FINITE CYCLES OF INDECOMPOSABLE MODULES

PIOTR MALICKI, JOSÉ ANTONIO DE LA PEÑA, AND ANDRZEJ SKOWROŃSKI

Dedicated to Raymundo Bautista on the occasion of his 70th birthday

ABSTRACT. We solve a long standing open problem concerning the structure of finite cycles in the category $\operatorname{mod} A$ of finitely generated modules over an arbitrary artin algebra A, that is, the chains of homomorphisms $M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{r-1} \xrightarrow{f_r} M_r = M_0$ between indecomposable modules in $\operatorname{mod} A$ which do not belong to the infinite radical of $\operatorname{mod} A$. In particular, we describe completely the structure of an arbitrary module category $\operatorname{mod} A$ whose all cycles are finite. The main structural results of the paper allow to derive several interesting combinatorial and homological properties of indecomposable modules lying on finite cycles. For example, we prove that for all but finitely many isomorphism classes of indecomposable modules M lying on finite cycles of a module category $\operatorname{mod} A$ the Euler characteristic of M is well defined and nonnegative. Moreover, new types of examples illustrating the main results of the paper are presented.

0. Introduction

Throughout the paper, by an algebra is meant an artin algebra over a fixed commutative artin ring K, which we shall assume (without loss of generality) to be basic and indecomposable. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules. The Jacobson radical rad_A of $\operatorname{mod} A$ is the ideal generated by all nonisomorphisms between modules in $\operatorname{ind} A$, and the infinite radical $\operatorname{rad}_A^\infty$ of $\operatorname{mod} A$ is the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. By a result of Auslander [6], $\operatorname{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, $\operatorname{ind} A$ admits only a finite number of pairwise nonisomorphic modules (see also [32] for an alternative proof of this result). On the other hand, if A is of infinite representation type then $(\operatorname{rad}_A^\infty)^2 \neq 0$, by a result proved in [19].

An important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A is its Auslander-Reiten quiver Γ_A. Recall that Γ_A is a valued translation quiver whose vertices are the isomorphism classes $\{X\}$ of modules X in $\operatorname{ind} A$, the arrows correspond to irreducible homomorphisms between modules in $\operatorname{ind} A$, and the translation is the Auslander-Reiten translation $\tau_A = D\mathrm{Tr}$. We shall not distinguish between a module in X in $\operatorname{ind} A$ and the corresponding vertex $\{X\}$ of Γ_A. If A is an algebra of finite representation type, then every nonzero nonisomorphism in $\operatorname{ind} A$ is a finite sum of composition of irreducible homomorphisms between modules in $\operatorname{ind} A$, and

1991 *Mathematics Subject Classification.* 16G10, 16G60, 16G70.

Key words and phrases. Cycles of modules, Generalized multicoil algebras, Generalized double tilted algebras, Auslander-Reiten quiver.

This work was completed with the support of the research grant DEC-2011/02/A/ST1/00216 of the Polish National Science Center and the CIMAT Guanajuato, México.

hence we may recover $\operatorname{mod} A$ from the translation quiver Γ_A. In general, Γ_A describes only the quotient category $\operatorname{mod} A/\operatorname{rad}_A^\infty$.

Let A be an algebra and M a module in $\operatorname{ind} A$. An important information concerning the structure of M is coded in the structure and properties of its support algebra $\operatorname{Supp}(M)$ defined as follows. Consider a decomposition $A = P_M \oplus Q_M$ of A in $\operatorname{mod} A$ such that the simple summands of the semisimple module $P_M/\operatorname{rad}P_M$ are exactly the simple composition factors of M. Then $\operatorname{Supp}(M) = A/t_A(M)$, where $t_A(M)$ is the ideal in A generated by the images of all homomorphisms from Q_M to A in $\operatorname{mod} A$. We note that M is an indecomposable module over $\operatorname{Supp}(M)$. Clearly, we may realistically hope to describe the structure of $\operatorname{Supp}(M)$ only for modules M having some distinguished properties.

A prominent role in the representation theory of algebras is played by cycles of indecomposable modules (see [42], [47], [60], [69]). Recall that a *cycle* in $\operatorname{ind} A$ is a sequence

$$M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{r-1} \xrightarrow{f_r} M_r = M_0$$

of nonzero nonisomorphisms in $\operatorname{ind} A$ [60], and such a cycle is said to be *finite* if the homomorphisms $f_1, \dots, f_r$ do not belong to $\operatorname{rad}_A^\infty$ (see [3], [4]). Following Ringel [60], a module M in $\operatorname{ind} A$ which does not lie on a cycle in $\operatorname{ind} A$ is called *directing*. The following two important results on directing modules were established by Ringel in [60]. Firstly, if A is an algebra with all modules in $\operatorname{ind} A$ being directing, then A is of finite representation type. Secondly, the support algebra $\operatorname{Supp}(M)$ of a directing module M over an algebra A is a tilted algebra $\operatorname{End}_H(T)$, for a hereditary algebra H and a tilting module T in $\operatorname{mod} H$, and M is isomorphic to the image $\operatorname{Hom}_H(T, I)$ of an indecomposable injective module I in $\operatorname{mod} H$ via the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} \operatorname{End}_H(T)$. In particular, it follows that, if A is an algebra of infinite representation type, then $\operatorname{ind} A$ always contains a cycle. Moreover, it has been proved independently by Peng and Xiao [49] and Skowroński [67] that the Auslander-Reiten quiver Γ_A of an algebra A admits at most finitely many τ_A-orbits containing directing modules. Hence, in order to obtain information on the support algebras $\operatorname{Supp}(M)$ of nondirecting modules in $\operatorname{ind} A$, it is natural to study properties of cycles in $\operatorname{ind} A$ containing M. A module M in $\operatorname{ind} A$ is said to be *cycle-finite* if M is nondirecting and every cycle in $\operatorname{ind} A$ passing through M is finite. Obviously, every indecomposable module over an algebra of finite representation type is cycle-finite. Examples of cycle-finite indecomposable modules over algebras of infinite representation type are provided by all indecomposable modules in the stable tubes of tame hereditary algebras [24], canonical algebras [60], [61], or more generally concealed canonical algebras [35]. Following Assem and Skowroński [3], [4], an algebra A is said to be *cycle-finite* if all cycles in $\operatorname{ind} A$ are finite. The class of cycle-finite algebras is wide and contains the following distinguished classes of algebras: the algebras of finite representation type, the tame tilted algebras [27], [31], [60], the tame double tilted algebras [57], the tame generalized double tilted algebras [58], the tubular algebras [60], [61], the iterated tubular algebras [55], the tame quasi-tilted algebras [36], [73], the tame generalized multicoil algebras [45], the algebras with cycle-finite derived categories [3], and the strongly simply connected algebras of polynomial growth [71]. We also mention that a selfinjective algebra A is cycle-finite if and only if A is of finite representation type [30]. On the other hand, frequently an algebra A admits a Galois covering $R \to R/G = A$, where R is a cycle-finite locally bounded category and G is an admissible group of automorphisms of R, which allows to reduce the representation theory of A to the representation theory of cycle-finite algebras being finite convex subcategories of R (see [53] and [72] for some general results). For example, every finite dimensional selfinjective algebra A of polynomial growth over an algebraically closed field K admits a canonical standard form $\overline{A}$ (geometric socle deformation of A) such that $\overline{A}$ has a Galois covering $R \to R/G = \overline{A}$, where R is a cycle-finite selfinjective locally bounded category and G is an admissible infinite cyclic group of automorphisms of R, the Auslander-Reiten quiver $\Gamma_{\overline{A}}$ of $\overline{A}$ is the orbit quiver Γ_R/G of Γ_R, and the stable Auslander-Reiten quivers of A and $\overline{A}$ are isomorphic (see [64], [77]). We refer to [12], [42], [70] for some general results on the structure of cycle-finite algebras and their module categories.

In the paper we are concerned with the problem of describing the support algebras of cycle-finite modules over arbitrary (artin) algebras. We note that this may be considered as a natural extension of the problem concerning the structure of support algebras of directing modules, solved by Ringel in [60]. Namely, the directing modules in $\operatorname{ind} A$ may be viewed as modules M in $\operatorname{ind} A$ for which every oriented cycle of nonzero homomorphisms in $\operatorname{ind} A$ containing M consists entirely of isomorphisms. The considered problem, initiated more than 25 years ago in [3], turned out to be very difficult, and many researchers involved to its solution resigned. The main obstacle for solution of this problem was the large complexity of finite cycles of indecomposable modules and the fact that all cycles of indecomposable modules over algebras of finite representation type are finite. The main results of the paper show that new classes of algebras and complete understanding of the structure of their module categories were necessary for the solution of the considered problem. We will outline now our approach towards solution of the problem.

Let A be an algebra and M be a cycle-finite module in $\operatorname{ind} A$. Then every cycle in $\operatorname{ind} A$ passing through M has a refinement to a cycle of irreducible homomorphisms in $\operatorname{ind} A$ containing M and consequently M lies on an oriented cycle in the Auslander-Reiten quiver Γ_A of A. Following Malicki and Skowroński [44], we denote by ${}_c\Gamma_A$ the *cyclic quiver* of A obtained from Γ_A by removing all acyclic vertices (vertices not lying on oriented cycles in Γ_A) and the arrows attached to them. Then the connected components of the translation quiver ${}_c\Gamma_A$ are said to be *cyclic components* of Γ_A. It has been proved in [44] that two modules X and Y in $\operatorname{ind} A$ belong to the same cyclic component of Γ_A if and only if there is an oriented cycle in Γ_A passing through X and Y. For a cyclic component Γ of ${}_c\Gamma_A$, we consider a decomposition $A = P_\Gamma \oplus Q_\Gamma$ of A in $\operatorname{mod} A$ such that the simple summands of the semisimple module $P_\Gamma/\operatorname{rad} P_\Gamma$ are exactly the simple composition factors of indecomposable modules in Γ, the ideal $t_A(\Gamma)$ in A generated by the images of all homomorphisms from Q_Γ to A in $\operatorname{mod} A$, and call the quotient algebra $\operatorname{Supp}(\Gamma) = A/t_A(\Gamma)$ the *support algebra* of Γ. Observe now that M belongs to a unique cyclic component $\Gamma(M)$ of Γ_A consisting entirely of cycle-finite indecomposable modules, and the support algebra $\operatorname{Supp}(M)$ of M is a quotient algebra of the support algebra $\operatorname{Supp}(\Gamma(M))$ of $\Gamma(M)$. A cyclic component Γ of Γ_A containing a cycle-finite module is said to be a cycle-finite cyclic component of Γ_A. We will prove that the support algebra $\operatorname{Supp}(\Gamma)$ of a cycle-finite cyclic component Γ of Γ_A is isomorphic to an algebra of the form $e_\Gamma A e_\Gamma$ for an idempotent e_Γ of A whose primitive summands correspond to the vertices of a convex subquiver of the valued quiver Q_A of A. On the other hand, the support algebra $\operatorname{Supp}(M)$ of a cycle-finite module M in $\operatorname{ind} A$ is not necessarily an algebra of the form eAe for an idempotent e of A (see Section 6).

The main results of the paper provide a conceptual description of the support algebras of cycle-finite cyclic components of Γ_A. The description splits into two cases. In the case when a cycle-finite cyclic component Γ of Γ_A is infinite, we prove that $\operatorname{Supp}(\Gamma)$ is a suitable gluing of finitely many generalized multicoil algebras (introduced by Malicki and Skowroński in [45]) and algebras of finite representation type, and Γ is the corresponding gluing of the associated cyclic generalized multicoils via finite translation quivers. In the second case when a cycle-finite cyclic component Γ is finite, we prove that $\operatorname{Supp}(\Gamma)$ is a generalized double tilted algebra (in the sense of Reiten and Skowroński [58]) and Γ is the core of the connecting component of this algebra.

We would like to mention that the generalized multicoil algebras form a prominent class of algebras of global dimension at most 3, containing the class of quasitilted algebras of canonical type, and are obtained by sophisticated gluings of concealed canonical algebras using admissible algebra operations, generalizing the coil operations proposed by Assem and Skowroński in [4]. The generalized double tilted algebras form a distinguished class of algebras, containing all tilted algebras and all algebras of finite representation type, and can be viewed as two-sided gluings of tilted algebras. The tilted algebras and quasitilted algebras of canonical type were under intensive investigation over the last two decades by many representation theory algebraists. Hence, the main results of the paper give a good understanding of the support algebras of cycle-finite cyclic components. On the other hand, the results and examples presented in the paper create new interesting open problems and research directions (see Section 1).

The paper is organized as follows. In Section 1 we present the main results of the paper and related background. In Section 2 we describe properties of cyclic components of the Auslander-Reiten quivers of algebras, applied in the proofs of the main theorems. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively. In Sections 5 and 6 we present new types of examples, illustrating the main results of the paper.

For basic background on the representation theory applied here we refer to [2], [9], [60], [62], [63], [79].

The main results of the paper have been proved during the visit of P. Malicki and A. Skowroński at the Centro de Investigación en Mathemáticas (CIMAT) in Guanajuato (November 2012), who would like to thank J. A. de la Peña and CIMAT for the warm hospitality and wonderful conditions for the successful realization of this joint research project. The results were presented by the first named author during the conferences

"Advances in Representation Theory of Algebras" (Guanajuato, December 2012) and "Perspectives of Representation Theory of Algebras" (Nagoya, November 2013).

1. Main results and related background

In order to formulate the main results of the paper we need special types of components of the Auslander-Reiten quivers of algebras and distinguished classes of algebras with separating families of Auslander-Reiten components.

Let A be an algebra. For a subquiver Γ of Γ_A, we denote by $\mathrm{ann}_A(\Gamma)$ the intersection of the annihilators $\mathrm{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all indecomposable modules X in Γ, and call the quotient algebra $B(\Gamma) = A/\mathrm{ann}_A(\Gamma)$ the *faithful algebra* of Γ. By a component of Γ_A we mean a connected component of the translation quiver Γ_A. A component $\mathcal{C}$ of Γ_A is called *regular* if $\mathcal{C}$ contains neither a projective module nor an injective module, and *semiregular* if $\mathcal{C}$ does not contain both a projective and an injective module. It has been shown in [37] and [82] that a regular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a *stable tube* (is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for a positive integer r). Moreover, Liu proved in [38] that a semiregular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a *ray tube* (obtained from a stable tube by a finite number (possibly zero) of ray insertions) or a *coray tube* (obtained from a stable tube by a finite number (possibly zero) of coray insertions). A component $\mathcal{C}$ of Γ_A is said to be *coherent* [44] (see also [23]) if the following two conditions are satisfied:

(C1) For each projective module P in $\mathcal{C}$ there is an infinite sectional path

$$P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots .$$

(C2) For each injective module I in $\mathcal{C}$ there is an infinite sectional path

$$\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I.$$

Further, a component $\mathcal{C}$ of Γ_A is said to be *almost cyclic* if its cyclic part ${}_c\mathcal{C}$ is a cofinite subquiver of $\mathcal{C}$. We note that the stable tubes, ray tubes and coray tubes of Γ_A are special types of almost cyclic coherent components. In general, it has been proved by Malicki and Skowroński in [44] that a component $\mathcal{C}$ of Γ_A is almost cyclic and coherent if and only if $\mathcal{C}$ is a *generalized multicoil*, obtained from a finite family of stable tubes by a sequence of admissible operations (ad 1)-(ad 5) and their duals (ad 1^*)-(ad 5^*). On the other hand, a component $\mathcal{C}$ of Γ_A is said to be *almost acyclic* if all but finitely many modules of $\mathcal{C}$ are acyclic. It has been proved by Reiten and Skowroński in [58] that a component $\mathcal{C}$ of Γ_A is almost acyclic if and only if $\mathcal{C}$ admits a multisection Δ. Moreover, for an almost acyclic component $\mathcal{C}$ of Γ_A, there exists a finite convex subquiver $c(\mathcal{C})$ of $\mathcal{C}$ (possibly empty), called the *core* of $\mathcal{C}$, containing all modules lying on oriented cycles in $\mathcal{C}$ (see [58] for details). A family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to be *generalized standard* if $\mathrm{rad}_A^\infty(X,Y) = 0$ for all modules X and Y in $\mathcal{C}$ [66], and *sincere* if every simple module in $\mathrm{mod}\, A$ occurs as a composition factor of a module in $\mathcal{C}$. Finally, following Assem, Skowroński and Tomé [5], a family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to be *separating* if the components in Γ_A split into three disjoint families $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$ such that:

(S1) $\mathcal{C}^A$ is a sincere generalized standard family of components;

(S2) $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\mathrm{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;

(S3) any morphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in $\operatorname{mod} A$ factors through the additive category $\operatorname{add}(\mathcal{C}^A)$ of $\mathcal{C}^A$.

We then say that $\mathcal{C}^A$ separates $\mathcal{P}^A$ from $\mathcal{Q}^A$ and write

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A.$$

We mention that then the families $\mathcal{P}^A$ and $\mathcal{Q}^A$ are uniquely determined by the separating family $\mathcal{C}^A$, and $\mathcal{C}^A$ is a faithful family of components in Γ_A, that is, $\operatorname{ann}_A(\mathcal{C}^A) = 0$.

In the representation theory of algebras an important role is played by the canonical algebras introduced by Ringel in [60] and [61]. Every canonical algebra Λ is of global dimension at most 2 and its Auslander-Reiten quiver Γ_Λ admits a canonical separating family $\mathcal{T}^\Lambda$ of stable tubes, so Γ_Λ admits a disjoint union decomposition $\Gamma_\Lambda = \mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda \cup \mathcal{Q}^\Lambda$. Then an algebra C of the form $\operatorname{End}_\Lambda(T)$, with T a tilting module in the additive category $\operatorname{add}(\mathcal{P}^\Lambda)$ of $\mathcal{P}^\Lambda$ is called a *concealed canonical algebra* of type Λ, and $\mathcal{T}^C = \operatorname{Hom}_\Lambda(T, \mathcal{T}^\Lambda)$ is a separating family of stable tubes in Γ_C, so we have a disjoint union decomposition $\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C$. It has been proved by Lenzing and de la Peña in [35] that an algebra A is a concealed canonical algebra if and only if Γ_A admits a separating family $\mathcal{T}^A$ of stable tubes. The concealed canonical algebras form a distinguished class of *quasitilted algebras*, which are the endomorphism algebras $\operatorname{End}_{\mathscr{H}}(T)$ of tilting objects T in abelian hereditary K-categories $\mathscr{H}$ [26]. By a result due to Happel, Reiten and Smalø proved in [26], an algebra A is a quasitilted algebra if and only if $\operatorname{gl.dim} A \leq 2$ and every module X in $\operatorname{ind} A$ satisfies $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$. Further, it has been proved by Happel and Reiten in [25] that the class of quasitilted algebras consists of the *tilted algebras* (the endomorphism algebras $\operatorname{End}_H(T)$ of tilting modules T over hereditary algebras H) and the *quasitilted algebras of canonical type* (the endomorphism algebras $\operatorname{End}_{\mathscr{H}}(T)$ of tilting objects T in abelian hereditary categories $\mathscr{H}$ whose derived category $D^b(\mathscr{H})$ is equivalent to the derived category $D^b(\operatorname{mod}\Lambda)$ of the module category $\operatorname{mod}\Lambda$ of a canonical algebra Λ). Moreover, it has been proved by Lenzing and Skowroński in [36] (see also [73]) that an algebra A is a quasitilted algebra of canonical type if and only if Γ_A admits a separating family $\mathcal{T}^A$ of semiregular tubes (ray and coray tubes), and if and only if A is a semiregular branch enlargement of a concealed canonical algebra C. We are now in position to introduce the class of generalized multicoil algebras [45], being sophisticated gluings of quasitilted algebras of canonical type, playing the fundamental role in first main result of the paper. It has been proved by Malicki and Skowroński in [45] that the Auslander-Reiten quiver Γ_A of an algebra A admits a separating family of almost cyclic coherent components if and only if A is a *generalized multicoil algebra*, that is, a generalized multicoil enlargement of a product $C = C_1 \times \ldots \times C_m$ of concealed canonical algebras $C_1, \ldots, C_m$ using modules from the separating families $\mathcal{T}^{C_1}, \ldots, \mathcal{T}^{C_m}$ of stable tubes of $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$ and a sequence of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1*)-(ad 5*). For a generalized multicoil algebra A, there is a unique quotient algebra $A^{(l)}$ of A which is a product of quasitilted algebras of canonical type having separating families of coray tubes (the *left quasitilted algebra* of A) and a unique quotient algebra $A^{(r)}$ of A which is a product of quasitilted algebras of canonical type having separating families of ray tubes (the *right quasitilted algebra* of A) such that Γ_A

has a disjoint union decomposition (see [45, Theorems C and E])

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A,$$

where

- $\mathcal{P}^A$ is the left part $\mathcal{P}^{A^{(l)}}$ in a decomposition $\Gamma_{A^{(l)}} = \mathcal{P}^{A^{(l)}} \cup \mathcal{T}^{A^{(l)}} \cup \mathcal{Q}^{A^{(l)}}$ of the Auslander-Reiten quiver $\Gamma_{A^{(l)}}$ of the left quasitilted algebra $A^{(l)}$ of A, with $\mathcal{T}^{A^{(l)}}$ a family of coray tubes separating $\mathcal{P}^{A^{(l)}}$ from $\mathcal{Q}^{A^{(l)}}$;
- $\mathcal{Q}^A$ is the right part $\mathcal{Q}^{A^{(r)}}$ in a decomposition $\Gamma_{A^{(r)}} = \mathcal{P}^{A^{(r)}} \cup \mathcal{T}^{A^{(r)}} \cup \mathcal{Q}^{A^{(r)}}$ of the Auslander-Reiten quiver $\Gamma_{A^{(r)}}$ of the right quasitilted algebra $A^{(r)}$ of A, with $\mathcal{T}^{A^{(r)}}$ a family of ray tubes separating $\mathcal{P}^{A^{(r)}}$ from $\mathcal{Q}^{A^{(r)}}$;
- $\mathcal{C}^A$ is a family of generalized multicoils separating $\mathcal{P}^A$ from $\mathcal{Q}^A$, obtained from stable tubes in the separating families $\mathcal{T}^{C_1}, \dots, \mathcal{T}^{C_m}$ of stable tubes of the Auslander-Reiten quivers $\Gamma_{C_1}, \dots, \Gamma_{C_m}$ of the concealed canonical algebras $C_1, \dots, C_m$ by a sequence of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1*)-(ad 5*), corresponding to the admissible operations leading from $C = C_1 \times \dots \times C_m$ to A;
- $\mathcal{C}^A$ consists of cycle-finite modules and contains all indecomposable modules of $\mathcal{T}^{A^{(l)}}$ and $\mathcal{T}^{A^{(r)}}$;
- $\mathcal{P}^A$ contains all indecomposable modules of $\mathcal{P}^{A^{(r)}}$;
- $\mathcal{Q}^A$ contains all indecomposable modules of $\mathcal{Q}^{A^{(l)}}$.

Moreover, in the above notation, we have

- $\operatorname{gl.dim} A \leq 3$;
- $\operatorname{pd}_A X \leq 1$ for any indecomposable module X in $\mathcal{P}^A$;
- $\operatorname{id}_A Y \leq 1$ for any indecomposable module Y in $\mathcal{Q}^A$;
- $\operatorname{pd}_A M \leq 2$ and $\operatorname{id}_A M \leq 2$ for any indecomposable module M in $\mathcal{C}^A$.

A generalized multicoil algebra A is said to be *tame* if $A^{(l)}$ and $A^{(r)}$ are products of tilted algebras of Euclidean types or tubular algebras. We also note that every tame generalized multicoil algebra is a cycle-finite algebra.

The following theorem is the first main result of the paper.

Theorem 1.1. *Let A be an algebra and Γ be a cycle-finite infinite component of ${}_c\Gamma_A$. Then there exist infinite full translation subquivers $\Gamma_1, \dots, \Gamma_r$ of Γ such that the following statements hold.*

(i) *For each $i \in \{1, \dots, r\}$, Γ_i is a cyclic coherent full translation subquiver of Γ_A.*
(ii) *For each $i \in \{1, \dots, r\}$, $\operatorname{Supp}(\Gamma_i) = B(\Gamma_i)$ and is a generalized multicoil algebra.*
(iii) *$\Gamma_1, \dots, \Gamma_r$ are pairwise disjoint full translation subquivers of Γ and $\Gamma^{cc} = \Gamma_1 \cup \dots \cup \Gamma_r$ is a maximal cyclic coherent and cofinite full translation subquiver of Γ.*
(iv) *$B(\Gamma \setminus \Gamma^{cc})$ is of finite representation type.*
(v) *$\operatorname{Supp}(\Gamma) = B(\Gamma)$.*

It follows from the above theorem that all but finitely many modules lying in an infinite cycle-finite component Γ of ${}_c\Gamma_A$ can be obtained from indecomposable modules in stable tubes of concealed canonical algebras by a finite sequence of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1*)-(ad 5*) (see [45, Section 3] for details). We refer also to [33] and [68] for some results on the composition factors of indecomposable modules lying in stable tubes of the Auslander-Reiten quivers of concealed canonical algebras, and to [48] for the structure of indecomposable modules lying in coils. We would like to stress that the cycle-finiteness assumption imposed on the infinite component Γ of ${}_c\Gamma_A$ is essential for the validity of the above theorem. Namely, it has been proved in [74], [76] that, for an arbitrary finite dimensional algebra B over a field K, a module M in $\operatorname{mod} B$, and a positive integer r, there exists a finite dimensional algebra A over K such that B is a quotient algebra of A, Γ_A admits a faithful generalized standard stable tube $\mathcal{T}$ of rank r, $\mathcal{T}$ is not cycle-finite, and M is a subfactor of all but finitely many indecomposable modules in $\mathcal{T}$. This shows that in general the problem of describing the support algebras of infinite cyclic components (even stable tubes) of Auslander-Reiten quivers is difficult.

In order to present the second main result of the paper, we need the class of generalized double tilted algebras introduced by Reiten and Skowroński in [58] (see also [1], [18] and [57]). A *generalized double tilted algebra* is an algebra B for which Γ_B admits a separating almost acyclic component $\mathcal{C}$.

For a generalized double tilted algebra B, the Auslander-Reiten quiver Γ_B has a disjoint union decomposition (see [58, Section 3])

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{C}^B \cup \mathcal{Q}^B,$$

where

- $\mathcal{C}^B$ is an almost acyclic component separating $\mathcal{P}^B$ from $\mathcal{Q}^B$, called a *connecting component* of Γ_B;
- There exist hereditary algebras $H_1^{(l)}, \dots, H_m^{(l)}$ and tilting modules $T_1^{(l)} \in \operatorname{mod} H_1^{(l)}, \dots, T_m^{(l)} \in \operatorname{mod} H_m^{(l)}$ such that the tilted algebras $B_1^{(l)} = \operatorname{End}_{H_1^{(l)}}(T_1^{(l)}), \dots, B_m^{(l)} = \operatorname{End}_{H_m^{(l)}}(T_m^{(l)})$ are quotient algebras of B and $\mathcal{P}^B$ is the disjoint union of all components of $\Gamma_{B_1^{(l)}}, \dots, \Gamma_{B_m^{(l)}}$ contained entirely in the torsion-free parts $\mathscr{Y}(T_1^{(l)}), \dots, \mathscr{Y}(T_m^{(l)})$ of $\operatorname{mod} B_1^{(l)}, \dots, \operatorname{mod} B_m^{(l)}$ determined by $T_1^{(l)}, \dots, T_m^{(l)}$;
- There exist hereditary algebras $H_1^{(r)}, \dots, H_n^{(r)}$ and tilting modules $T_1^{(r)} \in \operatorname{mod} H_1^{(r)}, \dots, T_n^{(r)} \in \operatorname{mod} H_n^{(r)}$ such that the tilted algebras $B_1^{(r)} = \operatorname{End}_{H_1^{(r)}}(T_1^{(r)}), \dots, B_n^{(r)} = \operatorname{End}_{H_n^{(r)}}(T_n^{(r)})$ are quotient algebras of B and $\mathcal{Q}^B$ is the disjoint union of all components of $\Gamma_{B_1^{(r)}}, \dots, \Gamma_{B_n^{(r)}}$ contained entirely in the torsion parts $\mathscr{X}(T_1^{(r)}), \dots, \mathscr{X}(T_n^{(r)})$ of $\operatorname{mod} B_1^{(r)}, \dots, \operatorname{mod} B_n^{(r)}$ determined by $T_1^{(r)}, \dots, T_n^{(r)}$;
- every indecomposable module in $\mathcal{C}^B$ not lying in the core $c(\mathcal{C}^B)$ of $\mathcal{C}^B$ is an indecomposable module over one of the tilted algebras $B_1^{(l)}, \dots, B_m^{(l)}$, $B_1^{(r)}, \dots, B_n^{(r)}$;
- every nondirecting indecomposable module in $\mathcal{C}^B$ is cycle-finite and lies in $c(\mathcal{C}^B)$;
- $\operatorname{pd}_B X \leq 1$ for all indecomposable modules X in $\mathcal{P}^B$;
- $\operatorname{id}_B Y \leq 1$ for all indecomposable modules Y in $\mathcal{Q}^B$;
- for all but finitely many indecomposable modules M in $\mathcal{C}^B$, we have $\operatorname{pd}_B M \leq 1$ or $\operatorname{id}_B M \leq 1$.

Then $B^{(l)} = B_1^{(l)} \times \ldots \times B_m^{(l)}$ is called the *left tilted algebra* of B and $B^{(r)} = B_1^{(r)} \times \ldots \times B_n^{(r)}$ is called the *right tilted algebra* of B. We note that the class of algebras of finite representation type coincides with the class of generalized double tilted algebras B with Γ_B being the connecting component $\mathcal{C}^B$ (equivalently, with the tilted algebras $B^{(l)}$ and $B^{(r)}$ being of finite representation type (possibly empty)). Finally, a generalized double tilted algebra is said to be *tame* if the tilted algebras $B^{(l)}$ and $B^{(r)}$ are generically tame in the sense of Crawley-Boevey [21], [22]. We note that every tame generalized double tilted algebra is a cycle-finite algebra. We would like to mention that there exist generalized double tilted algebras of infinite representation type of arbitrary global dimension $d \in \mathbb{N} \cup \{\infty\}$. We refer also to [28], [39], [65] for useful characterizations of tilted algebras.

The following theorem is the second main result of the paper.

Theorem 1.2. *Let A be an algebra and Γ be a cycle-finite finite component of ${}_c\Gamma_A$. Then the following statements hold.*

(i) $\operatorname{Supp}(\Gamma)$ *is a generalized double tilted algebra.*
(ii) Γ *is the core $c(\mathcal{C}^{B(\Gamma)})$ of a unique almost acyclic connecting component $\mathcal{C}^{B(\Gamma)}$ of $\Gamma_{B(\Gamma)}$.*
(iii) $\operatorname{Supp}(\Gamma) = B(\Gamma)$.

We would like to point that every finite cyclic component Γ of an Auslander-Reiten quiver Γ_A contains both a projective module and an injective module (see Corollary 2.6), and hence Γ_A admits at most finitely many finite cyclic components. We refer also to [34], [80], [81] for some results concerning double tilted algebras with connecting components containing nondirecting indecomposable modules.

An idempotent e of an algebra A is said to be *convex* provided e is a sum of pairwise orthogonal primitive idempotents of A corresponding to the vertices of a convex valued subquiver of the quiver Q_A of A (see Section 2 for definition). The following direct consequence of Theorems 1.1, 1.2 and Propositions 2.2, 2.3 provides a handy description of the faithful algebra of a cycle-finite component of ${}_c\Gamma_A$.

Corollary 1.3. *Let A be an algebra and Γ be a cycle-finite component of ${}_c\Gamma_A$. Then there exists a convex idempotent e_Γ of A such that $\operatorname{Supp}(\Gamma)$ is isomorphic to the algebra $e_\Gamma A e_\Gamma$.*

The third main result of the paper is a consequence of Theorems 1.1 and 1.2, and the results established in [46, Theorem 1.3].

Theorem 1.4. *Let A be an algebra. Then, for all but finitely many isomorphism classes of cycle-finite modules M in $\operatorname{ind} A$, the following statements hold.*

(i) $|\operatorname{Ext}_A^1(M,M)| \leq |\operatorname{End}_A(M)|$ *and* $\operatorname{Ext}_A^r(M,M) = 0$ *for* $r \geq 2$.
(ii) $|\operatorname{Ext}_A^1(M,M)| = |\operatorname{End}_A(M)|$ *if and only if there is a quotient concealed canonical algebra C of A and a stable tube $\mathcal{T}$ of Γ_C such that M is an indecomposable C-module in $\mathcal{T}$ of quasi-length divisible by the rank of $\mathcal{T}$.*

Here, $|V|$ denotes the length of a module V in $\operatorname{mod} K$. In particular, the above theorem shows that, for all but finitely many isomorphism classes of cycle-finite modules

M in a module category $\operatorname{ind} A$, the Euler characteristic

$$\chi_A(M) = \sum_{i=0}^{\infty} (-1)^i |\operatorname{Ext}_A^i(M, M)|$$

of M is well defined and nonnegative. We would like to mention that there are cycle-finite algebras A with indecomposable modules M lying in infinite cyclic components of Γ_A and the Euler characteristic $\chi_A(M)$ being an arbitrary given positive integer (see [52]).

Let A be an algebra and $K_0(A)$ the Grothendieck group of A. For a module M in $\operatorname{mod} A$, we denote by $[M]$ the image of M in $K_0(A)$. Then $K_0(A)$ is a free abelian group with a $\mathbb{Z}$-basis given by $[S_1], \dots, [S_n]$ for a complete family $S_1, \dots, S_n$ of pairwise nonisomorphic simple modules in $\operatorname{mod} A$. Thus, for modules M and N in $\operatorname{mod} A$, we have $[M] = [N]$ if and only if the modules M and N have the same composition factors including the multiplicities. In particular, it would be interesting to find sufficient conditions for a module M in $\operatorname{ind} A$ to be uniquely determined (up to isomorphism) by its composition factors (see [59] for a general result in this direction).

The next theorem provides information on the composition factors of cycle-finite modules, and is a direct consequence of Theorems 1.1, 1.2, 1.4 and the results established in [41, Theorems A and B].

Theorem 1.5. *Let A be an algebra. The following statements hold.*

(i) *There is a positive integer m such that, for any cycle-finite module M in* $\operatorname{ind} A$ *with $|\operatorname{End}_A(M)| \neq |\operatorname{Ext}_A^1(M, M)|$, the number of isomorphism classes of modules X in* $\operatorname{ind} A$ *with $[X] = [M]$ is bounded by m.*

(ii) *For all but finitely many isomorphism classes of cycle-finite modules M in* $\operatorname{ind} A$ *with $|\operatorname{End}_A(M)| = |\operatorname{Ext}_A^1(M, M)|$, there are infinitely many pairwise nonisomorphic modules X in* $\operatorname{ind} A$ *with $[X] = [M]$.*

Following Auslander and Reiten [7], one associates with each nonprojective module X in a module category $\operatorname{ind} A$ the number $\alpha(X)$ of indecomposable direct summands in the middle term

$$0 \to \tau_A X \to Y \to X \to 0$$

of the almost split sequence with the right term X. It has been proved by Bautista and Brenner [10] that, if A is an algebra of finite representation type and X a nonprojective module in $\operatorname{ind} A$, then $\alpha(X) \leq 4$, and if $\alpha(X) = 4$ then Y admits a projective-injective indecomposable direct summand P, and hence $X = P/\operatorname{soc}(P)$. In [40] Liu proved that the same is true for any indecomposable nonprojective module X lying on an oriented cycle of the Auslander-Reiten quiver Γ_A of any algebra A, and consequently for any nonprojective cycle-finite module in $\operatorname{ind} A$.

The following theorem is a direct consequence of Theorems 1.1 and 1.2, and [44, Corollary B], and provides more information on almost split sequences of cycle-finite modules.

Theorem 1.6. *Let A be an algebra. Then, for all but finitely many isomorphism classes of nonprojective cycle-finite modules M in* $\operatorname{ind} A$*, we have $\alpha(M) \leq 2$.*

In connection to Theorem 1.6, we would like to mention that, for a cycle-finite algebra A and a nonprojective module M in $\operatorname{ind} A$, we have $\alpha(M) \leq 5$, and if $\alpha(M) = 5$ then the middle term of the almost split sequence in $\operatorname{mod} A$ with the right term M admits a projective-injective indecomposable direct summand P, and hence $M = P/\operatorname{soc}(P)$ (see [13, Conjecture 1], [43] and [54]).

The next theorem describe the structure of the module category $\operatorname{ind} A$ of an arbitrary cycle-finite algebra A, and is a direct consequence of Theorems 1.1 and 1.2 as well as [42, Theorem 2.2] and its dual.

Theorem 1.7. *Let A be a cycle-finite algebra. Then there exist tame generalized multicoil algebras $B_1, \dots, B_p$ and tame generalized double tilted algebras $B_{p+1}, \dots, B_q$ which are quotient algebras of A and the following statements hold.*

(i) $\operatorname{ind} A = \bigcup_{i=1}^{q} \operatorname{ind} B_i$.
(ii) *All but finitely many isomorphism classes of modules in* $\operatorname{ind} A$ *belong to* $\bigcup_{i=1}^{p} \operatorname{ind} B_i$.
(iii) *All but finitely many isomorphism classes of nondirecting modules in* $\operatorname{ind} A$ *belong to generalized multicoils of* $\Gamma_{B_1}, \dots, \Gamma_{B_p}$.

The next theorem extends the homological characterization of strongly simply connected algebras of polynomial growth established in [51] to arbitrary cycle-finite algebras, and is a direct consequence of Theorem 1.4 and the properties of directing modules described in [60, 2.4(8)].

Theorem 1.8. *Let A be a cycle-finite algebra. Then, for all but finitely many isomorphism classes of modules M in* $\operatorname{ind} A$, *we have* $|\operatorname{Ext}_A^1(M, M)| \leq |\operatorname{End}_A(M)|$ *and* $\operatorname{Ext}_A^r(M, M) = 0$ *for* $r \geq 2$.

We end this section with some questions related to the results described above.

In [37], [38] Liu introduced the notions of left and right degrees of irreducible homomorphisms of modules and showed their importance for describing the shapes of the components of the Auslander-Reiten quivers of algebras. In particular, Liu pointed out in [37] that every cycle of irreducible homomorphisms between indecomposable modules in a module category $\operatorname{mod} A$ contains an irreducible homomorphism of finite left degree and an irreducible homomorphism of finite right degree. It would be interesting to describe the degrees of irreducible homomorphisms occurring in cycles of cycle-finite modules (see [14], [15], [16], [17] for some results in this direction).

In [50] de la Peña proved that the support algebra of a directing module over a tame algebra over an algebraically closed field is a tilted algebra being a gluing of at most two representation-infinite tilted algebras of Euclidean type. It would be interesting to know if the support algebra $\operatorname{Supp}(\Gamma)$ of a cycle-finite finite component Γ in the cyclic quiver ${}_c\Gamma_A$ of a cycle-finite algebra is a gluing of at most two representation-infinite tilted algebras of Euclidean type. In general, it is not clear how many tilted algebras may occur in the decompositions of the left tilted algebra and the right tilted algebra of the support algebra $\operatorname{Supp}(\Gamma)$ of a cycle-finite component Γ of the cyclic quiver ${}_c\Gamma_A$ of an algebra A (see Examples 6.1 and 6.2).

2. Cyclic components

In this section we recall some concepts and describe some properties of cyclic components of the Auslander-Reiten quivers of algebras. Let A be an algebra (basic, indecomposable) and $e_1, \dots, e_n$ be a set of pairwise orthogonal primitive idempotents of A with $1_A = e_1 + \cdots + e_n$. Then

- $P_i = e_iA$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} A$;
- $I_i = D(Ae_i)$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} A$;
- $S_i = \operatorname{top}(P_i) = e_iA/e_i\operatorname{rad}A$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$;
- $S_i = \operatorname{soc}(I_i)$, for any $i \in \{1, \dots, n\}$.

Moreover, $F_i = \operatorname{End}_A(S_i) \cong e_iAe_i/e_i(\operatorname{rad}A)e_i$, for $i \in \{1, \dots, n\}$, are division algebras. The *quiver* Q_A of A is the valued quiver defined as follows:

- the vertices of Q_A are the indices $1, \dots, n$ of the chosen set $e_1, \dots, e_n$ of primitive idempotents of A;
- for two vertices i and j in Q_A, there is an arrow $i \to j$ from i to j in Q_A if and only if $e_i(\operatorname{rad}A)e_j/e_i(\operatorname{rad}A)^2e_j \neq 0$. Moreover, one associates to an arrow $i \to j$ in Q_A the valuation (d_{ij}, d'_{ij}), so we have in Q_A the valued arrow
$$i \xrightarrow{(d_{ij},d'_{ij})} j,$$
with the valuation numbers are $d_{ij} = \dim_{F_j} e_i(\operatorname{rad}A)e_j/e_i(\operatorname{rad}A)^2e_j$ and $d'_{ij} = \dim_{F_i} e_i(\operatorname{rad}A)e_j/e_i(\operatorname{rad}A)^2e_j$.

It is known that Q_A coincides with the Ext-quiver of A. Namely, Q_A contains a valued arrow $i \xrightarrow{(d_{ij},d'_{ij})} j$ iff $\operatorname{Ext}^1_A(S_i, S_j) \neq 0$ and $d_{ij} = \dim_{F_j} \operatorname{Ext}^1_A(S_i, S_j)$, $d'_{ij} = \dim_{F_i} \operatorname{Ext}^1_A(S_i, S_j)$. An algebra A is called *triangular* provided its quiver Q_A is acyclic (there is no oriented cycle in Q_A). We shall identify an algebra A with the associated category A^* whose objects are the vertices $1, \dots, n$ of Q_A, $\operatorname{Hom}_{A^*}(i, j) = e_jAe_i$ for any objects i and j of A^*, and the composition of morphisms in A^* is given by the multiplication in A. For a module M in $\operatorname{mod} A$, we denote by $\operatorname{supp}(M)$ the full subcategory of $A = A^*$ given by all objects i such that $Me_i \neq 0$, and call the *support* of M. More generally, for a translation subquiver $-$ of Γ_A, we denote by $\operatorname{supp}(-)$ the full subcategory of A given by all objects i such that $Xe_i \neq 0$ for some indecomposable module X in $-$, and call it the *support* of $-$. We also mention that $\operatorname{supp}(\Gamma)$ is usually different from the support algebra $\operatorname{Supp}(\Gamma)$ of Γ. Then a module M in $\operatorname{mod} A$ (respectively, a family of components $\mathcal{C}$ in Γ_A) is said to be *sincere* if $\operatorname{supp}(M) = A$ (respectively, if $\operatorname{supp}(\mathcal{C}) = A$). Finally, a full subcategory B of A is said to be a *convex subcategory* of A if every path in Q_A with source and target in B has all vertices in B. Observe that, for a convex subcategory B of A, there is a fully faithful embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$ such that $\operatorname{mod} B$ is the full subcategory of $\operatorname{mod} A$ consisting of the modules M with $Me_i = 0$ for all objects i of A which are not objects of B.

An essential role in further considerations will be played by the following result proved in [44, Proposition 5.1].

Proposition 2.1. *Let A be an algebra and X, Y be modules in* $\operatorname{ind} A$*. Then X and Y belong to the same component of ${}_c\Gamma_A$ if and only if there is an oriented cycle in Γ_A passing through X and Y.*

We prove now the following property of cycle-finite cyclic components.

Proposition 2.2. *Let A be an algebra and Γ be a cycle-finite component of ${}_c\Gamma_A$. Then* $\operatorname{supp}(\Gamma)$ *is a convex subcategory of A.*

Proof. Let $C = \operatorname{supp}(\Gamma)$. Assume to the contrary that C is not a convex subcategory of A. Then Q_A contains a path

$$i = i_0 \xrightarrow{\left(d_{i_0 i_1}, d'_{i_0 i_1}\right)} i_1 \xrightarrow{\left(d_{i_1 i_2}, d'_{i_1 i_2}\right)} i_2 \to \cdots \to i_{s-1} \xrightarrow{\left(d_{i_{s-1} i_s}, d'_{i_{s-1} i_s}\right)} i_s = j,$$

with $s \geqslant 2$, i, j in C and $i_1, \ldots, i_{s-1}$ not in C. Since Q_A coincides with the Ext-quiver of A, we have $\operatorname{Ext}^1_A(S_{i_{t-1}}, S_{i_t}) \neq 0$ for $t \in \{1, \ldots, s\}$. Then there exist in $\operatorname{mod} A$ nonsplitable exact sequences

$$0 \to S_{i_t} \to L_t \to S_{i_{t-1}} \to 0,$$

for $t \in \{1, \ldots, s\}$. Clearly, $L_1, \ldots, L_s$ are indecomposable modules in $\operatorname{mod} A$ of length 2. In particular, we obtain nonzero nonisomorphisms $f_r : L_r \to L_{r-1}$ with $\operatorname{Im} f_r = S_{i_{r-1}}$, for $r \in \{2, \ldots, s\}$. Consider now the ideal J in A of the form

$$J = Ae_i(\operatorname{rad} A)e_{i_1}(\operatorname{rad} A) + (\operatorname{rad} A)e_{i_{s-1}}(\operatorname{rad} A)e_j A$$

and the quotient algebra $B = A/J$. Since i_1 and i_{s-1} do not belong to $C = \operatorname{supp}(\Gamma)$, for any module M in Γ, we have $Me_{i_1} = 0$ and $Me_{i_{s-1}} = 0$, and consequently $MJ = 0$. This shows that Γ is a cyclic component of Γ_B. Moreover, it follows from the definition of J that S_{i_1} is a direct summand of the radical $\operatorname{rad} P^*_i$ of the projective cover $P^*_i = e_i B$ of S_i in $\operatorname{mod} B$ and $S_{i_{s-1}}$ is a direct summand of the socle factor I^*_j/S_j of the injective envelope $I^*_j = D(Be_j)$ of S_j in $\operatorname{mod} B$. Further, since i and j are in C, there exist indecomposable modules X and Y in Γ such that S_i is a composition factor of X and S_j is a composition factor of Y. Then we infer that $\operatorname{Hom}_B(P^*_i, X) \neq 0$ and $\operatorname{Hom}_B(Y, I^*_j) \neq 0$, because Γ consists of C-modules, and hence B-modules. It follows from Proposition 2.1 that we have in Γ a path from X to Y. Therefore, we obtain in $\operatorname{ind} A$ a cycle of the form

$$X \to \cdots \to Y \to I^*_j \to S_{i_{s-1}} \to L_{s-1} \to \cdots \to L_2 \to S_{i_1} \to P^*_i \to X,$$

which is an infinite cycle, because X and Y belong to Γ but S_{i_1} and $S_{i_{s-1}}$ are not in Γ. This contradicts the cycle-finiteness of Γ. Hence $C = \operatorname{supp}(\Gamma)$ is indeed a convex subcategory of A. □

Let A be an algebra, Γ be a component of ${}_c\Gamma_A$, and $A = P_\Gamma \oplus Q_\Gamma$ a decomposition of A in $\operatorname{ind} A$ such that the simple summands of $P_\Gamma/\operatorname{rad} P_\Gamma$ are exactly the simple composition factors of the indecomposable modules in Γ. Then there exists an idempotent e_Γ of A such that $P_\Gamma = e_\Gamma A$, $Q_\Gamma = (1 - e_\Gamma)A$, $t_A(\Gamma) = A(1 - e_\Gamma)A$, and $e_\Gamma A e_\Gamma$ is isomorphic to the endomorphism algebra $\operatorname{End}_A(P_\Gamma)$. In follows from Proposition 2.2 that e_Γ is a convex idempotent of A. Observe also that $\mathrm{End}_A(P_\Gamma)$ is the algebra of the support category $\mathrm{supp}(\Gamma)$ of Γ. The next result gives another description of $\mathrm{End}_A(P_\Gamma)$ in case the component Γ of ${}_c\Gamma_A$ is cycle-finite.

Proposition 2.3. *Let A be an algebra and Γ be a cycle-finite component of ${}_c\Gamma_A$. Consider a decomposition $A = P_\Gamma \oplus Q_\Gamma$ of A in $\operatorname{mod} A$ such that the simple summands of $P_\Gamma/\mathrm{rad}P_\Gamma$ are exactly the simple composition factors of the indecomposable modules in Γ. Then the algebras $\mathrm{Supp}(\Gamma)$ and $\mathrm{End}_A(P_\Gamma)$ are isomorphic.*

Proof. Observe that the support algebra $\mathrm{Supp}(\Gamma) = A/t_A(\Gamma)$ is isomorphic to the endomorphism algebra $\mathrm{End}_A(P_\Gamma/P_\Gamma t_A(\Gamma))$. Moreover, $P_\Gamma t_A(\Gamma)$ is the right A-submodule of P_Γ generated by the images of all homomorphisms from Q_Γ to P_Γ in $\operatorname{mod} A$. For any homomorphism $f \in \mathrm{End}_A(P_\Gamma)$ we have the canonical commutative diagram in $\operatorname{mod} A$ of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_\Gamma t_A(\Gamma) & \longrightarrow & P_\Gamma & \longrightarrow & P_\Gamma/P_\Gamma t_A(\Gamma) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\
0 & \longrightarrow & P_\Gamma t_A(\Gamma) & \longrightarrow & P_\Gamma & \longrightarrow & P_\Gamma/P_\Gamma t_A(\Gamma) & \longrightarrow & 0,
\end{array}$$

where f' is the restriction of f to $P_\Gamma t_A(\Gamma)$ and $\overline{f}$ is induced by f. Clearly, by the projectivity of P_Γ in $\operatorname{mod} A$, every homomorphism $g \in \mathrm{End}_A(P_\Gamma/P_\Gamma t_A(\Gamma))$ is of the form $\overline{f}$ for some homomorphism $f \in \mathrm{End}_A(P_\Gamma)$. This shows that the assignment $f \to \overline{f}$ induces an epimorphism $\mathrm{End}_A(P_\Gamma) \to \mathrm{End}_A(P_\Gamma/P_\Gamma t_A(\Gamma))$ of algebras. Assume now that $\overline{f} = 0$ for a homomorphism $f \in \mathrm{End}_A(P_\Gamma)$. Then $\mathrm{Im} f \subseteq P_\Gamma t_A(\Gamma)$. On the other hand, it follows from the definition of $t_A(\Gamma)$ that there is an epimorphism $v : Q_\Gamma^m \to P_\Gamma t_A(\Gamma)$ in $\operatorname{mod} A$ for some positive integer m. Using the projectivity of P_Γ in A, we conclude that there is a homomorphism $u : P_\Gamma \to Q_\Gamma^m$ such that $f = vu$. But $f \neq 0$ implies that $u \neq 0$ and $v \neq 0$, and then a contradiction with the convexity of $\mathrm{Supp}(\Gamma)$ in $A = A^*$ established in Proposition 2.2. Hence $f = 0$. Therefore, the canonical epimorphism of algebras $\mathrm{End}_A(P_\Gamma) \to \mathrm{End}_A(P_\Gamma/P_\Gamma t_A(\Gamma))$ is an isomorphism, and so the algebras $\mathrm{End}_A(P_\Gamma)$ and $\mathrm{Supp}(\Gamma)$ are isomorphic. □

The following fact proved by Bautista and Smalø in [11] (see also [79, Corollary III. 11.3]) will be essential for our considerations.

Proposition 2.4. *Let A be an algebra and*

$$X = X_0 \to X_1 \to \cdots \to X_{r-1} \to X_r = X$$

a cycle in Γ_A. Then there exists $i \in \{2, \ldots, r\}$ such that $\tau_A X_i \cong X_{i-2}$.

Lemma 2.5. *Let A be an algebra and Γ be a cyclic component of Γ_A. Assume that*

$$X = X_0 \to X_1 \to \cdots \to X_{r-1} \to X_r = X$$

is a cycle in Γ. Then the following statements hold.

(i) *If all modules X_i, $i \in \{1, \ldots, r\}$, are nonprojective, then Γ contains a cycle of the form*

$$\tau_A X = \tau_A X_0 \to \tau_A X_1 \to \cdots \to \tau_A X_{r-1} \to \tau_A X_r = \tau_A X.$$

(ii) *If all modules* X_i, $i \in \{1, \dots, r\}$, *are noninjective, then* Γ *contains a cycle of the form*

$$\tau_A^{-1}X = \tau_A^{-1}X_0 \to \tau_A^{-1}X_1 \to \cdots \to \tau_A^{-1}X_{r-1} \to \tau_A^{-1}X_r = \tau_A^{-1}X.$$

Proof. It follows from Proposition 2.4 that there exists $i \in \{2, \dots, r\}$ such that $\tau_A X_i = X_{i-2}$, or equivalently, $X_i = \tau_A^{-1}X_{i-2}$. Hence, if all modules X_i, $i \in \{1, \dots, r\}$, are nonprojective, then we have in Γ_A a cycle

$$\tau_A X = \tau_A X_0 \to \tau_A X_1 \to \cdots \to \tau_A X_i \to \cdots \to \tau_A X_{r-1} \to \tau_A X_r = \tau_A X$$

with $\tau_A X_i = X_{i-2}$, and hence all modules of this cycle belong to the cyclic component Γ containing X_{i-2}. Similarly, if all modules X_i, $i \in \{1, \dots, r\}$, are noninjective, then we have in Γ_A a cycle

$$\tau_A^{-1}X = \tau_A^{-1}X_0 \to \tau_A^{-1}X_1 \to \cdots \to \tau_A^{-1}X_{i-2} \to \cdots \to \tau_A^{-1}X_{r-1} \to \tau_A^{-1}X_r = \tau_A^{-1}X$$

with $X_i = \tau_A^{-1}X_{i-2}$, and hence all modules of this cycle belong to the cyclic component Γ containing X_i. □

Corollary 2.6. *Let* A *be an algebra and* Γ *a finite cyclic component of* Γ_A. *Then* Γ *contains a projective and an injective module.*

Proof. Assume Γ does not contain a projective module. Then it follows from Lemma 2.5 that, for any indecomposable module X in Γ, $\tau_A X$ is also a module in Γ. Since Γ is a finite translation quiver, this implies that $\Gamma = \tau_A\Gamma$, and hence Γ is a component of Γ_A. Then there exists an indecomposable algebra B (a block of A) such that Γ is a component of Γ_B, and consequently $\Gamma = \Gamma_B$, by the well known theorem of Auslander (see [79, Theorem III. 10.2]). But this is a contradiction, because Γ_B contains projective modules. Therefore, Γ contains a projective module. The proof that Γ contains an injective module is similar. □

Let A be an algebra and $\mathcal{C}$ a component of Γ_A. We denote by ${}_l\mathcal{C}$ the *left stable part* of $\mathcal{C}$ obtained by removing in $\mathcal{C}$ the τ_A-orbits of projective modules and the arrows attached to them, and by ${}_r\mathcal{C}$ the *right stable part* of $\mathcal{C}$ obtained by removing in $\mathcal{C}$ the τ_A-orbits of injective modules and the arrows attached to them. We note that, if $\mathcal{C}$ is infinite, then ${}_l\mathcal{C}$ or ${}_r\mathcal{C}$ is nonempty.

The following proposition will be applied in the proofs of our main theorems.

Proposition 2.7. *Let* A *be an algebra,* $\mathcal{C}$ *a component of* Γ_A, *and* Σ *an infinite family of cycle-finite modules in* $\mathcal{C}$. *Then one of the following statements hold.*

(i) *The stable part* ${}_s\mathcal{C}$ *of* $\mathcal{C}$ *contains a stable tube* $\mathcal{D}$ *having infinitely many modules from* Σ.

(ii) *The left stable part* ${}_l\mathcal{C}$ *of* $\mathcal{C}$ *contains a component* $\mathcal{D}$ *with an oriented cycle and an injective module such that the cyclic part* ${}_c\mathcal{D}$ *of* $\mathcal{D}$ *contains infinitely many modules from* Σ.

(iii) *The right stable part* ${}_r\mathcal{C}$ *of* $\mathcal{C}$ *contains a component* $\mathcal{D}$ *with an oriented cycle and a projective module such that the cyclic part* ${}_c\mathcal{D}$ *of* $\mathcal{D}$ *contains infinitely many modules from* Σ.

Proof. (1) Assume first that there is a τ_A-orbit $\mathcal{O}$ in $\mathcal{C}$ containing infinitely many modules from Σ. Consider the case when $\mathcal{O}$ contains infinitely many left stable modules from Σ. Then there exist a module M in $\mathcal{O} \cap \Sigma$ and an infinite sequence $0 = r_0 < r_1 < r_2 < \ldots$ of integers such that the modules $\tau_A^{r_i} M$, $i \in \mathbb{N}$, belong to $\mathcal{O} \cap \Sigma$. Let $\mathcal{D}$ be the component of ${}_l\mathcal{C}$ containing the modules $\tau_A^{r_i} M$, $i \in \mathbb{N}$. We have two cases to consider.

Assume $\mathcal{D}$ contains an oriented cycle. Observe that $\mathcal{D}$ is not a stable tube, and hence does not contain a τ_A-periodic module, because $\mathcal{D}$ contains infinitely many modules from the τ_A-orbit $\mathcal{O}$. Hence, applying [38, Lemma 2.2 and Theorem 2.3], we conclude that $\mathcal{D}$ contains an infinite sectional path

$$\cdots \to \tau_A^t X_s \to \cdots \to \tau_A^t X_2 \to \tau_A^t X_1 \to X_s \to \cdots \to X_2 \to X_1,$$

where $t > s \geq 1$, X_i is an injective module for some $i \in \{1, \ldots, s\}$, and each module in $\mathcal{D}$ belongs to the τ_A-orbit of one of the modules X_i. Clearly, then there is a nonnegative integer m such that all modules $\tau_A^r M$, $r \geq m$, belong to the cyclic part ${}_c\mathcal{D}$ of $\mathcal{D}$. Therefore, the statement (ii) holds.

Assume $\mathcal{D}$ is acyclic. Then it follows from [38, Theorem 3.4] that there is an acyclic locally finite valued quiver Δ such that $\mathcal{D}$ is isomorphic to a full translation subquiver of $\mathbb{Z}\Delta$, which is closed under predecessors. But then there exists a positive integer i such that $\tau_A^{r_i} M$ is not a successor of a projective module in $\mathcal{C}$, and consequently does not lie on an oriented cycle in $\mathcal{C}$. On the other hand, $\tau_A^{r_i} M$ belongs to Σ, and then is a cycle-finite indecomposable module, so lying on a cycle in $\mathcal{C}$, a contradiction.

Similarly, if $\mathcal{O}$ contains infinitely many right stable modules from Σ, then the statement (iii) holds.

(2) Assume now that every τ_A-orbit in $\mathcal{C}$ contains at most finitely many modules from Σ. Since Σ is an infinite family of modules, we infer that there is an infinite component $\mathcal{D}$ of the stable part ${}_s\mathcal{C}$ of $\mathcal{C}$ containing infinitely many modules from Σ. We have two cases to consider.

Assume $\mathcal{D}$ contains an oriented cycle. Then it follows from [82, Corollary] (see also [37, Theorems 2.5 and 2.7]) that $\mathcal{D}$ is a stable tube. Thus the statement (i) holds.

Assume $\mathcal{D}$ is acyclic. Applying [82, Corollary] again, we conclude that there exists an infinite locally finite acyclic valued quiver Δ such that $\mathcal{D}$ is isomorphic to the translation quiver $\mathbb{Z}\Delta$. Let n be the rank of the Grothendieck group $K_0(A)$ of A. Then there is a module M in $\mathcal{D} \cap \Sigma$ such that the length of any walk in $\mathcal{C}$ from a nonstable module in $\mathcal{C}$ to a module in the τ_A-orbit $\mathcal{O}(M)$ of M is at least $2n$. Then it follows from [20, Lemma 1.5] (see also [67, Lemma 4]) that, for each positive integer s, there exists a path

$$M = X_0 \to X_1 \to \cdots \to X_t = \tau_A^s M$$

in $\operatorname{ind} A$ with all modules X_i in $\mathcal{C}$, and consequently a cycle in $\operatorname{ind} A$ passing through M and $\tau_A^s M$, because there is a path

$$\tau_A^s M = Y_0 \to Y_1 \to \cdots \to Y_r = M$$

of irreducible homomorphisms in $\operatorname{ind} A$. Moreover, M is a cycle-finite module, as a module from Σ. This shows that $\mathcal{C}$ contains oriented cycles passing through M and any module $\tau_A^s M$, $s \geq 1$. We also note that there is a component $\mathcal{D}'$ of the left stable part ${}_l\mathcal{C}$ of $\mathcal{C}$ containing all τ_A-orbits of $\mathcal{D}$. Then there is an infinite locally finite acyclic valued subquiver Δ' containing Δ as a full valued subquiver, such that $\mathcal{D}'$ is isomorphic to a full translation subquiver of $\mathbb{Z}\Delta'$, which is closed under predecessors. Then there exists a positive integer m such that the module $\tau_A^m M$ is not a successor of a projective module in $\mathcal{C}$, and then $\tau_A^m M$ does not lie on an oriented cycle in $\mathcal{C}$, a contradiction. □

Corollary 2.8. *Let A be an algebra and Γ be a cycle-finite infinite component of ${}_c\Gamma_A$. Then ${}_l\Gamma$ or ${}_r\Gamma$ admits a component $\mathcal{D}$ containing an oriented cycle and infinitely many modules of Γ.*

3. Proof of Theorem 1.1

Let A be an algebra and Γ be a cycle-finite infinite component of ${}_c\Gamma_A$. Consider the component $\mathcal{C}$ of Γ_A containing the translation quiver Γ. Since Γ is infinite and cyclic, we conclude from Corollary 2.8 that ${}_l\mathcal{C}$ or ${}_r\mathcal{C}$ contains a connected component Σ containing an oriented cycle and infinitely many modules of Γ. We claim that there exists a cyclic coherent full translation subquiver Ω of Γ containing all modules of the cyclic part ${}_c\Sigma$ of Σ. We have three cases to consider:

(1) Assume Σ is contained in the stable part ${}_s\mathcal{C} = {}_l\mathcal{C} \cap {}_r\mathcal{C}$ of $\mathcal{C}$. Then Σ is an infinite stable translation quiver containing an oriented cycle, and hence Σ is a stable tube, by the main result of [82]. Clearly, the stable tube Σ is a cyclic and coherent translation quiver. Since Σ is a component of ${}_l\mathcal{C}$ and a component of ${}_r\mathcal{C}$, we conclude that Γ contains a cyclic coherent full translation subquiver Ω such that Σ is obtained from Ω by removing all finite τ_A-orbits without τ_A-periodic modules.

(2) Assume Σ is a component of ${}_l\mathcal{C}$ containing at least one injective module. Then it follows from [38, Lemma 2.2 and Theorem 2.3] that Σ contains an infinite sectional path

$$\cdots \to \tau_A^r X_s \to \cdots \to \tau_A^r X_2 \to \tau_A^r X_1 \to X_s \to \cdots \to X_2 \to X_1,$$

where $r > s \geq 1$, X_i is an injective module for some $i \in \{1, \dots, s\}$, and each module in Σ belongs to the τ_A-orbit of one of the modules X_i. Observe that there exists an infinite sectional path in Σ

$$X_s \to \tau_A^{r-1} X_1 \to \cdots$$

starting from X_s. Let p be the minimal element in $\{1, \dots, s\}$ such that there exists an infinite sectional path in Σ starting from X_p. Then Γ contains a cyclic coherent full translation subquiver Ω such that Σ is obtained from Ω by removing the τ_A-orbits of projective modules P lying on infinite sectional paths in Σ of the forms

$$P \to \cdots \to X_j \to \cdots \to \tau_A^{r-j+p-1} X_1 \to \cdots$$

for some $j \in \{p, \dots, s\}$, or

$$P \to \cdots \to \tau_A^{mr} X_i \to \tau_A^{mr-1} X_{i+1} \to \cdots$$

for some $m \geq 1$ and $i \in \{1, \dots, s\}$.

(3) Assume Σ is a component of ${}_r\mathcal{C}$ containing at least one projective module. Then it follows from [38, duals of Lemma 2.2 and Theorem 2.3] that Σ contains an infinite sectional path

$$X_1 \to X_2 \to \cdots \to X_t \to \tau_A^{-m} X_1 \to \tau_A^{-m} X_2 \to \cdots \to \tau_A^{-m} X_t \to \cdots ,$$

where $m > t \geq 1$, X_j is an projective module for some $j \in \{1, \dots, t\}$, and each module in Σ belongs to the τ_A-orbit of one of the modules X_j. Observe that there exists an infinite sectional path in Σ

$$\cdots \to \tau_A^{-m+1} X_1 \to X_t$$

ending in X_t. Let q be the minimal element in $\{1, \dots, t\}$ such that there exists an infinite sectional path in Σ ending in X_q. Then Γ contains a cyclic coherent full translation subquiver Ω such that Σ is obtained from Ω by removing the τ_A-orbits of injective modules I lying on infinite sectional paths in Σ of the forms

$$\cdots \to \tau_A^{-m+t-i+1} X_1 \to \cdots \to X_i \to \cdots \to I$$

for some $i \in \{q, \dots, t\}$, or

$$\cdots \to \tau_A^{-ms+1} X_{j+1} \to \tau_A^{-ms} X_j \to \cdots \to I$$

for some $s \geq 1$ and $j \in \{1, \dots, t\}$.

Let $\Gamma_1, \dots, \Gamma_t$ be all maximal cyclic coherent pairwise different full translation subquivers of Γ. Clearly, $\Gamma_1, \dots, \Gamma_t$ are pairwise disjoint. For each $i \in \{1, \dots, t\}$, consider the support algebra $B^{(i)} = \operatorname{Supp}(\Gamma_i)$ of Γ_i.

Fix $i \in \{1, \dots, t\}$. We shall prove that $B^{(i)}$ is a generalized multicoil algebra and Γ_i is the cyclic part of a generalized multicoil Γ_i^* of $\Gamma_{B^{(i)}}$, and consequently Γ_i is a cyclic generalized multicoil full translation subquiver of $\Gamma_{B^{(i)}}$. Since Γ_i is a cyclic coherent full translation subquiver of the component $\mathcal{C}$ of Γ_A and of Γ, it follows from the proofs of Theorems A and F in [44] that Γ_i, considered as a translation quiver, is a generalized multicoil, and consequently can be obtained from a finite family $\mathcal{T}_1^{(i)}, \dots, \mathcal{T}_{p_i}^{(i)}$ of stable tubes by an iterated application of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1*)-(ad 5*). We note that all vertices of the stable tubes $\mathcal{T}_1^{(i)}, \dots, \mathcal{T}_{p_i}^{(i)}$ are indecomposable modules of Γ, and the stable tubes $\mathcal{T}_1^{(i)}, \dots, \mathcal{T}_{p_i}^{(i)}$ can be obtained from Γ by removing the modules of $\Gamma \setminus (\mathcal{T}_1^{(i)} \cup \ldots \cup \mathcal{T}_{p_i}^{(i)})$ and shrinking the corresponding sectional paths in Γ with the ends at the modules in $\mathcal{T}_1^{(i)} \cup \ldots \cup \mathcal{T}_{p_i}^{(i)}$ into the arrows. We claim now that Γ_i is a generalized standard full translation subquiver of Γ_A. Suppose that $\operatorname{rad}^\infty(X, Y) \neq 0$ for some indecomposable A-modules X and Y lying in Γ_i. Then, applying Proposition 2.1, we conclude that there is in $\operatorname{ind} A$ an infinite cycle

$$X \xrightarrow{f} Y \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2 \cdots \to Z_{t-1} \xrightarrow{f_t} Z_t = X$$

where $Z_1, \dots, Z_t = X, Y$ are modules in Γ_i, $f_1, \dots, f_t$ are irreducible homomorphisms and $0 \neq f \in \operatorname{rad}^\infty(X, Y)$, a contradiction with the cycle-finiteness of Γ. Similarly, there is no path in $\operatorname{ind} B^{(i)}$ of the form

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

with X and Y in Γ_i and Z not in Γ_i (external short path of Γ_i in $\operatorname{ind} B$ in the sense of [56]). Since Γ_i is a sincere cyclic coherent full translation subquiver of $\Gamma_{B^{(i)}}$, applying [45, Theorem A] (and its proof), we conclude that $B^{(i)}$ is a generalized multicoil algebra, Γ_i is the cyclic part of a generalized multicoil Γ_i^* of $\Gamma_{B^{(i)}}$, and $\operatorname{ann}_{B^{(i)}}(\Gamma_i) = \operatorname{ann}_{B^{(i)}}(\Gamma_i^*) = 0$, and hence $B^{(i)} = B(\Gamma_i) = B(\Gamma_i^*)$. For each $j \in \{1, \dots, p_i\}$, consider the quotient algebra $C_j^{(i)} = A/\operatorname{ann}_A(\mathcal{T}_j^{(i)})$ of A by the annihilator $\operatorname{ann}_A(\mathcal{T}_j^{(i)})$ of the family of indecomposable A-modules forming $\mathcal{T}_j^{(i)}$. Then $C_j^{(i)}$ is a concealed canonical algebra and $\mathcal{T}_j^{(i)}$ is a stable tube of $\Gamma_{C_j^{(i)}}$. We note that we may have $C_j^{(i)} = C_k^{(i)}$ for $j \neq k$ in $\{1, \dots, p_i\}$. Then denoting by $C^{(i)}$ the product of pairwise different algebras in the family $C_1^{(i)}, \dots, C_{p_i}^{(i)}$, with respect to the annihilators $\operatorname{ann}_A(\mathcal{T}_1^{(i)}), \dots, \operatorname{ann}_A(\mathcal{T}_{p_i}^{(i)})$ of $\mathcal{T}_1^{(i)}, \dots, \mathcal{T}_{p_i}^{(i)}$, we obtain that $B^{(i)}$ is a generalized multicoil enlargement of $C^{(i)}$ involving the stable tubes $\mathcal{T}_1^{(i)}, \dots, \mathcal{T}_{p_i}^{(i)}$ and admissible operations of types (ad 1)-(ad 5) and (ad 1*)-(ad 5*) corresponding to the translation quiver operations leading from the stable tubes $\mathcal{T}_1^{(i)}, \dots, \mathcal{T}_{p_i}^{(i)}$ to the generalized multicoil Γ_i^*. Further, by [45, Theorem C], we have the following additional properties of $B^{(i)}$:

(1) There is a unique factor algebra (not necessarily connected) $B_l^{(i)}$ of $B^{(i)}$ (the left part of $B^{(i)}$) obtained from $C^{(i)}$ by an iteration of admissible operations of type (ad 1*) and a family $\widehat{\mathcal{T}}_1^{(i)}, \dots, \widehat{\mathcal{T}}_{p_i}^{(i)}$ of coray tubes in $\Gamma_{B_l^{(i)}}$, obtained from the stable tubes $\mathcal{T}_1^{(i)}, \dots, \mathcal{T}_{p_i}^{(i)}$ by the corresponding coray insertions, such that $B^{(i)}$ is obtained from $B_l^{(i)}$ by an iteration of admissible operations of types (ad 1)-(ad 5) and Γ_i^* is obtained from the family $\widehat{\mathcal{T}}_1^{(i)}, \dots, \widehat{\mathcal{T}}_{p_i}^{(i)}$ by an iteration of admissible operations of types (ad 1)-(ad 5) corresponding to those leading from $B_l^{(i)}$ to $B^{(i)}$.

(2) There is a unique factor algebra (not necessarily connected) $B_r^{(i)}$ of $B^{(i)}$ (the right part of $B^{(i)}$) obtained from $C^{(i)}$ by an iteration of admissible operations of type (ad 1) and a family $\widetilde{\mathcal{T}}_1^{(i)}, \dots, \widetilde{\mathcal{T}}_{p_i}^{(i)}$ of ray tubes in $\Gamma_{B_r^{(i)}}$, obtained from the stable tubes $\mathcal{T}_1^{(i)}, \dots, \mathcal{T}_{p_i}^{(i)}$ by the corresponding ray insertions, such that $B^{(i)}$ is obtained from $B_r^{(i)}$ by an iteration of admissible operations of types (ad 1*)-(ad 5*) and Γ_i^* is obtained from the family $\widetilde{\mathcal{T}}_1^{(i)}, \dots, \widetilde{\mathcal{T}}_{p_i}^{(i)}$ by an iteration of admissible operations of types (ad 1*)-(ad 5*) corresponding to those leading from $B_r^{(i)}$ to $B^{(i)}$.

As a consequence, the generalized multicoil Γ_i^* of $\Gamma_{B^{(i)}}$ admits a left border $\Delta_l^{(i)}$ and a right border $\Delta_r^{(i)}$ having the following properties:

(a) $\Delta_l^{(i)}$ and $\Delta_r^{(i)}$ are disjoint and unions of finite sectional paths of Γ_i;
(b) Γ_i is the full translation subquiver of Γ_i^* consisting of all modules which are both successors of modules lying in $\Delta_l^{(i)}$ and predecessors of modules lying in $\Delta_r^{(i)}$;
(c) $\Gamma_i^* \setminus \Gamma_i$ consists of a finite number of directing $B^{(i)}$-modules;
(d) Every module in $\Gamma \setminus \Gamma_i$ which is a predecessor of a module in Γ_i is a predecessor of a module in $\Delta_l^{(i)}$;

(e) Every module in $\Gamma \setminus \Gamma_i$ which is a successor of a module in Γ_i is a successor of a module in $\Delta_r^{(i)}$;

(f) $B(\Delta_l^{(i)}) = \mathrm{Supp}(\Delta_l^{(i)})$ and is a product of tilted algebras of equioriented Dynkin types $\mathbb{A}_n$ and $\Delta_l^{(i)}$ is the union of sections of the connecting components of the indecomposable parts of $B(\Delta_l^{(i)})$;

(g) $B(\Delta_r^{(i)}) = \mathrm{Supp}(\Delta_r^{(i)})$ and is a product of tilted algebras of equioriented Dynkin types $\mathbb{A}_n$ and $\Delta_r^{(i)}$ is the union of sections of the connecting components of the indecomposable parts of $B(\Delta_r^{(i)})$.

We denote by Γ^{cc} the union of the translation subquivers $\Gamma_1, \dots, \Gamma_t$. We claim that $\Gamma \setminus \Gamma^{cc}$ consists of finitely modules and Γ^{cc} is a maximal cyclic coherent full translation subquiver of Γ. Suppose that infinitely many modules of Γ are not contained in Γ^{cc}. We have the following properties of modules in $\Gamma \setminus \Gamma^{cc}$. Since Γ is a connected component of ${}_c\Gamma_A$, by Proposition 2.1, for any modules M in $\Gamma \setminus \Gamma^{cc}$ and N in Γ^{cc}, there is an oriented cycle in Γ passing through M and N. Moreover, if N belongs to Γ_i, then every such a cycle is of the form

$$M \to \dots \to X \to \dots \to N \to \dots \to Y \to \dots \to M$$

with X in $\Delta_l^{(i)}$ and Y in $\Delta_r^{(i)}$. Applying Proposition 2.7 to the infinite family $\Sigma = \Gamma \setminus \Gamma^{cc}$ of cycle-finite modules, we obtain that the left stable part ${}_l\mathcal{C}$ or the right stable part ${}_r\mathcal{C}$ of $\mathcal{C}$ admits an infinite component Σ' containing an oriented cycle and infinitely many modules from $\Gamma \setminus \Gamma^{cc}$. Then, as in the first part of the proof, we infer that there exists a cyclic coherent full translation subquiver Ω' of Γ containing all modules of Σ'. Obviously, Ω' is disjoint with $\Gamma_1, \dots, \Gamma_t$, and this contradicts to our choice of $\Gamma_1, \dots, \Gamma_t$. Therefore, indeed, $\Gamma \setminus \Gamma^{cc}$ consists of finitely many modules.

Our next aim is to show that the algebra $B(\Gamma \setminus \Gamma^{cc}) = A/\mathrm{ann}_A(\Gamma \setminus \Gamma^{cc})$ is of finite representation type. We abbreviate $D = B(\Gamma \setminus \Gamma^{cc})$. Observe that, if every indecomposable module from $\mathrm{mod}\, D$ lies in $\Gamma \setminus \Gamma^{cc}$, then D is of finite representation type. Therefore, assume that $\mathrm{mod}\, D$ admits an indecomposable module Z which is not in $\Gamma \setminus \Gamma^{cc}$. Let M be the direct sum of all indecomposable A-modules lying in $\Gamma \setminus \Gamma^{cc}$. Moreover, let $D = P' \oplus P''$ be a decomposition of D in $\mathrm{mod}\, D$, where P' is the direct sum of all indecomposable projective D-modules lying in $\Gamma \setminus \Gamma^{cc}$ and P'' is the direct sum of the remaining indecomposable projective D-modules. Observe that M is a faithful module in $\mathrm{mod}\, D$ and hence we have a monomorphism of right D-modules $P'' \to M^t$, which then factors through a direct sum of modules lying on the sum $\Delta_r^{(1)} \cup \ldots \cup \Delta_r^{(t)}$ of the right parts $\Delta_r^{(1)}, \dots, \Delta_r^{(t)}$ of $\Gamma_1, \dots, \Gamma_t$, and consequently P'' is a module over the algebra $B(\Delta_r^{(1)}) \times \ldots \times B(\Delta_r^{(t)})$. Consider also a projective cover $\pi : P_D(Z) \to Z$ of Z in $\mathrm{mod}\, D$. Let $P_D(Z) = P'_D(Z) \oplus P''_D(Z)$, where $P'_D(Z)$ is a direct sum of direct summands of P' and $P''_D(Z)$ is a direct sum of direct summands of P'', and denote by $\pi' : P'_D(Z) \to Z$ and $\pi'' : P''_D(Z) \to Z$ the restrictions of π to $P'_D(Z)$ and $P''_D(Z)$, respectively. Then $\pi' : P'_D(Z) \to Z$ factors through a direct sum of modules lying on the sum $\Delta_l^{(1)} \cup \ldots \cup \Delta_l^{(t)}$ of the left parts $\Delta_l^{(1)}, \dots, \Delta_l^{(t)}$ of $\Gamma_1, \dots, \Gamma_t$, because Z does not belong to $\Gamma \setminus \Gamma^{cc}$. In particular, we obtain that $\pi'(P'_D(Z))$ is a module over the algebra $B(\Delta_l^{(1)}) \times \ldots \times B(\Delta_l^{(t)})$. Summing up, we conclude that $Z = \pi'(P'_D(Z)) + \pi''(P''_D(Z))$

is a module over the quotient algebra

$$\Lambda = B(\Delta_l^{(1)}) \times \ldots \times B(\Delta_l^{(t)}) \times B(\Delta_r^{(1)}) \times \ldots \times B(\Delta_r^{(t)}),$$

of A, which is an algebra of finite representation type as a product of tilted algebras of Dynkin types $\mathbb{A}_n$. Therefore, we obtain that every module from $\operatorname{ind} D$ which is not in $\Gamma \setminus \Gamma^{cc}$ is an indecomposable module in $\operatorname{ind} \Lambda$. Since $\Gamma \setminus \Gamma^{cc}$ is finite, we conclude that D is of finite representation type.

Finally, let $B = \operatorname{Supp}(\Gamma) = A/t_A(\Gamma)$. Then Γ is a sincere cycle-finite component of ${}_c\Gamma_B$ and $\operatorname{ann}_B(\Gamma) = \operatorname{ann}_A(\Gamma)/t_A(\Gamma)$. Hence, in order to show that $\operatorname{Supp}(\Gamma) = B(\Gamma)$, it is enough to prove that Γ is a faithful translation subquiver of Γ_B. Let $B = P \oplus Q$ be a decomposition in $\operatorname{mod} B$ such that Q is the direct sum of all indecomposable projective modules lying in Γ and P the direct sum of the remaining indecomposable projective right B-modules. Then P is a direct sum of indecomposable projective modules over the product $B^{(1)} \times \ldots \times B^{(t)}$ of generalized multicoil algebras $B^{(1)}, \ldots, B^{(t)}$. Since $B^{(i)} = \operatorname{Supp}(\Gamma_i) = B(\Gamma_i)$ for any $i \in \{1, \ldots, t\}$, we conclude that there is a monomorphism $P \to N^m$ for a module N in $\operatorname{mod} B$ being a direct sum of indecomposable modules lying in $\Gamma^{cc} = \Gamma_1 \cup \ldots \cup \Gamma_t$ and a positive integer m. Clearly, then there is a monomorphism in $\operatorname{mod} B$ of the form $B = P \oplus Q \to (N \oplus Q)^m$, and consequently Γ is a faithful component of Γ_B. Therefore, we obtain the equality $\operatorname{Supp}(\Gamma) = B(\Gamma)$.

4. Proof of Theorem 1.2

Let A be an algebra and Γ be a cycle-finite finite component of ${}_c\Gamma_A$. Moreover, let $B = A/t_A(\Gamma)$ be the support algebra of Γ. Observe that Γ is a sincere cycle-finite component of ${}_c\Gamma_B$. We will show that B is a generalized double tilted algebra, applying [75, Theorem]. Since Γ is a finite component of ${}_c\Gamma_B$, it follows from Corollary 2.6 that Γ contains a projective module and an injective module. Hence, applying Proposition 2.1, we conclude that there exists in Γ a path from an injective module to a projective module. Let

$$I = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m = P$$

be an arbitrary path in $\operatorname{ind} B$ from an indecomposable injective module I to an indecomposable projective module P. Since Γ is a sincere translation subquiver of Γ_B, there exist indecomposable modules M and N in Γ such that $\operatorname{Hom}_B(P, M) \neq 0$ and $\operatorname{Hom}_B(N, I) \neq 0$. Further, it follows from Proposition 2.1 that there exists a path in $\operatorname{ind} B$ from M to N. Therefore, we obtain in $\operatorname{ind} B$ a cycle of the form

$$M \to \cdots \to N \to X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m \to M,$$

and this is a finite cycle, because M and N belong to the cycle-finite component Γ of ${}_c\Gamma_B$. This shows that all the modules $X_0, X_1, \ldots, X_{m-1}, X_m$ belong to the finite translation quiver Γ of Γ_B. Then it follows from [75, Theorem] that B is a quasitilted algebra or a generalized double tilted algebra. Furthermore, by [20, Corollary (E)], the Auslander-Reiten quiver of a quasitilted algebra which is not a tilted algebra consists of semiregular components. Clearly, every tilted algebra is a generalized double tilted algebra [58]. Since the cyclic component Γ of Γ_B contains a path from an injective module to a projective module, we obtain that B is a generalized double tilted algebra.

Hence, it follows from [58, Section 3] that Γ_B admits an almost acyclic component $\mathcal{C}$ with a faithful multisection Δ. Recall that, following [58, Section 2], a full connected subquiver Δ of $\mathcal{C}$ is called a *multisection* if the following conditions are satisfied:

(i) Δ is almost acyclic.
(ii) Δ is convex in $\mathcal{C}$.
(iii) For each τ_B-orbit $\mathcal{O}$ in $\mathcal{C}$, we have $1 \le |\Delta \cap \mathcal{O}| < \infty$.
(iv) $|\Delta \cap \mathcal{O}| = 1$ for all but finitely many τ_B-orbits $\mathcal{O}$ in $\mathcal{C}$.
(v) No proper full convex subquiver of Δ satisfies (i)–(iv).

Moreover, for a multisection Δ of a component $\mathcal{C}$, the following full subquivers of $\mathcal{C}$ were defined in [58]:

$$\Delta'_l = \{X \in \Delta; \text{there is a nonsectional path in } \mathcal{C} \text{ from } X \text{ to a projective module } P\},$$
$$\Delta'_r = \{X \in \Delta; \text{there is a nonsectional path in } \mathcal{C} \text{ from an injective module } I \text{ to } X\},$$
$$\Delta''_l = \{X \in \Delta'_l; \tau_A^{-1}X \notin \Delta'_l\}, \qquad \Delta''_r = \{X \in \Delta'_r; \tau_A X \notin \Delta'_r\},$$
$$\Delta_l = (\Delta \setminus \Delta'_r) \cup \tau_A \Delta''_r, \quad \Delta_c = \Delta'_l \cap \Delta'_r, \quad \Delta_r = (\Delta \setminus \Delta'_l) \cup \tau_A^{-1} \Delta''_l.$$

Then Δ_l is called the *left part* of Δ, Δ_r the *right part* of Δ, and Δ_c the *core* of Δ.

The following basic properties of Δ have been established in [58, Proposition 2.4]:

(a) Every cycle of $\mathcal{C}$ lies in Δ_c.
(b) Δ_c is finite.
(c) Every indecomposable module X in $\mathcal{C}$ is in Δ_c, or a predecessor of Δ_l or a successor of Δ_r in $\mathcal{C}$.

It follows also from [58, Theorem 3.4, Corollary 3.5] and the known structure of the Auslander-Reiten quivers of tilted algebras (see [27], [31], [60], [63]) that every component of Γ_B different from $\mathcal{C}$ is a semiregular component. Hence the cyclic component Γ is a translation subquiver of $\mathcal{C}$, and consequently is contained in the core Δ_c of Δ. We also know from [58, Proposition 2.11] that, for another multisection Σ of $\mathcal{C}$, we have $\Sigma_c = \Delta_c$. Thus Δ_c is a uniquely defined core $c(\mathcal{C})$ of the connecting component $\mathcal{C}$ of Γ_B. We claim that $\Gamma = c(\mathcal{C})$. Let X be a module in $\Delta_c = \Delta'_l \cap \Delta'_r$. Then there are nonsectional paths in $\mathcal{C}$ from X to an indecomposable projective module P and from an indecomposable injective module I to X. Moreover, there exist indecomposable modules Y and Z in Γ such that $\operatorname{Hom}_B(P, Y) \neq 0$ and $\operatorname{Hom}_B(Z, I) \neq 0$, because Γ is a sincere translation subquiver of Γ_B. Further, by Proposition 2.1, we have in Γ a path from Y to Z. Hence we obtain in $\operatorname{ind} B$ a cycle of the form

$$X \to \cdots \to P \to Y \to \cdots \to Z \to I \to \cdots \to X,$$

which is a finite cycle because Y and Z belong to the cycle-finite component Γ of ${}_c\Gamma_B$. Therefore, there is in $\mathcal{C}$ a cycle passing through the modules X, Y and Z, and so X belongs to Γ. This shows that $\Gamma = \Delta_c = c(\mathcal{C})$.

Let $B^{(l)} = \operatorname{Supp}(\Delta_l)$ be the support algebra of the left part Δ_l of Δ (if Δ_l is nonempty) and $B^{(r)} = \operatorname{Supp}(\Delta_r)$ be the support algebra of the right part Δ_r of Δ (if Δ_r is nonempty). Then the following description of $\operatorname{ind} B$ follows from the results established in [58, Section 3]:

(1) $B^{(l)}$ is a tilted algebra (not necessarily indecomposable) such that Δ_l is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(l)}$ and the category of all predecessors of Δ_l in $\operatorname{ind} B$ coincides with the category of all predecessors of Δ_l in $\operatorname{ind} B^{(l)}$, or $B^{(l)}$ is empty in case Δ_l is empty.
(2) $B^{(r)}$ is a tilted algebra (not necessarily indecomposable) such that Δ_r is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(r)}$ and the category of all successors of Δ_r in $\operatorname{ind} B$ coincides with the category of all successors of Δ_r in $\operatorname{ind} B^{(r)}$, or $B^{(r)}$ is empty in case Δ_r is empty.
(3) Every indecomposable module in $\operatorname{ind} B$ is either in $\Gamma = c(\mathcal{C})$, a predecessor of Δ_l in $\operatorname{ind} B$, or a successor of Δ_r in $\operatorname{ind} B$.
(4) If Δ_l is nonempty, then Δ_l is a faithful subquiver of $\Gamma_{B^{(l)}}$, and hence $B^{(l)}$ is the faithful algebra $B(\Delta_l) = A/\operatorname{ann}_A(\Delta_l)$ of Δ_l.
(5) If Δ_r is nonempty, then Δ_r is a faithful subquiver of $\Gamma_{B^{(r)}}$, and hence $B^{(r)}$ is the faithful algebra $B(\Delta_r) = A/\operatorname{ann}_A(\Delta_r)$ of Δ_r.

We will prove now that B coincides with the faithful algebra $B(\Gamma) = A/\operatorname{ann}_A(\Gamma)$ of Γ. Observe that $t_A(\Gamma) \subseteq \operatorname{ann}_A(\Gamma)$ and $\operatorname{ann}_B(\Gamma) = \operatorname{ann}_A(\Gamma)/t_A(\Gamma)$. Therefore, it is sufficient to show that Γ is a faithful subquiver of Γ_B. Let M_Γ be the direct sum of all indecomposable B-modules lying in Γ. Then M_Γ is a sincere module in $\operatorname{mod} B$, by definition of B. In order to show that M_Γ is a faithful B-module, it is enough to prove that there is a monomorphism $B \to (M_\Gamma)^n$ for some positive integer n. Let $B = P^{(l)} \oplus P^{(c)}$ be a decomposition of B in $\operatorname{mod} B$ such that the indecomposable direct summands of $P^{(c)}$ are exactly the indecomposable projective B-modules lying in the core $\Gamma = c(\mathcal{C})$ of $\mathcal{C}$. Clearly, $P^{(c)}$ is a direct summands of M_Γ, and hence there is a monomorphism $P^{(c)} \to M_\Gamma$ in $\operatorname{mod} B$. On the other hand, the indecomposable direct summands of $P^{(l)}$ form a complete family of pairwise nonisomorphic indecomposable projective right modules over the left tilted algebra $B^{(l)}$ of B. Hence, if $P^{(l)} = 0$, or equivalently the left part Δ_l of Δ is empty, then $B = P^{(l)}$ and M_Γ is a faithful module in $\operatorname{mod} B$, as required. Therefore, assume that Δ_l is nonempty.

Let $\Gamma^{(l)}$ be the family of all indecomposable modules X in Γ such that there is an arrow $Y \to X$ in $\mathcal{C}$ with Y from Δ_l. We claim that, for any module X in $\Gamma^{(l)}$, there exists an indecomposable projective module P in Γ such that $\operatorname{Hom}_B(P, X) \neq 0$. We may assume that X is not projective. Then $\tau_B X$ is an indecomposable module not lying in Γ, because we have a path $\tau_B X \to Y \to X$ in $\mathcal{C}$, with X in the cyclic component Γ of Γ_B and Y not in Γ. Observe that then $\tau_B X \in \Delta_l$ because X is in $\Gamma = \Delta_c = \Delta_l' \cap \Delta_r'$. Consider now an oriented cycle in Γ

$$X = X_0 \to X_1 \to \cdots \to X_{r-1} \to X_r = X$$

passing through X. It follows from Proposition 2.4 that there exists $i \in \{2, \ldots, r\}$ such that $\tau_B X_i \cong X_{i-2}$. Since $\tau_B X$ does not belong to Γ, we then conclude that there is in Γ a sectional path

$$X_s \to X_{s+1} \to \cdots \to X_{r-1} \to X_r = X$$

with $X_s = P$ an indecomposable projective module. Hence we obtain that $\mathrm{Hom}_B(P, X) \neq 0$, because the composition of irreducible homomorphisms in $\mathrm{mod}\, B$ corresponding to arrows of a sectional path in Γ_B is nonzero, by a theorem of Bautista and Smalø [11] (see also [79, Theorem III.11.2]). Observe also that, for any module Y lying on Δ_l, we have $\mathrm{Hom}_B(P^{(c)}, Y) = 0$, because Y is a module in $\mathrm{mod}\, B^{(l)}$. This leads to the following property of modules in $\Gamma^{(l)}$: any irreducible homomorphism $f : Y \to X$ with X in $\Gamma^{(l)}$ and Y in Δ_l is a monomorphism.

Consider now the family $\Omega^{(l)}$ of all indecomposable modules Y in Δ_l such that there is an arrow in $\mathcal{C}$ from Y to a module X in Γ, and hence in $\Gamma^{(l)}$. Moreover, let $M^{(l)}$ be the direct sum of all indecomposable modules in $\Omega^{(l)}$. Observe that $M^{(l)}$ is a right $B^{(l)}$-module. Moreover, for any module Y in $\Omega^{(l)}$, there is an irreducible monomorphism $Y \to X$ in $\mathrm{mod}\, B$ with X lying in $\Gamma^{(l)}$. This implies that there is a monomorphism in $\mathrm{mod}\, B$ of the form $M^{(l)} \to (M_\Gamma)^m$ for some positive integer m. We will prove that $M^{(l)}$ is a faithful right $B^{(l)}$-module.

Let P be an indecomposable projective module in $\mathrm{mod}\, B^{(l)}$, or equivalently, an indecomposable direct summand of $P^{(l)}$. Since M_Γ is a sincere module in $\mathrm{mod}\, B$, we conclude that there is an indecomposable module Z in Γ such that $\mathrm{Hom}_B(P, Z) \neq 0$. Further, the radical $\mathrm{rad}\mathrm{End}_B(M_\Gamma)$ of the endomorphism algebra $\mathrm{End}_B(M_\Gamma)$ in nilpotent. Then there exist a path of irreducible homomorphisms

$$Z_{t+1} \xrightarrow{g_{t+1}} Z_t \xrightarrow{g_t} Z_{t-1} \to \cdots \to Z_2 \xrightarrow{g_2} Z_1 \xrightarrow{g_1} Z_0 = Z$$

and a homomorphism $v_{t+1} : P \to Z_{t+1}$ in $\mathrm{mod}\, B$ with $g_1 g_2 \dots g_t g_{t+1} v_{t+1} \neq 0$, $Z_0, Z_1, \dots, Z_t$ indecomposable modules in Γ and Z_{t+1} an indecomposable module in Δ_l (see [79, Proposition III.10.1]). This implies that $\mathrm{Hom}_B(P, M^{(l)}) \neq 0$ because Z_{t+1} is a direct summand of $M^{(l)}$. Therefore, $M^{(l)}$ is a sincere right $B^{(l)}$-module. We know also that $B^{(l)}$ is a tilted algebra and Δ_l is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(l)}$ and the category of all predecessors of Δ_l in $\mathrm{ind}\, B$ coincides with the category of all predecessors of Δ_l in $\mathrm{ind}\, B^{(l)}$. Then we conclude that, for any indecomposable module L in $\mathrm{ind}\, B^{(l)}$, we have

$$\mathrm{Hom}_{B^{(l)}}(L, M^{(l)}) = 0 \ \text{ or } \ \mathrm{Hom}_{B^{(l)}}(M^{(l)}, \tau_{B^{(l)}} L) = 0.$$

Summing up, we proved that $M^{(l)}$ is a sincere module in $\mathrm{mod}\, B^{(l)}$ which is not the middle of a short chain in the sense of [59] (see also [8]). Then it follows from [59, Corollary 3.2] that $M^{(l)}$ is a faithful module in $\mathrm{mod}\, B^{(l)}$. Hence, there exists a monomorphism $B^{(l)} \to (M^{(l)})^s$ in $\mathrm{mod}\, B^{(l)}$ for some positive integer s.

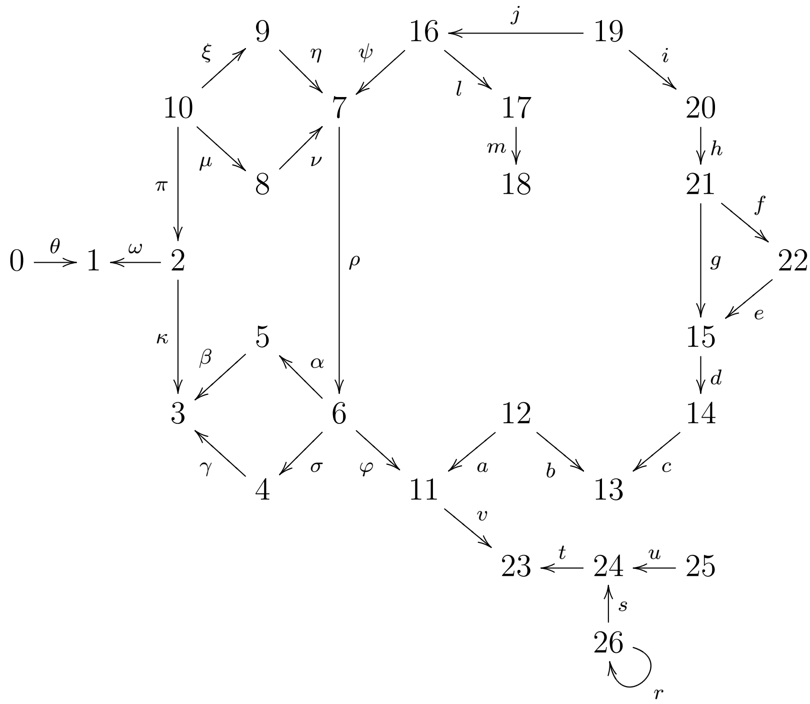
Finally, since there exist monomorphisms $M^{(l)} \to (M_\Gamma)^m$ and $P^{(l)} \to (M_\Gamma)$, and $B = P^{(l)} \oplus P^{(c)}$ with $P^{(l)} = B^{(l)}$ in $\mathrm{mod}\, B$, we obtain that there is a monomorphism in $\mathrm{mod}\, B$ of the form $B \to (M_\Gamma)^n$ for some positive integer n. Therefore, M_Γ is a faithful module in $\mathrm{mod}\, B$, and consequently $B = B(\Gamma)$. This finishes the proof of the theorem.

In connection with the final part of the above proof, we mention that, by a recent result proved by Jaworska, Malicki and Skowroński in [28], an algebra A is a tilted algebra if and only if there exists a sincere module M in $\mathrm{mod}\, A$ such that for any module X in $\mathrm{ind}\, A$, we have $\mathrm{Hom}_A(X, M) = 0$ or $\mathrm{Hom}_A(M, \tau_A X) = 0$. Moreover, all modules M in a module category $\mathrm{mod}\, A$ not being the middle of short chains have been described completely in [29].

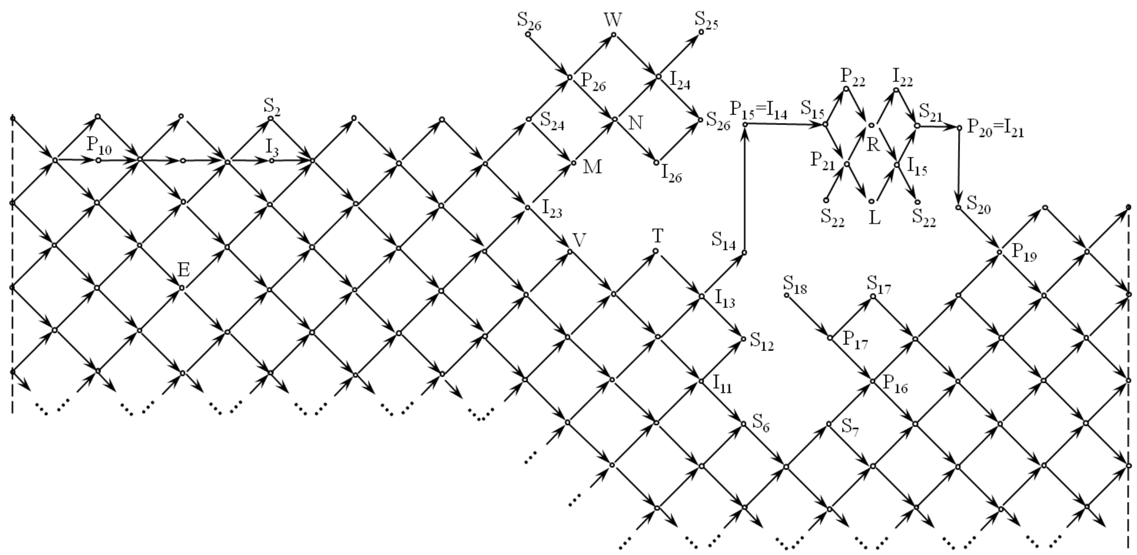
5. Examples: infinite cyclic components

In this section we present examples illustrating Theorem 1.1.

Example 5.1. Let K be a field and $A = KQ/I$ the bound quiver algebra given by the quiver Q of the form



and I the ideal in the path algebra KQ of Q over K generated by the elements $\alpha\beta-\sigma\gamma$, $\xi\eta-\mu\nu$, $\pi\kappa-\xi\eta\rho\alpha\beta$, $\rho\varphi$, $\psi\rho$, jl, dc, ed, gd, hg, hf, ih, av, rs, st, r^2. Then A is a cycle-finite algebra and Γ_A admits a component $\mathcal{C}$ of the form



The cyclic part ${}_c\mathcal{C}$ of $\mathcal{C}$ consists of one infinite component Γ and one finite component Γ' described as follows. The infinite cyclic component Γ is obtained by removing from $\mathcal{C}$ the modules $S_{12}, S_{17}, S_{18}, P_{17}$, S_{24}, M, S_{26}, P_{26}, N, I_{26}, W, I_{24}, S_{25}, and the arrows attached to them. The finite cyclic component Γ' is the full translation subquiver of $\mathcal{C}$ given by the vertices $S_{26}, P_{26}, N, I_{26}, W, I_{24}$. The maximal cyclic coherent part Γ^{cc} of Γ is the full translation subquiver of $\mathcal{C}$ obtained by removing from $\mathcal{C}$ the modules S_{12}, I_{13}, T, S_{14}, $P_{15} = I_{14}$, S_{15}, P_{21}, S_{22}, L, P_{22}, R, I_{15}, I_{22}, S_{21}, $P_{20} = I_{21}$, S_{20}, S_{17}, P_{17}, S_{18}, S_{24}, M, S_{26}, P_{26}, N, I_{26}, W, I_{24}, S_{25}, and the arrows attached to them. Further, Γ^{cc} is the cyclic part of the maximal almost cyclic coherent full translation subquiver Γ^* of $\mathcal{C}$ obtained by removing from $\mathcal{C}$ the modules $P_{15} = I_{14}$, S_{15}, P_{21}, S_{22}, L, P_{22}, R, I_{15}, I_{22}, S_{21}, $P_{20} = I_{21}$, S_{26}, P_{26}, I_{26}, W, N, I_{24} and the arrows attached to them, and shrinking the sectional path $M \to N \to I_{24} \to S_{25}$ to the arrow $M \to S_{25}$.

Let $B = A/\mathrm{ann}_A(\Gamma)$. Then $B = A/\mathrm{ann}_A(\Gamma^*)$, because $\mathrm{ann}_A(\Gamma) = \mathrm{ann}_A(\Gamma^*)$. Observe that $B = KQ_B/I_B$, where Q_B is the full subquiver of Q given by all vertices of Q except 15, 21, 22, 26, and $I_B = I \cap KQ_B$. We claim that B is a tame generalized multicoil algebra. Consider the path algebra $C = K\Sigma$ of the full subquiver Σ of Q given by the vertices 4, 5, 6, 7, 8, 9. Then C is a hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_5$, and hence a tame concealed algebra. It is known that Γ_C admits an infinite family $\mathcal{T}^C_\lambda$, $\lambda \in \Lambda(C)$, of pairwise orthogonal generalized standard stable tubes, having a unique stable tube, say $\mathcal{T}^C_1$, of rank 3 with the mouth formed by the modules $S_6 = \tau_C S_7$, $S_7 = \tau_C E$, $E = \tau_C S_6$, where E is the unique indecomposable C-module with the dimension vector $\underline{\dim} E = \begin{smallmatrix} 1 & \\ & 1 \\ 1 & \\ 1 & \\ & 1 \\ 1 & \end{smallmatrix}$, (see [24, Section 6] and [62, Theorem XIII 2.9]).

Then B is the generalized multicoil enlargement of C, obtained by applications of the following admissible operations:

- two admissible operations of types (ad 1^*) with the pivots S_6 and S_{12}, creating the vertices 11, 12, 13, 14 and the arrows φ, a, b, c;
- two admissible operations of types (ad 1^*) with the pivots E and S_2, creating the vertices 3, 2, 1, 0 and the arrows β, γ, κ, ω, θ;
- two admissible operations of types (ad 1) with the pivots S_7 and S_{16}, creating the vertices 16, 17, 18, 19, 20 and the arrows ψ, l, m, j, i;
- one admissible operation of type (ad 3) with the pivot the radical of P_{10}, creating the vertex 10 and the arrows ξ, μ, π;
- one admissible operation of type (ad 1^*) with the pivot V being the unique indecomposable module of dimension 2 having S_{11} as the socle and S_6 as the top, creating the vertices 23, 24, 25 and the arrows v, t, u.

Then the left part $B^{(l)}$ of B is the convex subcategory of B (and of A) given by the vertices 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 23, 24, 25, and is a tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_{16}$ with the connecting postprojective component $\mathcal{P}^{B^{(l)}}$ containing all indecomposable projective $B^{(l)}$-modules. The right part $B^{(r)}$ of B is the convex subcategory of B (and of A) given by the vertices 0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 17, 18, 19, 20, and is a tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_{14}$ with the connecting preinjective component $\mathcal{Q}^{B^{(r)}}$ containing all indecomposable injective $B^{(r)}$-modules. We also note that the left border Δ_l of the generalized multicoil Γ^* of Γ_B is given by the quivers $P_{17} \to S_{17}$ and S_{20}, and the right border Δ_r of Γ^* is given by the quivers $T \to I_{13} \to S_{12}$ and $S_{24} \to M$. Further, the algebra $B(\Gamma \setminus \Gamma^{cc}) = A/\mathrm{ann}_A(\Gamma \setminus \Gamma^{cc})$ is the disjoint union of three representation-finite convex subcategories of A: D_1 given by the vertices 12, 13, 14, 15, 20, 21, 22, D_2 given by the vertices 17, 18, and D_3 given by the vertices 24, 25, 26. We note that D_3 is the faithful algebra $B(\Gamma')$ of the finite cyclic component Γ'. It follows from [45, Theorems C and F] that the Auslander-Reiten quiver Γ_B of the generalized multicoil enlargement B of C is of the form

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{C}^B \cup \mathcal{Q}^B,$$

where $\mathcal{P}^B = \mathcal{P}^{B^{(l)}}$, $\mathcal{Q}^B = \mathcal{Q}^{B^{(r)}}$, and $\mathcal{C}^B$ is the family $\mathcal{C}^B_\lambda$, $\lambda \in \Lambda(C)$, of pairwise orthogonal generalized multicoils such that $\mathcal{C}^B_1 = \Gamma^*$ and $\mathcal{C}^B_\lambda = \mathcal{T}^C_\lambda$ for all $\lambda \in \Lambda(C)\backslash\{1\}$. Hence Γ_A is of the form

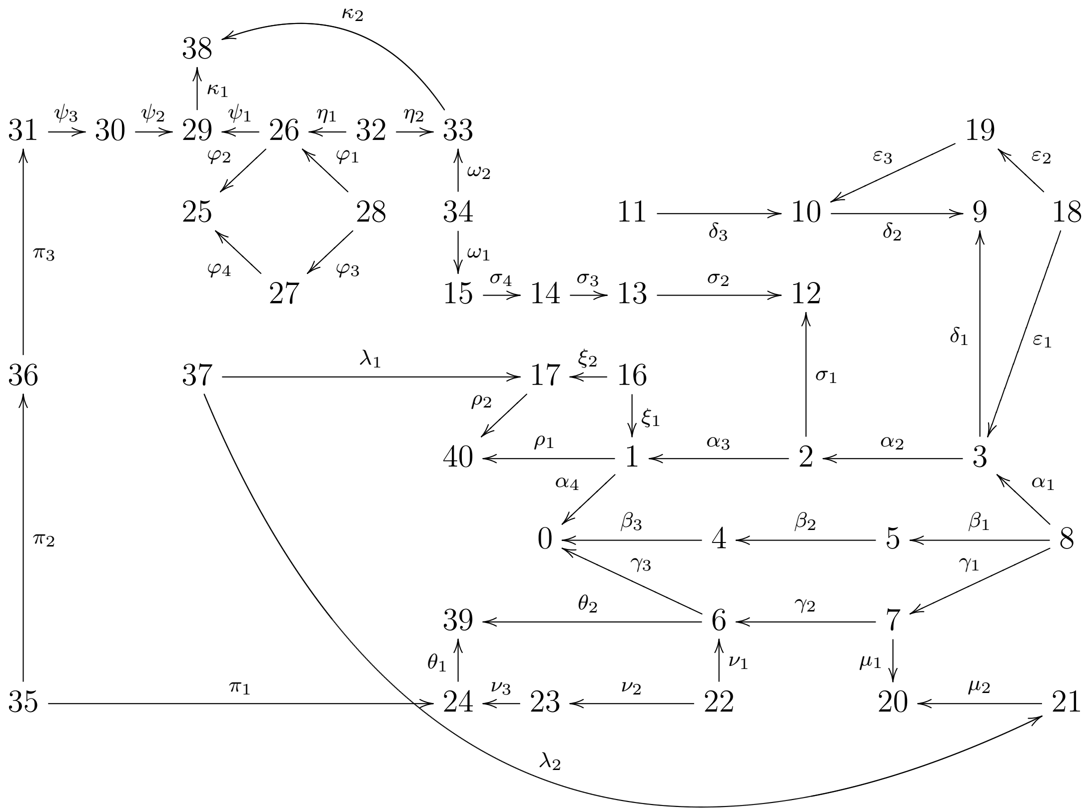
$$\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A,$$

where $\mathcal{P}^A = \mathcal{P}^{B^{(l)}}$, $\mathcal{Q}^A = \mathcal{Q}^{B^{(r)}}$, and $\mathcal{C}^A$ is the family $\mathcal{C}^A_\lambda$, $\lambda \in \Lambda(C)$, of pairwise orthogonal generalized standard components such that $\mathcal{C}^A_1 = \mathcal{C}$, $\mathcal{C}^A_\lambda = \mathcal{T}^C_\lambda$ for all $\lambda \in \Lambda(C) \setminus \{1\}$. Moreover, we have

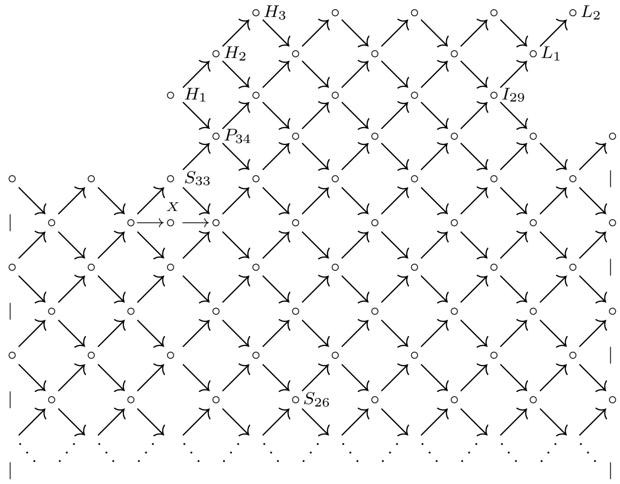
$$\mathrm{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0, \mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0, \mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0.$$

In particular, A is a cycle-finite algebra with $(\mathrm{rad}^\infty_A)^3 = 0$.

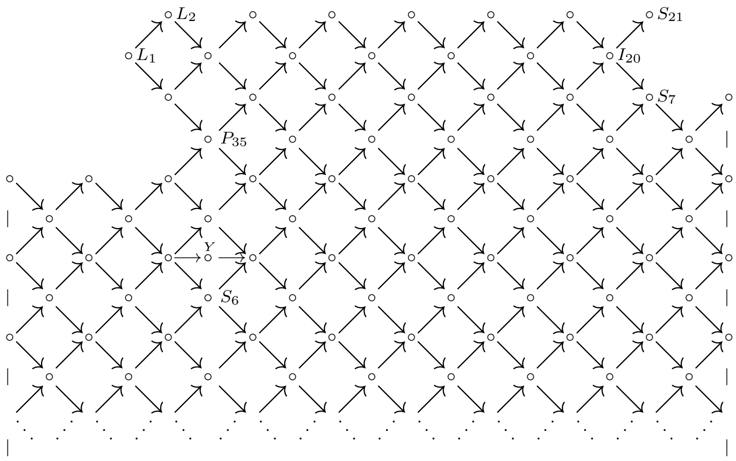
Example 5.2. Let K be a field and $B = KQ/J$ the bound quiver algebra given by the quiver Q of the form



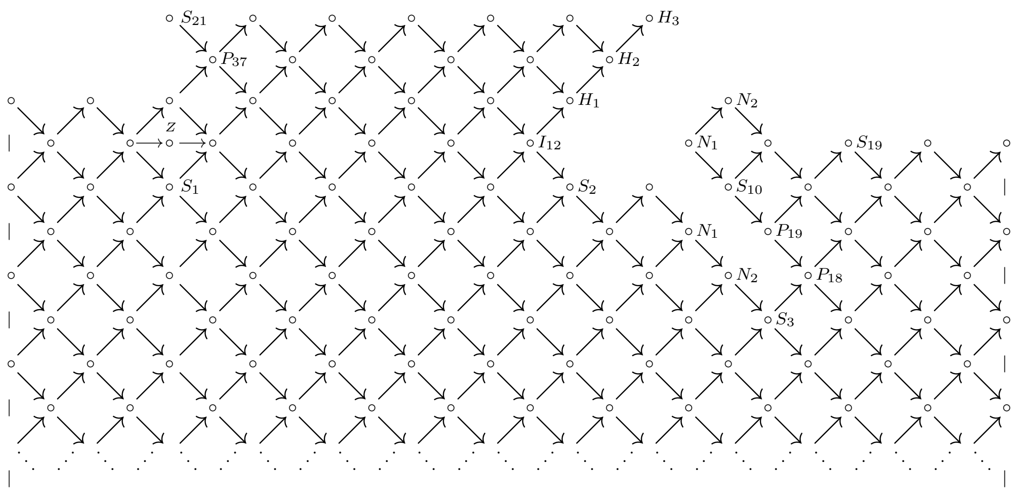
and J the ideal in the path algebra KQ of Q over K generated by the elements $\varphi_1\psi_1$, $\eta_1\varphi_2$, $\omega_1\sigma_4\sigma_3\sigma_2$, $\pi_3\psi_3\psi_2$, $\lambda_2\mu_2$, $\eta_2\kappa_2 - \eta_1\psi_1\kappa_1$, $\psi_2\kappa_1$, $\omega_2\kappa_2$, $\gamma_2\theta_2$, $\pi_1\theta_1$, $\nu_1\theta_2 - \nu_2\nu_3\theta_1$, $\alpha_3\rho_1$, $\lambda_1\rho_2$, $\xi_1\rho_1 - \xi_2\rho_2$, $\alpha_1\alpha_2\alpha_3\alpha_4 + \beta_1\beta_2\beta_3 + \gamma_1\gamma_2\gamma_3$, $\alpha_1\delta_1$, $\alpha_2\sigma_1$, $\xi_1\alpha_4$, $\varepsilon_1\delta_1$, $\varepsilon_1\alpha_2$, $\varepsilon_3\delta_2$, $\gamma_1\mu_1$, $\nu_1\gamma_3$. Denote by P_k, I_k, S_k the indecomposable projective module, the indecomposable injective module, and the simple module in $\mathrm{mod}\, B$ at the vertex k of Q. Then Γ_B admits a cyclic component $\mathcal{C}$ obtained by identification the sectional paths $H_1 \to H_2 \to H_3$, $L_1 \to L_2$, $N_1 \to N_2$ and the module S_{21} occurring in the following three translation quivers: $\mathcal{C}_1$ of the form



$\mathcal{C}_2$ of the form



and $\mathcal{C}_3$ of the form

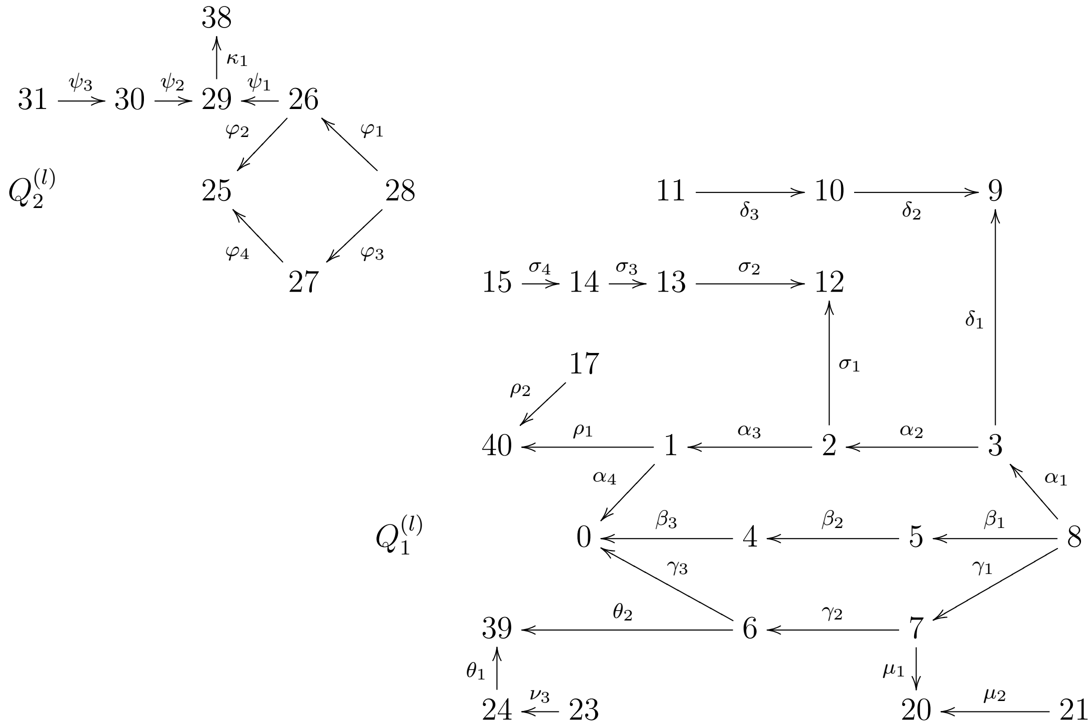


where $X = I_{38} = P_{32}$, $Y = I_{39} = P_{22}$, $Z = I_{40} = P_{16}$, $N_2 = I_9$ and the vertical dashed lines have to be identified in order to obtain the translation quivers $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. We claim that B is a generalized multicoil algebra. Denote by Q_C the full subquiver of Q given by the vertices $0, 1, 2, 3, 4, 5, 6, 7, 8$. Consider the bound quiver algebra $C = KQ_C/J_C$ with J_C the ideal in KQ_C generated by $\alpha_1\alpha_2\alpha_3\alpha_4 + \beta_1\beta_2\beta_3 + \gamma_1\gamma_2\gamma_3$ and the path algebra $D = KQ_D$ of the full subquiver Q_D of Q given by the vertices $25, 26, 27, 28$. Then C is a canonical algebra of wild type and D is a canonical algebra of Euclidean type $\widetilde{\mathbb{A}}_3$. It is known that Γ_C admits an infinite family $\mathcal{T}^C_\lambda$, $\lambda \in \Lambda(C)$, of pairwise orthogonal stable tubes, having a unique stable tube, say $\mathcal{T}^C_1$, of rank 4 with the mouth formed by the modules $S_1 = \tau_C S_2$, $S_2 = \tau_C S_3$, $S_3 = \tau_C E$, $E = \tau_C S_1$, where E is the unique indecomposable C-module with the dimension vector $\underline{\dim} E = \begin{smallmatrix} & 0 & & 0 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & & 1 & & 1 & & \end{smallmatrix}$, and a stable tube, say $\mathcal{T}^C_2$, of rank 3 with the mouth formed by the modules $S_6 = \tau_C S_7$, $S_7 = \tau_C F$, $F = \tau_C S_6$, where F is the unique indecomposable C-module with the dimension vector $\underline{\dim} F = \begin{smallmatrix} & 1 & & 1 & & 1 & \\ 1 & & 1 & & 1 & & 1 \\ & & 0 & & 0 & & \end{smallmatrix}$ (see [60, (3.7)]). Moreover, Γ_D admits an infinite family $\mathcal{T}^D_\mu$, $\mu \in \Lambda(D)$, of pairwise orthogonal stable tubes, having a stable tube, say $\mathcal{T}^D_1$, of rank 2 with the mouth formed by the modules $S_{26} = \tau_D G$, $G = \tau_D S_{26}$, where G is the unique indecomposable D-module with the dimension vector $\underline{\dim} G = \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix}$. Denote by $C_i = KQ_{C_i}/J_{C_i}$ the bound quiver algebra, where Q_{C_i} is the full subquiver of Q given by the vertices $0, 1, 2, \dots, i$, $i \geq 8$ ($C_8 = C$), $J_{C_i} = J \cap KQ_{C_i}$, and by $D_j = KQ_{D_j}/J_{D_j}$ the bound quiver algebra, where Q_{D_j} is the full subquiver of Q given by the vertices $25, 26, 27, \dots, j$, $j \geq 28$ ($D_{28} = D$), $J_{D_j} = J \cap KQ_{D_j}$. Moreover, for each $k \in \{8, 9, \dots, 39\}$ (respectively, $k \in \{28, 29, \dots, 33\}$) and $l \in \{0, 1, \dots, 39\}$ (respectively, $l \in \{28, 29, \dots, 33\}$), we denote by $P^{C_k}_l, I^{C_k}_l, S^{C_k}_l$ (respectively, $P^{D_k}_l, I^{D_k}_l, S^{D_k}_l$) the indecomposable projective module, the indecomposable injective module, and the simple module in $\operatorname{mod} C_k$ (respectively, in $\operatorname{mod} D_k$) at the vertex l of Q_{C_k} (respectively, of Q_{D_k}). Then B is the generalized multicoil enlargement of $C \times D$, obtained by applications of the following admissible operations:

- one admissible operation of type (ad 1*) with the pivot $S^{C_8}_3$, creating the vertices 9, 10, 11 and the arrows δ_1, δ_2, δ_3;
- one admissible operation of type (ad 1*) with the pivot $S^{C_{11}}_2$, creating the vertices 12, 13, 14, 15 and the arrows σ_1, σ_2, σ_3, σ_4;
- one admissible operation of type (ad 1) with the pivot $S^{C_{15}}_1$, creating the vertices 16, 17 and the arrows ξ_1, ξ_2;
- one admissible operation of type (ad 4) with the pivot $S^{C_{17}}_3$ and the finite sectional path $S^{C_{17}}_{10} \to I^{C_{17}}_{10}$, creating the vertices 18, 19 and the arrows ε_1, ε_2, ε_3;
- one admissible operation of type (ad 1*) with the pivot $S^{C_{19}}_7$, creating the vertices 20, 21 and the arrows μ_1, μ_2;
- one admissible operation of type (ad 1) with the pivot $S^{C_{21}}_6$, creating the vertices 22, 23, 24 and the arrows ν_1, ν_2, ν_3;
- one admissible operation of type (ad 1*) with the pivot $S^{D_{28}}_{26}$, creating the vertices 29, 30, 31 and the arrows ψ_1, ψ_2, ψ_2;

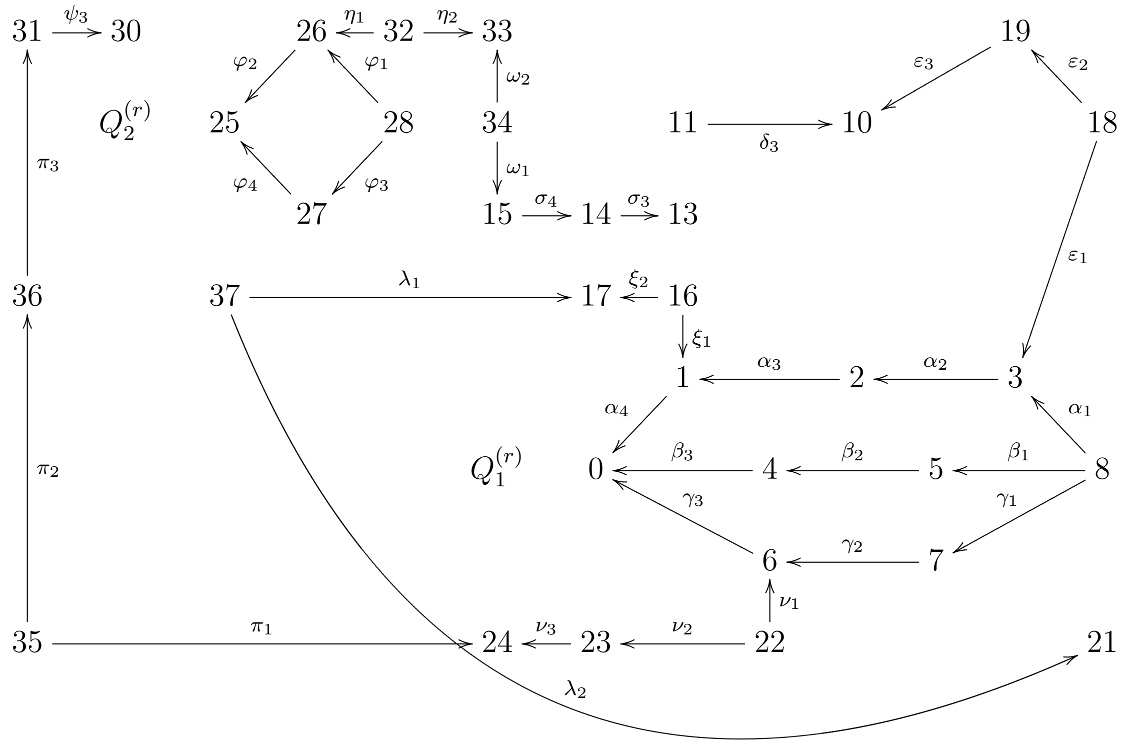
- one admissible operation of type (ad 1) with the pivot W being the unique indecomposable module of dimension 2 having $S_{29}^{D_{31}}$ as the socle and $S_{26}^{D_{31}}$ as the top, creating the vertices 32, 33 and the arrows η_1, η_2;
- one admissible operation of type (ad 4) with the pivot $S_{33}^{D_{33}}$ and the finite sectional path $I_{13}^{C_{24}} \to I_{14}^{C_{24}} \to S_{15}^{C_{24}}$, creating the vertex 34 and the arrows ω_1, ω_2;
- one admissible operation of type (ad 4) with the pivot $S_{24}^{C_{34}}$ and the finite sectional path $I_{30}^{C_{34}} \to S_{31}^{C_{34}}$, creating the vertices 35, 36 and the arrows π_1, π_2, π_3;
- one admissible operation of type (ad 4) with the pivot $S_{17}^{C_{36}}$ and the module $S_{21}^{C_{36}}$, creating the vertex 37 and the arrows λ_1, λ_2;
- one admissible operation of type (ad 2*) with the pivot $P_{32}^{C_{37}}$, creating the vertex 38 and the arrows κ_1, κ_2.
- one admissible operation of type (ad 2*) with the pivot $P_{22}^{C_{38}}$, creating the vertex 39 and the arrows θ_1, θ_2.
- one admissible operation of type (ad 2*) with the pivot $P_{16}^{C_{39}}$, creating the vertex 40 and the arrows ρ_1, ρ_2.

Then the left part $B^{(l)}$ of B is the convex subcategory of B being the product $B^{(l)} = B_1^{(l)} \times B_2^{(l)}$, where $B_1^{(l)} = KQ_1^{(l)}/J_1^{(l)}$ is the branch coextension of the canonical algebra C and $B_2^{(l)} = KQ_2^{(l)}/J_2^{(l)}$ is the branch coextension of the canonical algebra D given by the quivers



and the ideals $J_1^{(l)} = KQ_1^{(l)} \cap J$ in $KQ_1^{(l)}$ and $J_2^{(l)} = KQ_2^{(l)} \cap J$ in $KQ_2^{(l)}$. The right part $B^{(r)}$ of B is the convex subcategory of B being the product $B^{(r)} = B_1^{(r)} \times B_2^{(r)}$, where $B_1^{(r)} = KQ_1^{(r)}/J_1^{(r)}$ is the branch extension of the canonical algebra C and $B_2^{(r)} =$

$KQ_2^{(r)}/J_2^{(r)}$ is the branch extension of the canonical algebra D given by the quivers



and the ideals $J_1^{(r)} = KQ_1^{(r)} \cap J$ in $KQ_1^{(r)}$ and $J_2^{(r)} = KQ_2^{(r)} \cap J$ in $KQ_2^{(r)}$. It follows from [45, Theorems C and F] that the Auslander-Reiten quiver Γ_B of the generalized multicoil enlargement B of $C \times D$ is of the form

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{C}^B \cup \mathcal{Q}^B,$$

where $\mathcal{P}^B, \mathcal{C}^B, \mathcal{Q}^B$ are of the following families of components:

- $\mathcal{C}^B$ is a family of pairwise orthogonal generalized multicoils consisting of the faithful cyclic component $\mathcal{C}$ (described above), the family $\mathcal{T}_\lambda^C$, $\lambda \in \Lambda(C)\setminus\{1,2\}$, of stable tubes of Γ_C, and the family $\mathcal{T}_\mu^D$, $\mu \in \Lambda(D) \setminus \{1\}$, of stable tubes of Γ_D;
- $\mathcal{P}^B = \mathcal{P}^{B^{(l)}}$ and consists of the unique postprojective component $\mathcal{P}(A_1^{(l)})$ of the wild concealed algebra $A_1^{(l)}$ being the convex subcategory of $B_1^{(l)}$ given by all object of $B_1^{(l)}$ except 8, the unique postprojective component $\mathcal{P}(B_2^{(l)}) = \mathcal{P}^{B_2^{(l)}}$ of the tilted algebra $B_2^{(l)}$ of Euclidean type $\widetilde{\mathbb{A}}_7$, one component with the stable part $\mathbb{Z}\mathbb{A}_\infty$ containing the indecomposable projective $B_1^{(l)}$-module at the vertex 8, and infinitely many regular components of the form $\mathbb{Z}\mathbb{A}_\infty$;
- $\mathcal{Q}^B = \mathcal{Q}^{B^{(r)}}$ and consists of the unique preinjective component $\mathcal{Q}(A_1^{(r)})$ of the wild concealed algebra $A_1^{(r)}$ being the convex subcategory of $B_1^{(r)}$ given by all object of $B_1^{(r)}$ except 0, the unique preinjective component $\mathcal{Q}(B_2^{(r)}) = \mathcal{Q}^{B_2^{(r)}}$ of the tilted algebra $B_2^{(r)}$ of Euclidean type $\widetilde{\mathbb{A}}_9$, one component with the stable part $\mathbb{Z}\mathbb{A}_\infty$ containing the indecomposable injective $B_1^{(r)}$-module at the vertex 0, and infinitely many regular components of the form $\mathbb{Z}\mathbb{A}_\infty$.

Moreover, we have

$$\mathrm{Hom}_B(\mathcal{C}^B, \mathcal{P}^B) = 0, \mathrm{Hom}_B(\mathcal{Q}^B, \mathcal{C}^B) = 0, \mathrm{Hom}_B(\mathcal{Q}^B, \mathcal{P}^B) = 0.$$

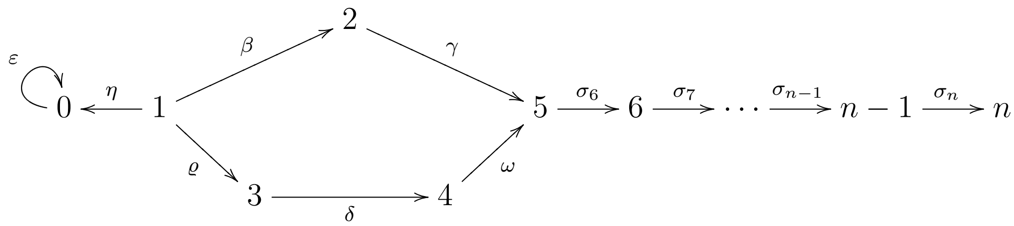
We also note that B is not a cycle-finite algebra, because Γ_B contains regular components of the form $\mathbb{Z}\mathbb{A}_\infty$ (see [67, Lemma 3]).

Finally, we mention that the cyclic component $\mathcal{C}$ of Γ_B is the cyclic generalized multicoil obtained from the stable tubes $\mathcal{T}_1^C$, $\mathcal{T}_2^C$ of Γ_C and the stable tube $\mathcal{T}_1^D$ of Γ_D by the 14 translation quiver admissible operations [44, Section 2] corresponding to the 14 admissible algebra operations leading from $C \times D$ to B, described above. We also point that the cyclic component $\mathcal{C}$ has a Möbius strip configuration obtained by identifying in $\mathcal{C}_3$ two sectional paths $N_1 \to N_2$.

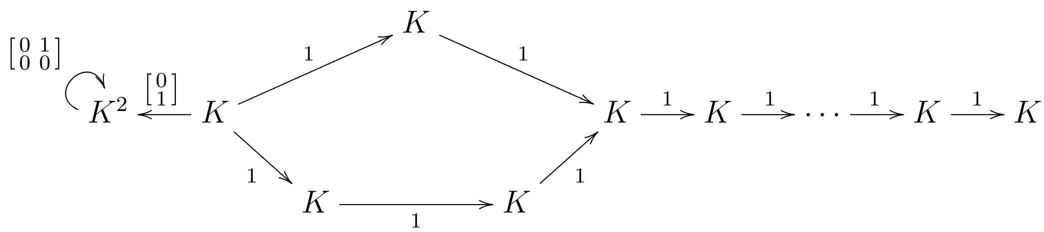
6. Examples: finite cyclic components

In this section we present examples illustrating Theorem 1.2 and showing faithful almost acyclic Auslander-Reiten components of new types.

Example 6.1. Let K be a field, $n \geq 7$ a natural number, and $A_n = KQ_n/I_n$ the bound quiver algebra given by the quiver Q_n of the form



and I_n the ideal in the path algebra KQ_n of Q_n over K generated by the elements $\varepsilon^2, \eta\varepsilon$ and $\beta\gamma - \varrho\delta\omega$. Then the category $\mathrm{mod}\, A_n$ is equivalent to the category $\mathrm{rep}_K(Q_n, I_n)$ of the K-linear representations of the bound quiver (Q_n, I_n). Consider the indecomposable module M_n in $\mathrm{mod}\, A_n$ corresponding to the indecomposable representation in $\mathrm{rep}_K(Q_n, I_n)$ of the form



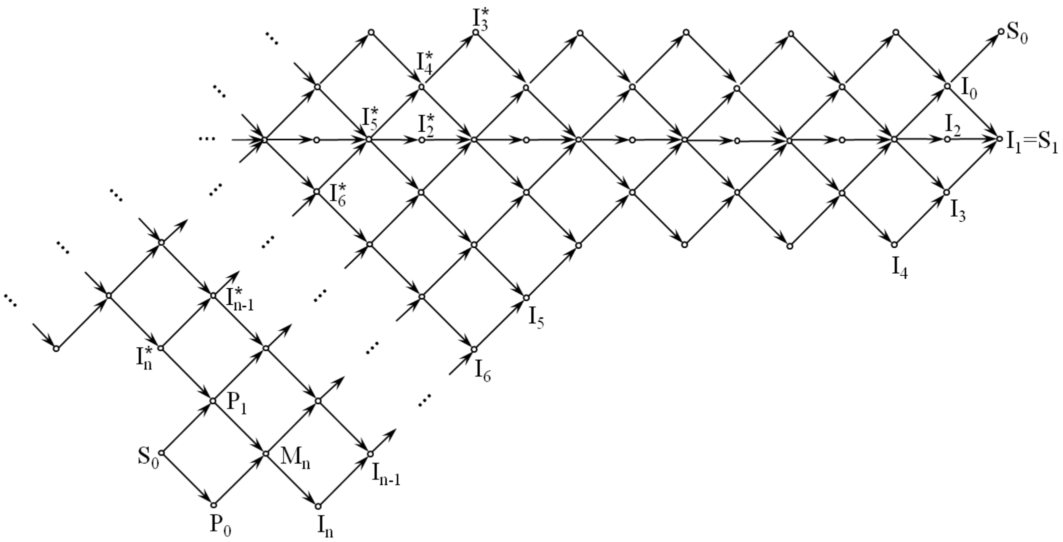
We note that M_n is a faithful A_n-module, and hence $B(M_n) = A_n$. Let Ω_n be the full subquiver of Q_n given by the vertices $2, 3, 4, 5, 6, \ldots, n-1, n$ and the arrows $\gamma, \delta, \omega, \sigma_6, \sigma_7, \ldots, \sigma_{n-1}, \sigma_n$, and $H_n = K\Omega_n$ the associated path algebra. Then H_n is a hereditary algebra. Observe that H_7, H_8, H_9 are hereditary algebras of Dynkin types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ (respectively), H_{10} is a hereditary algebra of Euclidean type $\widetilde{\mathbb{E}}_8$, and, for $n \geq 11$, H_n is a hereditary algebra of wild type. For each $i \in \{0, 1, \ldots, n-1, n\}$, we denote by P_i, I_i, S_i the indecomposable projective module, the indecomposable injective module, the simple module in $\mathrm{mod}\, A_n$ at the vertex i of Q_n. Moreover, for each $j \in \{2, 3, 4, \ldots, n-1, n\}$, we denote by I_j^* the indecomposable injective module in $\operatorname{mod} H_n$ at the vertex j of Ω_n. Further, let $\Lambda = K[\varepsilon]/(\varepsilon^2)$. Then P_0 is the indecomposable projective module in $\operatorname{mod}\Lambda$ and S_0 is its top. Finally, observe that A_n is the one-point extension algebra

$$\begin{bmatrix} \Lambda \times H_n & 0 \\ S_0 \oplus I_n^* & K \end{bmatrix}$$

of $\Lambda \times H_n$ by the module $S_0 \oplus I_n^*$, with the extension vertex 1. Since I_n^* is the indecomposable injective module in $\operatorname{mod} H_n$ and H_n is a hereditary algebra, we conclude that

$$\operatorname{Hom}_{\Lambda\times H_n}(S_0 \oplus I_n^*, \tau_{H_n} X) = \operatorname{Hom}_{H_n}(I_n^*, \tau_{H_n} X) = 0$$

for any module X in $\operatorname{ind} H_n$. Then, applying [63, Corollary XV.1.7] (see also [78, Lemma 5.6]), we conclude that every almost split sequence in $\operatorname{mod} H_n$ is an almost split sequence in $\operatorname{mod} A_n$. This implies that the Auslander-Reiten quiver Γ_{H_n} of H_n is a full translation subquiver of the Auslander-Reiten quiver Γ_{A_n} of A_n. In particular, we obtain that the preinjective component $\mathcal{Q}(H_n)$ of Γ_{H_n}, containing the indecomposable injective modules I_j^* , $j \in \{2, 3, 4, \ldots, n-1, n\}$, is a full translation subquiver of a component $\mathcal{C}_n$ of Γ_{A_n} which is closed under predecessors. Then the direct calculation shows that $\mathcal{C}_n$ is a component of the form



Observe that $\mathcal{C}_n$ is an almost acyclic component of Γ_{A_n}, contains the faithful module M_n, and is closed under successors in $\operatorname{ind} A_n$. Hence $\mathcal{C}_n$ is a faithful, almost acyclic, generalized standard component of Γ_A. Then it follows from [58, Theorem 3.1] that A_n is a generalized double tilted algebra and $\mathcal{C}_n$ is its unique connecting component. We also note that $\mathcal{C}_n$ admits a unique multisection $\Delta = \Delta_n$ consisting of all indecomposable modules in $\mathcal{C}_n$ which lie on oriented cycles passing through the simple module S_0. Moreover, we have $\Delta_l' = \Delta = \Delta_r'$, and hence $\Delta = \Delta_c$. Further, the left part Δ_l of Δ coincides with $\tau_{A_n}\Delta_r''$ and consists of the indecomposable modules I_j^*, for $j \in \{2, 3, 4, \ldots, n-1, n\}$. Similarly, the right part Δ_r of Δ coincides with $\tau_{A_n}^{-1}\Delta_l''$ and consists of the indecomposable injective modules I_1, I_2, I_3, I_4. Therefore, the left tilted part $A_n^{(l)}$ of A_n is the hereditary algebra H_n and the right tilted part $A_n^{(r)}$ of A_n is the path algebra $K\Sigma$ of the quiver Σ of the form

$$2 \xleftarrow{\beta} 1 \xrightarrow{\varrho} 3 \xrightarrow{\delta} 4.$$

Observe now that $\Delta = \Delta_c$ is a cycle-finite finite component of ${}_c\Gamma_{A_n}$ containing the faithful indecomposable module M_n, because $\mathcal{C}_n$ is a generalized standard component of Γ_A closed under successors in $\operatorname{ind} A_n$. In particular, we conclude that Δ is the cyclic component $\Gamma(M_n)$ of the module M_n and $B(\Gamma(M_n)) = B(M_n) = A_n$ is a generalized double tilted algebra. We also mention that A_7, A_8, A_9 are of finite representation type with $\Gamma_{A_7} = \mathcal{C}_7$, $\Gamma_{A_8} = \mathcal{C}_8$, $\Gamma_{A_9} = \mathcal{C}_9$, and hence are cycle-finite algebras. Further, A_{10} is a cycle-finite algebra of infinite representation type whose Auslander-Reiten quiver has the disjoint union decomposition

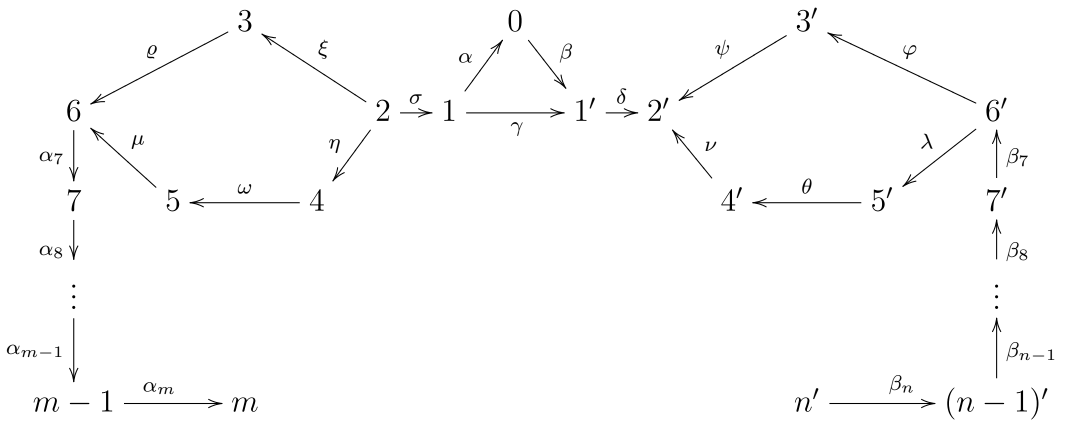
$$\Gamma_{A_{10}} = \mathcal{P}(H_{10}) \cup \mathcal{T}^{H_{10}} \cup \mathcal{C}_{10},$$

where $\mathcal{P}(H_{10})$ is the postprojective component and $\mathcal{T}^{H_{10}}$ an infinite family of pairwise orthogonal generalized standard stable tubes of $\Gamma_{H_{10}}$. On the other hand, the algebras A_n, for $n \geq 11$, are not cycle-finite because their Auslander-Reiten quivers admit regular components of Γ_{H_n} being of the form $\mathbb{Z}\mathbb{A}_\infty$, and hence consisting of indecomposable modules lying on infinite cycles (see [67, Theorem]). More precisely, for $n \geq 11$, the Auslander-Reiten quiver of A_n has the disjoint union decomposition

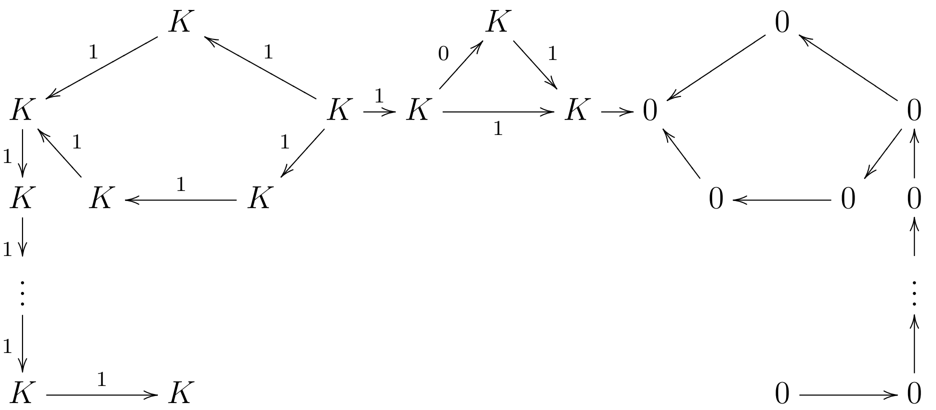
$$\Gamma_{A_n} = \mathcal{P}(H_n) \cup \mathcal{R}(H_n) \cup \mathcal{C}_n,$$

where $\mathcal{P}(H_n)$ is the postprojective component and $\mathcal{R}(H_n)$ is an infinite family of regular components of the form $\mathbb{Z}\mathbb{A}_\infty$ in Γ_{H_n}. We also mention that the algebras A_n, for $n \geq 7$, are of infinite global dimension, because the simple module S_0 is of infinite projective dimension.

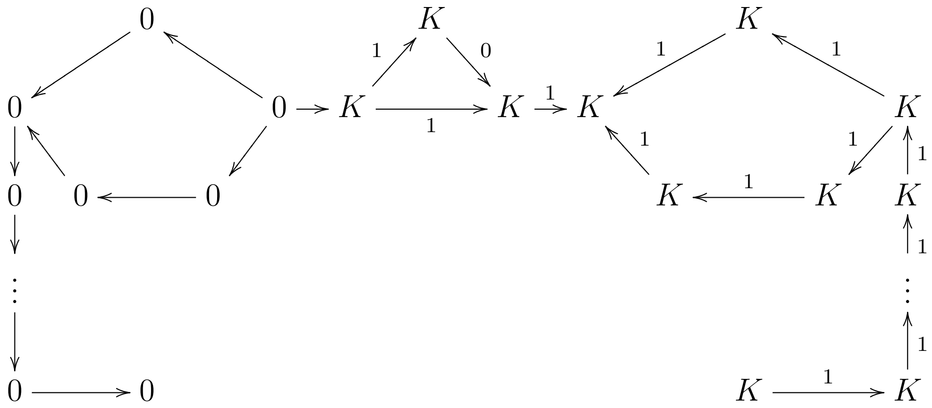
Example 6.2. Let K be a field, $m, n \geq 8$ natural numbers, and $B_{m,n} = KQ_{m,n}/I_{m,n}$ the bound quiver algebra given by the quiver $Q_{m,n}$ of the form



and $I_{m,n}$ the ideal in the path algebra $KQ_{m,n}$ of $Q_{m,n}$ over K generated by the elements $\alpha\beta$, $\sigma\alpha$, $\beta\delta$, $\sigma\gamma\delta$, $\xi\varrho - \eta\omega\mu$, $\varphi\psi - \lambda\theta\nu$. Then the category $\operatorname{mod} B_{m,n}$ is equivalent to the category $\operatorname{rep}_K(Q_{m,n}, I_{m,n})$ of the K-linear representations of the bound quiver $(Q_{m,n}, I_{m,n})$. Consider the indecomposable module M_m in $\operatorname{mod} B_{m,n}$ corresponding to the indecomposable representation in $\operatorname{rep}_K(Q_{m,n}, I_{m,n})$ of the form

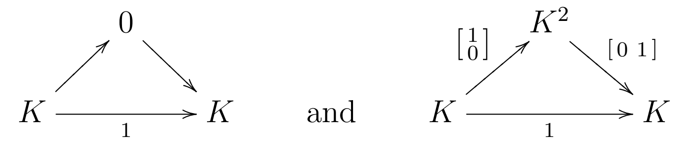


and the indecomposable module N_n in $\operatorname{mod} B_{m,n}$ corresponding to the indecomposable representation in $\operatorname{rep}_K(Q_{m,n}, I_{m,n})$ of the form



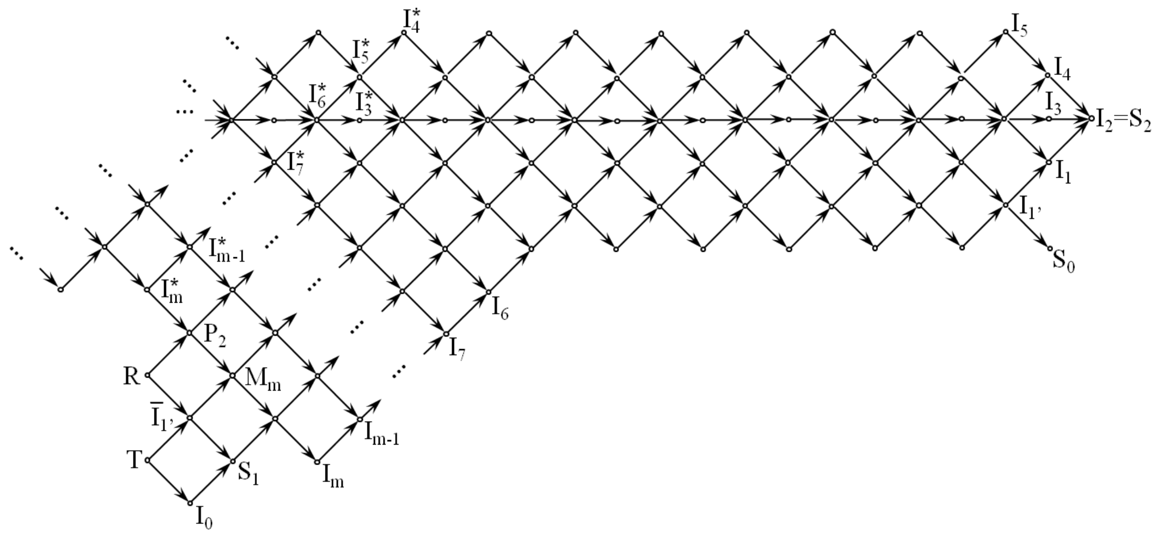
We note that $M_m \oplus N_n$ is a faithful $B_{m,n}$-module.

Let Ω_m be the subquiver of $Q_{m,n}$ given by the vertices $3, 4, 5, 6, 7, \ldots, m-1, m$ and the arrows ϱ, ω, μ, α_7, $\alpha_8, \ldots, \alpha_{m-1}, \alpha_m$, and $H_m = K\Omega_m$ the path algebra of Ω_m over K. Similarly, let Ω'_n be the subquiver of $Q_{m,n}$ given by the vertices $3', 4', 5', 6', 7', \ldots, (n-1)', n'$ and the arrows φ, θ, λ, β_7, $\beta_8, \ldots, \beta_{n-1}, \beta_n$, and $H'_n = K\Omega'_n$ the path algebra of Ω'_n over K. Then H_m and H'_n are hereditary algebras. Moreover, H_8 and H'_8 are of Dynkin type $\mathbb{E}_6$, H_9 and H'_9 are of Dynkin type $\mathbb{E}_7$, H_{10} and H'_{10} are of Dynkin type $\mathbb{E}_8$, H_{11} and H'_{11} are of Euclidean type $\widetilde{\mathbb{E}}_8$, and H_m and H'_n, for $m, n \geq 12$, are of wild type. For each $i \in \{3, 4, \ldots, m-1, m\}$, we denote by I^*_i the indecomposable injective H_m-module at the vertex i. Similarly, for each $j' \in \{3', 4', \ldots, (n-1)', n'\}$, we denote by $P^*_{j'}$ the indecomposable projective H'_n-module at the vertex j'. Furthermore, for each vertex i of $Q_{m,n}$, we denote by P_i, I_i, S_i the indecomposable projective module, the indecomposable injective module, and the simple module in $\operatorname{mod} B_{m,n}$ at the vertex i. Finally, we denote by Σ the subquiver of $Q_{m,n}$ given by the vertices $0, 1, 1'$ and the arrows α, β, γ, and $\Lambda = K\Sigma/J$ the bound quiver algebra with J the ideal in the path algebra $K\Sigma$ of Σ over K generated by $\alpha\beta$. We denote by R and T the indecomposable modules in $\operatorname{mod}\Lambda$ corresponding to the representations

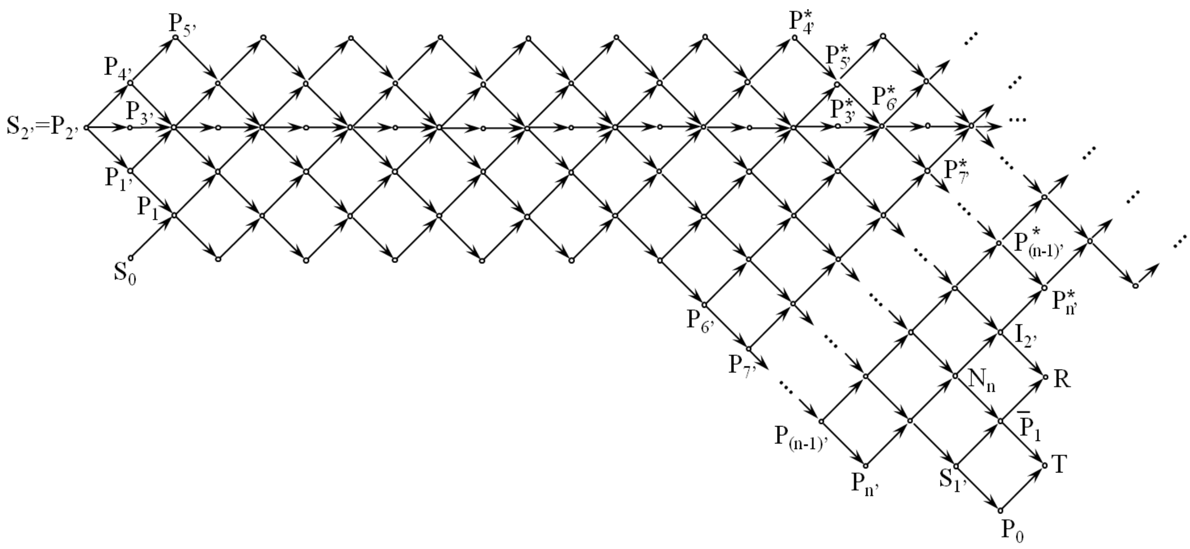


in $\operatorname{rep}_K(\Sigma, J)$, respectively. Moreover, denote by $\overline{P}_1$ the indecomposable projective Λ-module at the vertex 1 and by $\overline{I}_{1'}$ the indecomposable injective Λ-module at the vertex $1'$, and observe that P_0 is the indecomposable projective Λ-module at the vertex 0 and I_0 is the indecomposable injective Λ-module at the vertex 0.

We claim that $B_{m,n}$ is a generalized double tilted algebra and the indecomposable modules M_m and N_n belong to a cycle-finite cyclic component $\Gamma_{m,n}$, and hence $B(\Gamma_{m,n}) = B_{m,n}$. More precisely, we will show that $\Gamma_{m,n}$ is the cyclic part of the almost acyclic generalized standard component $\mathcal{C}_{m,n}$ of $\Gamma_{B_{m,n}}$ obtained by identification the modules R, T and S_0 occurring in the following two translation quivers: $\mathcal{C}_m^-$ of the form



and $\mathcal{C}_n^+$ of the form



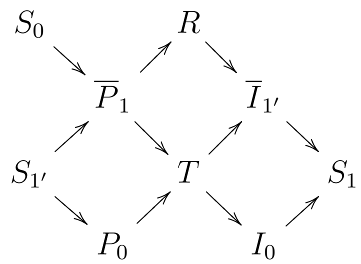
Let Q_m be the subquiver of $Q_{m,n}$ given by the vertices $1'$, 0, 1, 2, 3, 4, 5, 6, $7, \dots, m-1, m$ and the arrows α, β, γ, ξ, ϱ, η, ω, μ, α_7, $\alpha_8, \dots, \alpha_{m-1}, \alpha_m$, J_m the ideal in the path algebra KQ_m of Q_m over K generated by $\alpha\beta$, $\sigma\alpha$, $\xi\varrho - \eta\omega\mu$, and $C_m = KQ_m/J_m$ the associated bound quiver algebra. Then C_m is the one-point extension algebra

$$\begin{bmatrix} \Lambda \times H_m & 0 \\ R \oplus I_m^* & K \end{bmatrix}$$

of $\Lambda \times H_m$ by the module $R \oplus I_m^*$, with the extension vertex 2. Since I_m^* is the indecomposable injective module over the hereditary algebra H_m, we conclude that

$$\operatorname{Hom}_{\Lambda \times H_m}(R \oplus I_m^*, \tau_{H_m} X) = \operatorname{Hom}_{H_m}(I_m^*, \tau_{H_m} X) = 0$$

for any indecomposable module X in $\operatorname{mod} H_m$. Then, applying [63, Corollary XV.1.7] (or [78, Lemma 5.6]), we conclude that every almost split sequence in $\operatorname{mod} H_m$ is an almost split sequence in $\operatorname{mod} C_m$. This implies that the Auslander-Reiten quiver Γ_{H_m} of H_m is a full translation subquiver of the Auslander-Reiten quiver Γ_{C_m} of C_m. Moreover, a direct calculation shows that the component $\mathcal{C}_m$ of Γ_{C_m}, containing the indecomposable injective H_m-modules I_i^*, $i \in \{3, 4, 5, 6, 7, \dots, m-1, m\}$, is the translation quiver obtained from the translation quiver $\mathcal{C}_m^-$ and the translation quiver below



by identifying the common modules R, $\overline{I}_{1'}$, S_1, T, I_0 and S_0. We observe that the Auslander-Reiten quiver Γ_{C_m} consists of the component $\mathcal{C}_m$ and the components of Γ_{H_m} different from the preinjective component. We also note that the indecomposable module M_m is a unique sincere module in $\operatorname{ind} C_m$, and M_m is not a faithful module in $\operatorname{mod} C_m$.

Dually, let Q'_n be the subquiver of $Q_{m,n}$ given by the vertices 1, 0, $1'$, $2'$, $3'$, $4'$, $5'$, $6'$, $7', \dots, (n-1)', n$ and the arrows α, β, γ, δ, ψ, φ, ν, θ, λ, β_7, $\beta_8, \dots, \beta_{n-1}, \beta_n$, J'_n the ideal in the path algebra KQ'_n of Q'_n over K generated by $\alpha\beta$, $\beta\delta$, $\varphi\psi - \lambda\theta\nu$, and $C'_n = KQ'_n/J'_n$ the associated bound quiver algebra. Then C'_n is the one-point coextension algebra

$$\begin{bmatrix} K & 0 \\ \operatorname{Hom}_K(R \oplus P_{n'}^*, K) & \Lambda \times H'_n \end{bmatrix}$$

of $\Lambda \times H'_n$ by the module $R \oplus P_{n'}^*$, with the coextension vertex $2'$. Since $P_{n'}^*$ is the indecomposable projective module over the hereditary algebra H'_n, we conclude that

$$\operatorname{Hom}_{\Lambda \times H'_n}(\tau_{H'_n}^{-1} Y, R \oplus P_{n'}^*) = \operatorname{Hom}_{H'_n}(\tau_{H'_n}^{-1} Y, P_{n'}^*) = 0$$

for any indecomposable module Y in $\operatorname{mod} H'_n$. Then, applying the dual of [63, Corollary XV.1.7] (or [78, Lemma 5.6]), we conclude that every almost split sequence in $\operatorname{mod} H'_n$ is an almost split sequence in $\operatorname{mod} C'_n$. This implies that the Auslander-Reiten quiver $\Gamma_{H'_n}$ of H'_n is a full translation subquiver of the Auslander-Reiten quiver $\Gamma_{C'_n}$ of C'_n.

Moreover, a direct calculation shows that the component $\mathcal{C}'_n$ of $\Gamma_{C'_n}$, containing the indecomposable projective H'_n-modules $P^*_{j'}$, $j' \in \{3', 4', 5', 6', 7', \dots, (n-1)', n'\}$, is the translation quiver obtained from the translation quiver $\mathcal{C}^+_n$ and the translation quiver below

$$
\begin{array}{ccccccccc}
 & & & & R & & & & S_0 \\
 & & & \nearrow & & \searrow & & \nearrow & \\
 & & \overline{P}_1 & & & & \overline{I}_{1'} & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
S_{1'} & & & & T & & & & S_1 \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & P_0 & & & & I_0 & &
\end{array}
$$

by identifying the common modules $S_{1'}$, P_0, $\overline{P}_1$, R, T and S_0. We observe that the Auslander-Reiten quiver $\Gamma_{C'_n}$ consists of the component $\mathcal{C}'_n$ and the components of $\Gamma_{H'_n}$ different from the postprojective component. We also note that the indecomposable module N_n is a unique sincere module in $\operatorname{ind} C'_n$, and N_n is not a faithful module in $\operatorname{mod} C'_n$.

Further, we observe that the algebra $B_{m,n} = KQ_{m,n}/I_{m,n}$ is the one-point extension algebra

$$\begin{bmatrix} C'_n \times H_m & 0 \\ R \oplus I^*_m & K \end{bmatrix}$$

of $C'_n \times H_m$ by the module $R \oplus I^*_m$, with the extension vertex 2. It follows from the structure of the Auslander-Reiten quiver $\Gamma_{C'_n}$ of C'_n that, for any indecomposable module Z in $\operatorname{mod} C'_n$ nonisomorphic to the simple module S_0, we have

$$\operatorname{Hom}_{C'_n \times H_m}(R \oplus I^*_m, \tau_{C'_n} Z) = \operatorname{Hom}_{C'_n}(R, \tau_{C'_n} Z) = 0.$$

Then, applying [63, Corollary XV.1.7] (or [78, Lemma 5.6]) again, we conclude that every almost split sequence in $\operatorname{mod} C'_n$ with the right term nonisomorphic to S_0 is an almost split sequence in $\operatorname{mod} B_{m,n}$. This shows that the translation quiver obtained from $\Gamma_{C'_n}$ by removing the module S_0 and two arrows attached to it is a full translation subquiver of $\Gamma_{B_{m,n}}$. In particular, we conclude that the almost split sequence in $\operatorname{mod} C'_n$ with the left term S_0 is an almost split sequence in $\operatorname{mod} B_{m,n}$.

Finally, we observe that the algebra $B_{m,n} = KQ_{m,n}/I_{m,n}$ is also the one-point coextension algebra

$$\begin{bmatrix} K & 0 \\ \operatorname{Hom}_K(R \oplus P^*_{n'}, K) & C_m \times H'_n \end{bmatrix}$$

of $C_m \times H'_n$ by the module $R \oplus P^*_{n'}$, with the coextension vertex $2'$. It follows also from the structure of the Auslander-Reiten quiver Γ_{C_m} of C_m that, for any indecomposable module Z in $\operatorname{mod} C_m$ nonisomorphic to the simple module S_0, we have

$$\operatorname{Hom}_{C_m \times H'_n}(\tau^{-1}_{C_m} Z, R \oplus P^*_{n'}) = \operatorname{Hom}_{C_m}(\tau^{-1}_{C_m} Z, R) = 0.$$

Then, applying the dual of [63, Corollary XV.1.7] (or [78, Lemma 5.6]) again, we conclude that every almost split sequence in $\operatorname{mod} C_m$ with the left term nonisomorphic to S_0 is an almost split sequence in $\operatorname{mod} B_{m,n}$. This shows that the translation quiver obtained from Γ_{C_m} by removing the module S_0 and two arrows attached to it is a full translation subquiver of $\Gamma_{B_{m,n}}$. In particular, we obtain that the almost split sequence in $\operatorname{mod} C_m$ with the right term S_0 is also an almost split sequence in $\operatorname{mod} B_{m,n}$.

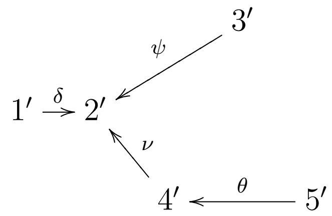
Summing up, we proved that $\Gamma_{B_{m,n}}$ contains the component $\mathcal{C}_{m,n}$ of the required form, containing the preinjective component $\mathcal{Q}(H_m)$ of Γ_{H_m} as a full translation subquiver closed under predecessors and the postprojective component $\mathcal{P}(H'_n)$ of $\Gamma_{H'_n}$ as a full translation subquiver closed under successors. Moreover, the Auslander-Reiten quiver $\Gamma_{B_{m,n}}$ of $B_{m,n}$ has a disjoint union decomposition

$$\Gamma_{B_{m,n}} = \mathcal{P}_{m,n} \cup \mathcal{C}_{m,n} \cup \mathcal{Q}_{m,n}$$

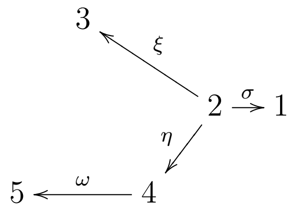
such that

- $\mathcal{P}_{m,n}$ is empty for $m \in \{8, 9, 10\}$;
- $\mathcal{P}_{11,n}$ consists of the postprojective component $\mathcal{P}(H_{11})$ of Euclidean type $\widetilde{\mathbb{E}}_8$ and an infinite family $\mathcal{T}^{H_{11}}$ of pairwise orthogonal generalized standard stable tubes in $\Gamma_{H_{11}}$;
- $\mathcal{P}_{m,n}$, for $m \geq 12$, consists of the postprojective component $\mathcal{P}(H_m)$ of wild type and an infinite family of regular components of the form $\mathbb{Z}\mathbb{A}_\infty$ in Γ_{H_m};
- $\mathcal{Q}_{m,n}$ is empty for $n \in \{8, 9, 10\}$;
- $\mathcal{Q}_{m,11}$ consists of the preinjective component $\mathcal{Q}(H'_{11})$ of Euclidean type $\widetilde{\mathbb{E}}_8$ and an infinite family $\mathcal{T}^{H'_{11}}$ of pairwise orthogonal generalized standard stable tubes in $\Gamma_{H'_{11}}$;
- $\mathcal{Q}_{m,n}$, for $n \geq 12$, consists of the preinjective component $\mathcal{Q}(H'_n)$ of wild type and an infinite family of regular components of the form $\mathbb{Z}\mathbb{A}_\infty$ in $\Gamma_{H'_n}$.

Finally, observe that $\mathcal{C}_{m,n}$ is an almost acyclic component of $\Gamma_{B_{m,n}}$ whose cyclic part $\Gamma_{m,n}$ is connected and consists of all indecomposable modules in $\mathcal{C}_{m,n}$ which lie on oriented cycles passing through the simple module S_0. In fact, $\Gamma_{m,n}$ is the unique multisection Δ of $\mathcal{C}_{m,n}$, and so $\Delta_c = \Gamma_{m,n}$. Further, $\Gamma_{m,n}$ contains the indecomposable modules M_m and N_n. Since $M_m \oplus N_n$ is a faithful module in $\operatorname{mod} B_{m,n}$, we conclude that $\Gamma_{m,n}$ is a faithful cyclic component of $\Gamma_{B_{m,n}}$, and hence $B_{m,n} = B(\Gamma_{m,n}) = B_{m,n}/\operatorname{ann}_{B_{m,n}}(\Gamma_{m,n})$. Observe also that $B_{m,n} = \operatorname{Supp}(\Gamma_{m,n})$. In particular, $\mathcal{C}_{m,n}$ is a faithful component of $\Gamma_{B_{m,n}}$. Moreover, the left part Δ_l of Δ coincides with $\tau_{B_{m,n}}\Delta''_r$ and consists of the indecomposable modules I^*_i, $i \in \{3, 4, \dots, m-1\}$, and the indecomposable modules $P_{1'}$, $\tau^{-1}_{B_{m,n}} P_{2'}$, $\tau^{-1}_{B_{m,n}} P_{3'}$, $\tau^{-1}_{B_{m,n}} P_{4'}$, $\tau^{-1}_{B_{m,n}} P_{5'}$. Similarly, the right part Δ_r of Δ coincides with $\tau^{-1}_{B_{m,n}}\Delta''_l$ and consists of the indecomposable modules $P^*_{j'}$, $j' \in \{3', 4', \dots, (n-1)'\}$, and the indecomposable modules I_1, $\tau_{B_{m,n}} I_2$, $\tau_{B_{m,n}} I_3$, $\tau_{B_{m,n}} I_4$, $\tau_{B_{m,n}} I_5$. Observe also that $\mathcal{Q}(H_m)$ is a generalized standard component of Γ_{H_m}, $\mathcal{P}(H'_n)$ is a generalized standard component of $\Gamma_{H'_n}$, and $\operatorname{Hom}_{B_{m,n}}(P, Q) = 0$ for any indecomposable modules $P \in \mathcal{P}(H'_n)$ and $Q \in \mathcal{Q}(H_m)$. This shows that $\mathcal{C}_{m,n}$ is a generalized standard component of $\Gamma_{B_{m,n}}$. Then it follows from [58, Theorem 3.1] that $B_{m,n}$ is a generalized double tilted algebra. Moreover, the left tilted part $B^{(l)}_{m,n}$ is the product $H_m \times H'$ of H_m and the path algebra $H' = K\Omega'$ of the quiver Ω' of the form



and Dynkin type $\mathbb{D}_5$, while the right tilted part $B^{(r)}_{m,n}$ is the product $H'_n \times H$ of H'_n and the path algebra $H = K\Omega$ of the quiver Ω of the form



and Dynkin type $\mathbb{D}_5$. In particular, we obtain that $B_{m,n}$ is a tame generalized double tilted algebra (equivalently, cycle-finite algebra) if and only if $m, n \in \{8, 9, 10, 11\}$. Clearly, $B_{m,n}$ is of finite representation type if and only if $m, n \in \{8, 9, 10\}$. Finally, we note that the algebras $B_{m,n}$, for all $m, n \geq 8$, are of global dimension three, with the simple module S_1 having the projective dimension three.

References

[1] *I. Assem* and *F. U. Coelho*, Two-sided gluings of tilted algebras, J. Algebra **269** (2003), 456–479.

[2] *I. Assem*, *D. Simson* and *A. Skowroński*, Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Mathematical Society Student Texts **65**, Cambridge University Press, Cambridge, 2006.

[3] *I. Assem* and *A. Skowroński*, Algebras with cycle-finite derived categories, Math. Ann. **280** (1988), 441–463.

[4] *I. Assem* and *A. Skowroński*, Minimal representation-infinite coil algebras, Manuscr. Math. **67** (1990), 305–331.

[5] *I. Assem*, *A. Skowroński* and *B. Tomé*, Coil enlargements of algebras, Tsukuba J. Math. **19** (1995), 453–479.

[6] *M. Auslander*, Representation theory of artin algebras II, Comm. Algebra **1** (1974), 269–310.

[7] *M. Auslander* and *I. Reiten*, Uniserial functors, in: Representation Theory II, in: Lecture Notes in Math. **832**, Springer-Verlag, Berlin-Heidelberg, 1980, pp. 1–47.

[8] *M. Auslander* and *I. Reiten*, Modules determined by their composition factors, Illinois J. Math. **29** (1985), 280-301.

[9] *M. Auslander*, *I. Reiten* and *S. O. Smalø*, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics **36**, Cambridge University Press, Cambridge, 1995.

[10] *R. Bautista* and *S. Brenner*, On the number of terms in the middle of an almost split sequence, in: Representations of Algebras, in: Lecture Notes in Math. **903**, Springer-Verlag, Berlin-Heidelberg, 1981, pp. 1–8.

[11] *R. Bautista* and *S. O. Smalø*, Nonexistent cycles, Comm. Algebra **11** (1983), 1755–1767.

[12] *J. Białkowski*, *A. Skowroński*, *A. Skowyrski* and *P. Wiśniewski*, Cycle-finite algebras of semiregular type, Colloq. Math. **129** (2012), 211–247.

[13] *S. Brenner* and *M. C. R. Butler*, Wild subquivers of the Auslander-Reiten quiver of a tame algebra, in: Trends in the Representation Theory of Finite Dimensional Algebras, Contemp. Math. **229**, Amer. Math. Soc., Providence, RI, 1998, pp. 29–48.

[14] *C. Chaio*, Degrees of irreducible morphisms in standard components, J. Pure Appl. Algebra **214** (2010), 1063–1075.

[15] *C. Chaio*, *P. Malicki* and *A. Skowroński*, Degrees of irreducible morphisms in coherent almost cyclic Auslander-Reiten components, J. Algebra **391** (2013), 125–151.

[16] *C. Chaio*, *P. Le Meur* and *S. Trepode*, Degrees of irreducible morphisms and finite-representation type, J. London Math. Soc. **84** (2011), 35–57.

[17] *C. Chaio* and *S. Trepode*, The composite of irreducible morphisms in standard components, J. Algebra **323** (2010), 1000–1011.

[18] *F. U. Coelho* and *M. Lanzilotta*, Algebras with small homological dimensions, Manuscripta Math. **100** (1999), 1–11.

[19] *F. U. Coelho*, *E. M. Marcos*, *H. A. Merklen* and *A. Skowroński*, Module categories with infinite radical square zero are of finite type, Comm. Algebra **22** (1994), 4511–4517.

[20] *F. U. Coelho* and *A. Skowroński*, On Auslander-Reiten components for quasitilted algebras, Fund. Math. **149** (1996), 67–82.

[21] *W. Crawley-Boevey*, Tame algebras and generic modules, Proc. London Math. Soc. **63** (1991), 241–265.

[22] *W. Crawley-Boevey*, Modules of finite length over their endomorphism rings, in: Representations of Algebras and Related Topics, in: London Math. Soc. Lecture Note Series **168**, Cambridge University Press, Cambridge, 1992, pp. 127–184.

[23] *G. D'Este* and *C. M. Ringel*, Coherent tubes, J. Algebra **87** (1984), 150–201.

[24] *V. Dlab* and *C. M. Ringel*, Indecomposable representations of graphs and algebras, Memoirs Amer. Math. Soc. **6** no. 173 (1976).

[25] *D. Happel* and *I. Reiten*, Hereditary abelian categories with tilting object over arbitrary base fields, J. Algebra **256** (2002), 414–432.

[26] *D. Happel*, *I. Reiten* and *S. O. Smalø*, Tilting in abelian categories and quasitilted algebras, Memoirs Amer. Math. Soc. **120** no. 575 (1996).

[27] *D. Happel* and *C. M. Ringel*, Tilted algebras, Trans. Amer. Math. Soc. **274** (1982), 399–443.

[28] *A. Jaworska*, *P. Malicki* and *A. Skowroński*, Tilted algebras and short chains of modules, Math. Z. **273** (2013), 19–27.

[29] *A. Jaworska*, *P. Malicki* and *A. Skowroński*, Modules not being the middle of short chains, Quart. J. Math., 2012. Oxford Math. Press doi: 10.1093/qmath/has034.

[30] *A. Jaworska* and *A. Skowroński*, The component quiver of a self-injective artin algebra, Colloq. Math. **122** (2011), 233–239.

[31] *O. Kerner*, Tilting wild algebras, J. London Math. Soc. **39** (1989), 29–47.

[32] *O. Kerner* and *A. Skowroński*, On module categories with nilpotent infinite radical, Compositio Math. **77** (1991), 313-333.

[33] *O. Kerner* and *A. Skowroński*, Quasitilted one-point extensions of wild hereditary algebras, J. Algebra **244** (2001), 785-827.

[34] *M. Lanzilotta* and *D. Smith*, Laura algebras and quasi-directed components, Colloq. Math. **105** (2006), 179–196.

[35] *H. Lenzing* and *J. A. de la Peña*, Concealed-canonical algebras and separating tubular families, Proc. London Math. Soc. **78** (1999), 513–540.

[36] *H. Lenzing* and *A. Skowroński*, Quasi-tilted algebras of canonical type, Colloq. Math. **71** (1996), 161–181.

[37] *S. Liu*, Degrees of irreducible maps and the shapes of the Auslander-Reiten quivers, J. London Math. Soc. **45** (1992), 32–54.

[38] *S. Liu*, Semi-stable components of an Auslander-Reiten quiver, J. London Math. Soc. **47** (1993), 405–416.

[39] *S. Liu*, Tilted algebras and generalized standard Auslander-Reiten components, Archiv. Math. (Basel) **61** (1993), 12–19.

[40] *S. Liu*, Almost split sequences for non-regular modules, Fund. Math. **143** (1993), 183–190.

[41] *P. Malicki*, On the composition factors of indecomposable modules in almost cyclic coherent Auslander-Reiten components, J. Pure Appl. Algebra **207** (2006), 469–490.

[42] *P. Malicki*, *J. A. de la Peña* and *A. Skowroński*, Cycle-finite module categories, in: Algebras, Quivers and Representations - Abel Symposium 2011. Abel Symposia **8**, Springer-Verlag, 2013, pp. 209–252.

[43] *P. Malicki*, *J. A. de la Peña* and *A. Skowroński*, On the number of terms in the middle of almost split sequences over cycle-finite artin algebras, Centr. Eur. J. Math. **12** (2014), 39–45.

[44] *P. Malicki* and *A. Skowroński*, Almost cyclic coherent components of an Auslander-Reiten quiver, J. Algebra **229** (2000), 695–749.

[45] *P. Malicki* and *A. Skowroński*, Algebras with separating almost cyclic coherent Auslander-Reiten components, J. Algebra **291** (2005), 208–237.

[46] *P. Malicki* and *A. Skowroński*, On the indecomposable modules in almost cyclic coherent Auslander-Reiten components, J. Math. Soc. Japan **63** (2011), 1121–1154.

[47] *P. Malicki* and *A. Skowroński*, Algebras with separating Auslander-Reiten components, in: Representations of Algebras and Related Topics, European Math. Soc. Series Congress Reports, European Math. Soc. Publ. House, Zürich, 2011, pp. 251–353.

[48] *P. Malicki*, *A. Skowroński* and *B. Tomé*, Indecomposable modules in coils, Colloq. Math. **93** (2002), 67–130.

[49] *L.G. Peng* and *J. Xiao*, On the number of DTr-orbits containing directing modules, Proc. Amer. Math. Soc. **118** (1993), 753–756.

[50] *J. A. de la Peña*, Tame algebras with sincere directing modules, J. Algebra **161** (1993), 171–185.

[51] *J. A. de la Peña* and *A. Skowroński*, Geometric and homological characterizations of polynomial growth strongly simply connected algebras, Invent. Math. **126** (1996), 287–296.

[52] *J. A. de la Peña* and *A. Skowroński*, The Tits and Euler forms of a tame algebra, Math. Ann. **315** (1999), 37–59.

[53] *J. A. de la Peña* and *A. Skowroński*, Algebras with cycle-finite Galois coverings, Trans. Amer. Math. Soc. **363** (2011), 4309–4336.

[54] *J. A. de la Peña* and *M. Takane*, On the number of terms in the middle of almost split sequences over tame algebras, Trans. Amer. Math. Soc. **351** (1999), 3857–3868.

[55] *J. A. de la Peña* and *B. Tomé*, Iterated tubular algebras, J. Pure Appl. Algebra **64** (1990), 303–314.

[56] *I. Reiten* and *A. Skowroński*, Sincere stable tubes, J. Algebra **232** (2000), 64–75.

[57] *I. Reiten* and *A. Skowroński*, Characterizations of algebras with small homological dimensions, Advances Math. **179** (2003), 122–154.

[58] *I. Reiten* and *A. Skowroński*, Generalized double tilted algebras, J. Math. Soc. Japan **56** (2004), 269–288.

[59] *I. Reiten*, *A. Skowroński* and *S.O. Smalø*, Short chains and short cycles of modules, Proc. Amer. Math. Soc. **117** (1993), 343-354.

[60] *C. M. Ringel*, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math. **1099**, Springer-Verlag, Berlin-Heidelberg, 1984.

[61] *C. M. Ringel*, The canonical algebras, with an appendix by W. Crawley-Boevey in: Topics in Algebra, Part 1: Rings and Representations of Algebras, Banach Center Publ. **26**, PWN, Warsaw, 1990, pp. 407–432.

[62] *D. Simson* and *A. Skowroński*, Elements of the Representation Theory of Associative Algebras 2: Tubes and Concealed Algebras of Euclidean Type, London Mathematical Society Student Texts **71**, Cambridge University Press, Cambridge, 2007.

[63] *D. Simson* and *A. Skowroński*, Elements of the Representation Theory of Associative Algebras 3: Representation-Infinite Tilted Algebras, London Mathematical Society Student Texts **72**, Cambridge University Press, Cambridge, 2007.

[64] *A. Skowroński*, Selfinjective algebras of polynomial growth, Math. Ann. **285** (1989), 177–199.

[65] *A. Skowroński*, Generalized standard Auslander-Reiten components without oriented cycles, Osaka J. Math. **30** (1993), 515–527.

[66] *A. Skowroński*, Generalized standard Auslander-Reiten components, J. Math. Soc. Japan **46** (1994), 517–543.

[67] *A. Skowroński*, Regular Auslander-Reiten components containing directing modules, Proc. Amer. Math. Soc. **120** (1994), 19–26.

[68] *A. Skowroński*, On the composition factors of periodic modules, J. London Math. Soc. **46** (1994), 477–492.

[69] *A. Skowroński*, Cycles in module categories, in: Finite Dimensional Algebras and Related Topics, NATO ASI Series, Series C: Math. and Phys. Sciences **424**, Kluwer Acad. Publ., Dordrecht, 1994, pp. 309-345.
[70] *A. Skowroński*, Cycle-finite algebras, J. Pure Appl. Algebra **103** (1995), 105–116.
[71] *A. Skowroński*, Simply connected algebras of polynomial growth, Compositio Math. **109** (1997), 99-133.
[72] *A. Skowroński*, Tame algebras with strongly simply connected Galois coverings, Colloq. Math. **72** (1997), 335–351.
[73] *A. Skowroński*, Tame quasi-tilted algebras, J. Algebra **203** (1998), 470–490.
[74] *A. Skowroński*, Generalized canonical algebras and standard stable tubes, Colloq. Math. **90** (2001), 77–93.
[75] *A. Skowroński*, On artin algebras with almost all indecomposable modules of projective or injective dimension at most one, Centr. Eur. J. Math. **1** (2003), 108–122.
[76] *A. Skowroński*, A construction of complex syzygy periodic modules over symmetric algebras, Colloq. Math. **103** (2005), 61-69.
[77] *A. Skowroński*, Selfinjective algebras: finite and tame type, in: Trends in Representation Theory of Algebras and Related Topics, Contemporary Math. **406**, Amer. Math. Soc., Providence, RI, 2006, pp. 169–238.
[78] *A. Skowroński* and *K. Yamagata*, Socle deformations of self-injective algebras, Proc. London Math. Soc. **72** (1996), 545-566.
[79] *A. Skowroński* and *K. Yamagata*, Frobenius Algebras I. Basic Representation Theory, European Mathematical Society Textbooks in Mathematics, European Math. Soc. Publ. House, Zürich, 2011.
[80] *D. Smith*, On generalized standard Auslander-Reiten components having only finitely many non-directing modules, J. Algebra **279** (2004), 493–513.
[81] *D. Smith*, Almost laura algebras, J. Algebra **319** (2008), 432–456.
[82] *Y. Zhang*, The structure of stable components, Canad. J. Math. **43** (1991), 652–672.

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland
E-mail address: `pmalicki@mat.uni.torun.pl`

Centro de Investigación en Mathemáticas (CIMAT), Guanajuato, México
E-mail address: `jap@cimat.mx`

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland
E-mail address: `skowron@mat.uni.torun.pl`