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Marcin Fałdziński
Nicolaus Copernicus University in Toruń

Application of Modified POT Method with Volatility Model for Estimation of Risk Measures

A b s t r a c t. The main aim of this paper is the presentation and empirical analysis of the new approach which combines volatility models with Peaks over Threshold method that comes from extreme value theory. The new approach is applied for estimation of risk measures (VaR and ES) in financial time series. For the empirical analysis the financial risk model evaluation was conducted. In this paper the POT method was compared with standard volatility models (GARCH and SV) in case of the conditional modeling.

K e y w o r d s: Extreme Value Theory, Peaks over Threshold, Value-at-Risk, Expected Shortfall.

1. Introduction

Current literature in the area of risk management is very extensive (Szegö, 2004) and results are frequently ambiguous. The existing approaches for estimating the profit/loss distribution of a portfolio of financial instruments can be schematically divided into three groups: nonparametric historical simulation methods, parametric methods based on volatility models (GARCH type models) and methods based on the Extreme Value Theory. This work focuses on methods based on volatility models and Peaks over Threshold (POT) method. Previous results from papers which applied the POT method in risk management were the main motivation to deal with a problem of extremes in financial time series. The main aim of this paper is to propose and analyse more complex approach taking into account extremes and non-extremes in risk management of financial time series. Most of the attention is focused on the new approach proposed in this paper.

2. The POT Method and Volatility Models

In the Peaks over Threshold method we are interested in excesses over a high threshold value u. Detailed description of POT method can be found in:

Embrechts, Klüppelberg, Mikosch (2003) or Osińska, Fałdziński (2008). Here will be presented McNeil and Frey's approach, which joint volatility models and POT method. We assume that X_t is a time series representing daily observations of log return on a financial asset price. We assume that dynamics of X process is given as:

$$X_t = \mu_t + \sigma_t Z_t , \tag{1}$$

where innovations Z_t are the white noise process with zero mean and unit variance. We assume that μ_t is the expected return and σ_t is the volatility of the return, where both are measurable to information set F_{t-1} at time $t-1$. To implement an estimation procedure for the process (1), we need to choose a dynamic conditional mean as well as a conditional variance model. Many volatility models were proposed in econometric literature, from ARCH/GARCH models, and their different modifications and generalization, to SV models. McNeil and Frey defined simple risk measures forms for one day horizon with relation to process (1) as:

$$VaR_q^t = \mu_{t+1} + \sigma_{t+1} VaR(Z)_q , \tag{2}$$

$$ES_q^t = \mu_{t+1} + \sigma_{t+1} ES(Z)_q , \tag{3}$$

where $VaR_q^t(Z)$ is the Value-at-Risk of Z_t process, and $ES_q^t(Z)$ is the corresponding Expected Shortfall. The method proposed by them demands minimal assumptions for innovations distribution and focus on modeling distribution tails using Extreme Value Theory. Generally we can say, that we use two stage approach, which can be presented in the following steps:

1. Fit a GARCH-type model (generally volatility model) to return series. Estimate μ_{t+1} and σ_{t+1} using fitted model and calculate standardized residuals. It means, that fitted model is used to estimate one-day ahead predictions of μ_{t+1} and σ_{t+1}.
2. EVT is used to estimate $VaR(Z)_q$ and $ES(Z)_q$ with application of POT method for mentioned residuals.

A Value-at-Risk in the EVT for the Peaks over Threshold method is equal:

$$VaR(\alpha) = u + \frac{\hat{\sigma}}{\hat{\gamma}}\left(\left(\frac{n}{N_u}\alpha\right)^{-\hat{\gamma}} - 1\right), \tag{4}$$

where α is a tolerance level, u is a threshold, $\hat{\gamma}, \hat{\sigma}$ are estimated parameters from Generalised Pareto Distribution (GPD), n is the total number of realizations and N_u is number of extremes. Because of VaR drawbacks the alternative risk measure was developed, which is called Expected Shortfall (ES), and was proposed by Artzner et al. (1997; 1999). The Expected Shortfall for the Peaks over Threshold method is given by:

$$ES(\alpha) = \frac{VaR(\alpha)}{1-\hat{\gamma}} + \frac{\hat{\sigma} - \hat{\gamma}u}{1-\hat{\gamma}}, \tag{5}$$

In the literature we can find comparisons of models which estimate VaR and ES where extreme value theory is applied (i.e. Brooks, Clare, Dalle Molle, Persand, 2005; Harmantzis, Miao, Chien, 2006; Kuester, Mittnik, Paolella, 2006; Osińska, Fałdziński, 2008; Fałdziński, 2008). In all of these papers authors find, that EVT is good or very good approach to estimate risk measures. Empirical results presented in these papers show that volatility models with application of Extreme Value Theory more accurately estimate expected future values of asset returns, particularly in case of extremely rare events (i.e. extremes). Standard volatility models better fit to mean values in financial time series in contrast to models with EVT. Therefore the new approach is based on such combination, that extremes are estimated using POT method, and non-extremes are estimated using standard volatility models. This combination is an attempt to identify extremes in financial time series. The new approach can be written in the following:

$$N\text{-}GARCH\text{-}POT = \begin{cases} GARCH\text{-}POT & \mu_{t+1} + \sigma_{t+1} \geq u_t \\ GARCH & \mu_{t+1} + \sigma_{t+1} < u_t \end{cases} \tag{6}$$

This new hybrid of volatility model and POT method is based on conditional volatility forecast σ_{t+1}, conditional mean μ_{t+1} and on threshold u_t. If sum of μ_{t+1} and σ_{t+1} forecasts are higher than threshold then we apply McNeil and Frey's approach, in other case we have standard GARCH model. Threshold u_t could be constant or time-varying. Switching mechanism is formulated to identify whether the forecast of time series return is an extreme value, or not. Forecasts of asset returns are very important, the more precise forecast of time series is, the extremes are better identified, which is connected to proper switching mechanism (6). The idea of new approach is completely based on the forecast capability from volatility model for time series (in this case it is GARCH model but we also used SV model).

3. Backtesting

A key part for risk measures is necessity to check correctness of estimation and simultaneously choose the most precise method for their estimates. Risk models need to be validated and backtesting is the class of quantitative methods used to e.g. rank a group of models against each other (Dowd, 2005; Alexander, 2008). For backtesting we used three binominal tests: the failure test LR_{uc}, the mixed Kupiec-test $LR_{ind}K$ (Haas, 2001) and the test of independence $LR_{ind}CH$

(Christoffersen, 1998). Unfortunately presented tests have weak power, and additionally these methods do not give the opportunity to create a ranking of the models. Angelidis and Degiannakis (2006) presented modified Lopez's (Lopez, 1999) loss function:

$$\Psi_{1,t+1} = \begin{cases} \left| y_{t+1} - ES_{t+1|t} \right| & \text{if violation occurs} \\ 0 & \text{else} \end{cases} \tag{7}$$

$$\Psi_{2,t+1} = \begin{cases} \left(y_{t+1} - ES_{t+1|t} \right)^2 & \text{if violation occurs} \\ 0 & \text{else} \end{cases} \tag{8}$$

To judge which model is the best, we compute the mean absolute error $MAE = \sum_{t=1}^{\tilde{T}} \Psi_{1,t} / \tilde{T}$, and the mean squared error $MSE = \sum_{t=1}^{\tilde{T}} \Psi_{2,t} / \tilde{T}$, where $\tilde{T}$ is the number of the forecasts, and total loss (TL) is the sum of these errors (Angelidis, Degiannakis, 2006). The loss function approach is based on calculating magnitude of violations (or exceedances), i.e. cases when the risk measure underestimates a future value of asset. As we can see the presented loss functions measure only the underestimation of risk measure. A perfect situation is when an estimated risk measure does not underestimate and overestimate[1] too much a future value of asset. For example, if we would have two estimated risk measures and both of them have almost equal value of standard loss functions, then the better risk measure is that which has the lower overestimation. An overestimation of risk measures was proposed to measure (Fałdziński, 2009)**.** The overestimation of loss functions are given in the following:

$$\Phi_{1,t+1} = \begin{cases} \left| ES_{t+1} - y_{t+1} \right| & \text{if } \ 0 < y_{t+1} < ES_{t+1} \\ 0 & \text{if } \ y_{t+1} \le 0 \end{cases} \tag{17}$$

$$\Phi_{2,t+1} = \begin{cases} \left(ES_{t+1} - y_{t+1} \right)^2 & \text{if } \ 0 < y_{t+1} < ES_{t+1} \\ 0 & \text{if } \ y_{t+1} \le 0 \end{cases} \tag{18}$$

We also compute the mean absolute error of overestimation $MAE_{over} = \sum_{t=1}^{d} \Phi_{1,t} / d$, and the mean squared error $MSE_{over} = \sum_{t=1}^{d} \Phi_{2,t} / d$, where $d = \sum_{i=1}^{\tilde{T}} \mathbf{1}$, $\mathbf{1} = \begin{cases} 1 & \text{if } \ 0 < y_{t+1} < ES_{t+1} \\ 0 & \text{if } \ y_{t+1} \le 0 \end{cases}$ is the number of the positive fore-

[1] Cases when estimated risk measure is higher than value of asset

casts (i.e. larger then 0, but smaller than the given risk measure). Similarly we could construct the total loss of overestimation $(OTL = MAE_{over} + MSE_{over})$. The OTL could also be computed for VaR and SRM, but then we have to change ES_{t+1} for another estimated risk measure in formula (17), (18) and d.

4. Empirical Analysis

The subject of the empirical analysis is the comparison of the estimated Value-at-Risk and Expected Shortfall measures for the new approach with volatility models. The comparison is based on selection of the best model for total loss *TL* proposed by Angelidis and Degiannakis and total loss of overestimation (*OTL*). In the analysis the SV model with Gaussian distribution and the GARCH model with Gaussian and t-Student error distributions were used. We have chosen SV and GARCH models because they represent the most standard volatility models. The parameters were estimated with the maximum likelihood method in case of GARCH models and the *quasi*-maximum likelihood method in the case of the SV models. The time series used in the analysis comprise 3000 observations of log returns (daily data: 07.11.1994 – 31.10.2006). We used 5 financial time series (WIG, SP500, DAX, FTSE100, NIKKEI225) that represent the stock market returns. For each time series a thousand VaRs and ESs were estimated for backtesting purposes. To compute the ES for the volatility models we used Dowd's approach (Dowd, 2005). Also we used the time-varying threshold u, as a result of defining the number of extremes on 10% level for all observations in time series. This 10% level is a common standard in similar analyses.

Results for the binominal tests were computed besides their drawbacks. In short we can say that findings were very diverse and it was very difficult to make general conclusions that is why we decided no to show them. They are available upon request. Based on findings contained in Table 1 we can see that SV-POT model has the lowest value for total loss *TL* (next are N-SV-POT and SV models). It should not be a surprise because SV models more take into account the extremes than standard GARCH model. After the class of SV models (i.e. SV, SV-POT and N-SV-POT) we have four variant of GARCH-POT models with relatively higher values of *TL*. The end of the total loss ranking contains N-GARCH-POT and GARCH models alternately, but we should point out that N-GARCH-POT models are relatively better. On the other hand the class of SV models has the highest value of the total loss of overestimation (*OTL*) for value-at-risk and this is the consequence of the same property mentioned before. The lowest value of *OTL* for VaR is obtained for GARCH TD model. We have this kind of result because the standard GARCH model do not take into consideration the extremes like the other models in the analysis and that is why the overestimation is the lowest. The next in the ranking are GARCH and N-GARCH-POT models alternately.

Table 1. Backtesting results for WIG20 and SP500

WIG20 α=0.05						
Model	TL	Rank TL	OTL VaR	Rank OTL VaR	OTL ES	Rank OTL ES
GARCH	0.0998	11	2.788	3	7.446	7
GARCH TD	0.2019	15	2.680	1	10.111	14
AR-GARCH	0.0927	10	2.980	7	7.778	9
AR-GARCH TD	0.1874	14	2.812	4	10.416	15
SV	0.0281	3	4.420	13	7.428	6
GARCH-POT	0.0498	4	3.219	12	5.913	3
GARCH-POT TD	0.0525	5	3.115	9	5.733	2
AR-GARCH-POT	0.0734	7	3.118	10	6.817	4
AR-GARCH-POT TD	0.0530	6	3.101	8	5.634	1
SV-POT	0.0069	1	7.631	15	9.425	12
N-GARCH-POT	0.0823	8	2.909	5	6.905	5
N-GARCH-POT TD	0.1622	13	2.775	2	9.031	11
N-AR-GARCH-POT	0.0857	9	3.126	11	7.679	8
N-AR-GARCH-POT TD	0.1609	12	2.922	6	9.957	13
N-SV-POT	0.0162	2	6.486	14	8.754	10
SP500 α=0.05						
Model	TL	Rank LF	OTL VaR	Rank OTL VaR	OTL ES	Rank OTL ES
GARCH	0.0554	11	1.890	3	5.025	8
GARCH TD	0.1261	15	1.807	1	6.827	13
AR-GARCH	0.0495	10	2.080	11	5.379	11
AR-GARCH TD	0.0997	14	2.003	5	7.168	15
SV	0.0102	3	2.344	13	3.850	5
GARCH-POT	0.0194	7	2.038	7	3.450	1
GARCH-POT TD	0.0191	6	2.038	8	3.461	3
AR-GARCH-POT	0.0191	5	2.050	9	3.453	2
AR-GARCH-POT TD	0.0175	4	2.051	10	3.489	4
SV-POT	0.0015	1	4.048	15	5.108	9
N-GARCH-POT	0.0435	8	1.890	4	4.762	7
N-GARCH-POT TD	0.0854	12	1.822	2	6.222	12
N-AR-GARCH-POT	0.0440	9	2.084	12	5.245	10
N-AR-GARCH-POT TD	0.0880	13	2.006	6	6.873	14
N-SV-POT	0.0098	2	2.959	14	4.315	6

Note: N-AR-GARCH-POT TD - means the new approach proposed in this paper with AR(1)-GARCH(1,1) model and Peak over Threshold method where t-distribution was applied. Respectively other abbreviation are constructed.

It means that the switching mechanism takes into account non-extremes rather than extremes. In case of *OTL* for VaR, GARCH-POT models are better than the class of the SV models, but relatively worse than the other models. The lowest values of *OTL* for Expected Shortfall have been obtained for GARCH-POT models. Generally we can say that N-GARCH-POT models are relatively better than GARCH models in case of *OTL* for ES.

Table 2. Ranking according to total loss *TL* for indices

Model	WIG	WIG20	SP500	DAX	FTSE100	NIKKEI225
GARCH	10	11	11	9	11	10
GARCH TD	15	15	15	15	15	15
AR-GARCH	11	10	10	8	8	11
AR-GARCH TD	14	14	14	12	14	14
SV	3	3	3	3	3	3
GARCH-POT	6	4	7	5	4	7
GARCH-POT TD	7	5	6	4	5	4
AR-GARCH-POT	4	7	5	7	6	5
AR-GARCH-POT TD	5	6	4	6	7	6
SV-POT	1	1	1	1	1	1
N-GARCH-POT	9	8	8	11	10	9
N-GARCH-POT TD	13	13	12	14	12	13
N-AR-GARCH-POT	8	9	9	10	9	8
N-AR-GARCH-POT TD	12	12	13	13	13	12
N-SV-POT	2	2	2	2	2	2

Table 3. Ranking according to total loss of overestimation *OTL* for VaR

Model	WIG	WIG20	SP500	DAX	FTSE100	NIKKEI225
GARCH	3	3	3	4	8	11
GARCH TD	1	1	1	2	2	8
AR-GARCH	6	7	11	12	12	12
AR-GARCH TD	4	4	5	11	10	7
SV	13	13	13	13	13	13
GARCH-POT	10	12	7	7	4	4
GARCH-POT TD	8	9	8	8	3	1
AR-GARCH-POT	12	10	9	5	5	3
AR-GARCH-POT TD	9	8	10	6	6	2
SV-POT	15	15	15	15	15	15
N-GARCH-POT	5	5	4	3	7	9
N-GARCH-POT TD	2	2	2	1	1	5
N-AR-GARCH-POT	11	11	12	10	11	10
N-AR-GARCH-POT TD	7	6	6	9	9	6
N-SV-POT	14	14	14	14	14	14

If we compare models at the same class based on total loss *TL* (Table 2) we can state that new approach is placed between McNeil and Frey's method and standard volatility models. Similar conclusion can be deduced from the analysis of total loss of overestimation *OTL* for expected shortfall (Table 4). It means that the new approach is as good as the specific volatility model. To be accurate, if the forecast of conditional mean and conditional volatility for volatility model is more precise, then the new approach is better than two other methods. In case of total loss of overestimation *OTL* for Value-at-Risk (Table 3), we can see that volatility models are the best. It should not be a surprise, because VaR better fits to small and mean values of financial time series in comparison with ES. The new approach which connects volatility models and POT method, like before is placed between two other methods, but sometimes this method is the best.

Table 4. Ranking according to total loss of overestimation *OTL* for ES

Model	WIG	WIG20	SP500	DAX	FTSE100	NIKKEI225
GARCH	8	7	8	8	10	8
GARCH TD	13	14	13	11	14	14
AR-GARCH	9	9	11	9	11	9
AR-GARCH TD	15	15	15	13	15	15
SV	5	6	5	12	5	5
GARCH-POT	3	3	1	4	2	4
GARCH-POT TD	1	2	3	3	1	2
AR-GARCH-POT	4	4	2	2	3	3
AR-GARCH-POT TD	2	1	4	1	4	1
SV-POT	10	12	9	15	7	11
N-GARCH-POT	7	5	7	5	8	6
N-GARCH-POT TD	12	11	12	7	12	13
N-AR-GARCH-POT	11	8	10	6	9	7
N-AR-GARCH-POT TD	14	13	14	10	13	12
N-SV-POT	6	10	6	14	6	10

Generally we can say, that the new approach, which is a hybrid of standard volatility models and McNeil and Frey's method, is as precise as the specific volatility model and is able to forecast the financial time series.

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Zastosowanie zmodyfikowanej metody POT z modelami zmienności do szacowania miar ryzyka

Z a r y s t r e ś c i. Celem artykułu jest prezentacja nowego podejścia mającego na celu połączenie modeli zmienności z metodą Peaks over Threshold (POT), wywodzącą się z teorii wartości ekstremalnych. Podejście to opiera się na możliwości szacowania ekstremów na podstawie metody POT, natomiast wartości średnich na podstawie modeli zmienności. Nowe podejście jest zastosowane do estymacji miar ryzyka (VaR i ES) dla finansowych szeregów czasowych. Do oceny nowego podejścia wykorzystano procedury testowania wstecznego. W pracy zastosowano metodę POT dla stóp zwrotu indeksów rynków finansowych przefiltrowanych za pomocą modeli GARCH oraz SV, które porównano z wynikami otrzymanymi tylko za pomocą modeli GARCH i SV.

S ł o w a k l u c z o w e: teoria wartości ekstremalnych, Peaks over Threshold, miary ryzyka