complete intersection.*

It follows from Theorems 3.1 and 3.3 that the regular components of the Auslander-Reiten quivers of cycle-finite algebras are stable tubes and their supports are tame concealed or tubular algebras. Then the following result from [16, Theorem 1] describes the geometry of modules from the additive categories of regular components of cycle-finite algebras.

Theorem 10.9. *Let A be a tame concealed or tubular algebra and $\mathbf{d}$ the dimension vector of a module in* mod A *which is periodic with respect to the action of τ_A. Then the affine variety* mod$_A(\mathbf{d})$ *is irreducible, normal, a complete intersection,* $\dim \text{mod}_A(\mathbf{d}) = \dim G(\mathbf{d}) - q_A(\mathbf{d})$ *and the maximal $G(\mathbf{d})$-orbits in* mod$_A(\mathbf{d})$ *form the open sheet consisting of nonsingular points. Moreover, a module M in* mod$_A(\mathbf{d})$ *is a nonsingular point if and only if* $\text{Ext}_A^2(M, M) = 0$.

We refer also to [15] for the geometry of indecomposable modules over tame quasi-tilted algebras, which give information on the geometry of indecomposable nondirecting modules of semiregular tubes of cycle-finite algebras (see Theorems 3.6 and 3.7).

We end this section with the following consequence of [52, Theorems A and B] and Theorem 6.3, extending [57, Theorem A] from strongly simply connected algebras of polynomial growth to coherent cycle-finite algebras.

Theorem 10.10. *Let A be a coherent cycle-finite algebra, M a module in* ind A *and* $\mathbf{d} = \mathbf{dim}\,M$. *Then the following statements hold.*

(i) *M is a nonsingular point of* mod$_A(\mathbf{d})$.

(ii) $q_A(\mathbf{d}) \geq \chi_A(\mathbf{d}) = \dim_K \mathrm{End}_A(M) - \dim_K \mathrm{Ext}^1_A(M,M) \geq 0.$
(iii) $\dim_M \mathrm{mod}_A(\mathbf{d}) = \dim G(\mathbf{d}) - \chi_A(\mathbf{d}).$

We note that there are indecomposable modules M over coherent cycle-finite algebras A with arbitrary large $\chi_A(\mathbf{dim}\, M)$ (see [58, (5.3)]).

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