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Application of the Family of Sign RCA Models for Obtaining the Selected Risk Measures†

A b s t r a c t. Accurate modelling of risk is very important in finance. There are many alternative risk measures, however none of them is dominating. This paper proposes to use the family of Sign RCA models to obtain the Value-at-Risk (VaR) and Expected Shortfall (ES) measures. For models from the family of Sign RCA models and AR-GARCH model the one-step forecasts of VaR were calculated based on rolling estimates from the given model using different window sizes. To obtain the VaR and ES measures the filtered historical simulation was used in new version proposed by Christoffersen. The results were verified using backtesting and the loss function.

K e y w o r d s: Family of Sign RCA Models, risk measures, Value at Risk, Expected Shortfall.

1. Introduction

Random coefficient autoregressive models (RCA) are the straightforward generalization of the constant coefficient autoregressive models. A full description of this class of models including their properties, estimation methods and some application was originally presented by Nicholls and Quinn (1982). In later years, these models have been not so popular like GARCH models (Bollerslev, 1986; Engle, 1982) in general. GARCH models are easy to understand and estimate and they could describe a non-linear dynamics of financial time series. However, in the last decade one can see that RCA models gained more interest again. As a result some of RCA model were produced.

The aim of this paper is to apply the family of Sign RCA models to obtain the selected risk measures for daily and weekly data. Risk measures through different calculation method are obtained.

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2. The Family of Sign RCA Models

In the Table 1 equations of individual models from the family of Sign RCA models and their names were presented.

Table 1. The family of Sign RCA models (without conditions)

Model	Model equations	Equation
RCA(1)	$y_t = (\phi + \delta_t) y_{t-1} + \varepsilon_t$	I
Sign RCA(1)	$y_t = (\phi + \delta_t + \Phi s_{t-1}) y_{t-1} + \varepsilon_t$	II
RCA(1)-MA(1)	$y_t = (\phi + \delta_t) y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$	III
Sign RCA(1)-MA(1)	$y_t = (\phi + \delta_t + \Phi s_{t-1}) y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$	IV
RCA(1)-GARCH(1,1)	$y_t = (\phi + \delta_t) y_{t-1} + \varepsilon_t$, $\varepsilon_t = \sqrt{h_t} z_t$, $h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}$	V
Sign RCA(1)-GARCH(1,1)	$y_t = (\phi + \delta_t + \Phi s_{t-1}) y_{t-1} + \varepsilon_t$, $\varepsilon_t = \sqrt{h_t} z_t$, $h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}$	VI

Note: s_t – sign function is described by equation (3); ϕ, θ, Φ, α_i, β_1 – model parameters.

To ensure the existence of the I-VI models the following assumption must be satisfied:

$$\begin{pmatrix} \delta_t \\ \varepsilon_t \end{pmatrix} \sim^{iid} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\delta^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{pmatrix} \right), \tag{1}$$

$$\phi^2 + \sigma_\delta^2 < 1 . \tag{2}$$

The sign function, described by following formula

$$s_t = \begin{cases} 1 & \text{dla } y_t > 0 \\ 0 & \text{dla } y_t = 0 \\ -1 & \text{dla } y_t < 0 \end{cases}, \tag{3}$$

has the interpretation: if $\phi + \delta_t > |\Phi|$, the negative value of Φ means that the negative (positive) observation values at time $t-1$ correspond to a decrease (increase) of observation values at time t. In the case of stock returns it would suggest (for returns) that after a decrease of stock returns the higher decrease of stock returns occurs than expected, and in the case of the increase of stock returns the lower increase in stock returns occurs than expected.

Condition (2) is necessary and sufficient for the second-order stationarity of process described by equation I, however conditions (1)-(2) ensure the strict stationarity of this process. If conditions (1)-(2) are satisfied, then processes described by equations II-IV are stationary in mean. Theoretical properties of processes described by equations I-VI, satisfying conditions (1)-(2) can be found in several articles (Appadoo, Thavaneswaran, Singh, 2006; Aue, 2004; Górka, 2008; Thavaneswaran, Appadoo, Bector, 2006; Thavaneswaran, Appadoo, 2006).

Residuals from the RCA model can be described by the GARCH model (Thavaneswaran, Peiris, Appadoo, 2008; Thavaneswaran, Appadoo, Ghahramani, 2009). Then, the RCA(1)-GARCH(p,q) model described by equation V, where $z_t \sim N(0, \sigma_z^2)$, $\alpha_0 > 0$, $\alpha_i \geq 0$ and $\beta_j \geq 0$, is obtained.

When the sign function is added to the RCA-GARCH model, then the process described by equation VI is obtained (Thavaneswaran, Appadoo, Ghahramani, 2009). The conditions ensuring the positive value of conditional variance of this process are following: $z_t \sim N(0, \sigma_z^2)$, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $|\Phi| \leq \alpha_0$.

3. The Selected Risk Measures

In this paper two tools for measuring market risk were used, i. e. Value-at-Risk (VaR) and Expected Shortfall (ES). Value-at-Risk is the maximum loss over a target horizon such that there is a low, prespecified probability that the actual loss will be larger. Expected Shortfall is a coherent alternative to Value-at-Risk (Acerbi, Tasche, 2002). It is the expected loss conditional on exceeding VaR.

One-step-ahead conditional forecasts of Value-at-Risk are calculated in two ways. Firstly, Value-at-Risk is calculated by formula:

$$VaR_{t+1}^{l}(\alpha) = \mu_{t+1|t} + \sigma_{t+1|t} z_\alpha,$$

where $\mu_{t+1|t}$, $\sigma_{t+1|t}$ are one-step-ahead conditional forecasts of mean and volatility respectively.

Secondly, the formula proposed by Christoffersen (2009) is used:

$$VaR_{t+1}^{l}(\alpha) = \mu_{t+1|t} + \sigma_{t+1|t} q_\alpha^{u_t},$$

where $q_\alpha^{u_t}$ denotes the percentile of the set of standardization historical shocks u_t.

Expected Shortfall for the one-step forecast can be calculated as:

- average of values exceeding VaR (YY) (Yamai, Yoshiba, 2002),
- weighted average of values exceeding VaR (Dowd) (Dowd, 2002),

– filtered historical simulation (FHS), in the version proposed by Christoffersen (2009), i. e.

$$ES_{t+1}(\alpha) = \sigma_{t+1|t} \frac{1}{\alpha T} \sum_{t=1}^{T} u_t \cdot 1\left(u_t < \frac{VaR_{t+1}^{l}(\alpha)}{\sigma_{t+1|t}} \right),$$

where $1(*)$ denotes the indicator function returning a unit if the argument is true, and zero otherwise; α is tolerance level.

To check the accuracy of risk measures the backtesting of VaR and ES using the traditional VaR tests and the loss function was conducted.

The traditional VaR tests used to compare results are following:

– proportion of failures test[1] – LR_{POF},
– independence test – LR_{IND},
– time between failures test – LR_{TBF},

The loss functions used to compare results are following:

– regulatory loss function – RL
– firm's loss function – FL
– loss function with the expected loss proposed by Angelidis, Degiannakis (2006). It can be described as $LF_{ES} = MAE + MSE$ where $MAE = \frac{1}{N}\sum_{t=1}^{N} f_{1|t}$, $MSE = \frac{1}{N}\sum_{t=1}^{N} f_{2|t}$ while:

$$f_{1|t+1} = \begin{cases} 0 & r_{t+1} > -VaR_{r,t}, \\ \left| r_{t+1} + ES_{r,t} \right| & r_{t+1} \le -VaR_{r,t}, \end{cases}$$

$$f_{2|t+1} = \begin{cases} 0 & r_{t+1} > -VaR_{r,t}, \\ \left(r_{t+1} + ES_{r,t} \right)^2 & r_{t+1} \le -VaR_{r,t}. \end{cases}$$

4. Empirical Results

The data used in the empirical application are eight stock exchange indexes and thirty three share prices of the Polish firms' from the Warsaw Stock Exchange. It gives forty one time series. The data were obtained from bossa.pl for the period from November 30, 1998 to November 4, 2008, what gives 2490 daily percentage log returns and 493 weekly percentage log returns.

The calculations were carried out in the Gauss and Microsoft Excel.

[1] Other name of this test is the Kupiec test.

Firstly, for each returns series the descriptive statistics and some tests were calculated. All series have positive kurtosis (leptokurtic). Some of returns series are autocorrelated.

Secondly, parameters of six models from the Sign RCA family were estimated using maximum likelihood (MLE). The number of models from the family Sign RCA models with statistically significant parameters for the total sample is presented in the Table 2.

Table 2. The number of models with significant parameters from the family of Sign RCA models for percentage log returns (2490 daily data and 493 weekly data)

Model	α = 5%		α = 10%	
	daily data	weekly data	daily data	weekly data
AR(1)	26	9	27	10
RCA(1)	24	5	24	10
Sign RCA(1)	1	2	3	2
RCA(1)-MA(1)	15	21	19	22
Sign RCA(1)-MA(1)	5	3	7	4
RCA(1)-GARCH(1,1)	25	2	25	2
Sign RCA(1)-GARCH(1,1)	-	-	1	1

It is seen that the models like AR(1), RCA(1), RCA(1)-MA(1) and RCA(1)-GARCH(1,1) were found in about 50 percentage of cases. For smaller samples, similar results are obtained (see the Table 3). Models with sign function occur very seldom in empirical time series (see also the Table 3) both at the 5% and 10% significance level, and also for different size of sample and different level of data aggregation (both daily and weekly data).

Table 3. The number of models with significant parameters from the family of Sign RCA models for percentage log returns (1500 daily data and 300 weekly data)

Model	α = 5%		α = 10%	
	daily data	weekly data	daily data	weekly data
AR(1)	23	12	25	13
RCA(1)	22	10	24	11
Sign RCA(1)	2	1	3	3
RCA(1)-MA(1)	20	24	22	25
Sign RCA(1)-MA(1)	2	7	3	10
RCA(1)-GARCH(1,1)	18	-	20	-
Sign RCA(1)-GARCH(1,1)	1	-	2	1

For example, the RCA models for the selected indexes are presented in the Table 4.

On the basis of models from the family of Sign RCA models and AR-GARCH model fitted to the different window size, i.e, N =250, 500 and 1500 the one-step ahead forecasts of VaR and ES were made forecasting one-step-ahead from the end of window till the next 500 observations which were hold out. It should be pointed out that the last observation of each sample (N =250, 500 and 1500) is placed at the same point at time, hence these samples can be treated as overlapping. Each one-step ahead forecast was generated from estimates of the given model[2] using a sequence of rolling windows (with window size of 250, 500 and 1000 observations) which were moved 500 times by one observation on time axis.

Table 4. The RCA model for the selected indexes

	WIG-BUDOW		WIG-SPOZYW	
	daily data	weekly data	daily data	weekly data
ϕ	0.131	0.133	0.119	0.186
$S(\phi)$	0.024	0.053	0.026	0.059
σ_ε^2	1.799	12.441	1.300	6.835
σ_δ^2	0.199	0.135	0.291	0.357
ln L	-4480.33	-1348.43	-4146.82	-1244.88
Q(3)	4.810	5.784	19.371*	2.728
ARCH(3)	73.649*	11.911*	90.960*	24.364*
AIC	8966.66	2702.86	8299.64	2495.76
BIC	8984.12	2715.46	8317.10	2508.36

Note: Q(3) – the value of the Ljung-Box Q-statistic up to 3 lags, ARCH(3) – the value of the Engle ARCH test statistics up to 3 lags, AIC – Akaike information criterion, BIC – Bayesian information criterion.

The backtesting of forecasts of VaR and ES measures were carried out (the example results were presented in Table 5–7).

The empirically determined probability for the proportion of failures test for different window size is presented in Table 5. Almost all forecasts of VaR are underestimated. Only for RCA-MA with FHS method the VaR forecasts are overestimated. It is seen that as the window size decreases the scale of underestimation decreases, too.

The results of traditional VaR tests and loss function for the VaR forecasts were carried out from rolling estimation of models using window size of 250 observations for one of selected index (see Table 6).

[2] Forecasts were carried out form models with statistically significant parameters obtained in the basic sample.

Table 5. The empirically determined probability for the proportion of failures test for different window size (WIG-spozyw)

Model	Empirically determined probability N = 1500	N = 500	N = 250
AR-GARCH	6.8%	6.8%	6.6%
AR-GARCH (FHS)	10.8%	10.8%	8.8%
RCA	11,0%	9,0%	6.6%
RCA-GARCH	6.8%	6.6%	6.6%
RCA-GARCH (FHS)	10.8%	10,0%	8.8%
RCA (FHS)	13.4%	10.6%	7.4%
RCA-MA	11.8%	8.2%	6.4%
RCA-MA (FHS)	3.8%	3.6%	2.2%
Sign RCA	11.4%	9.2%	6.8%
Sign RCA-GARCH	6.8%	6.8%	7.2%
Sign RCA-GARCH (FHS)	10.8%	10,0%	8.4%
Sign RCA (FHS)	13.8%	10.6%	7.4%
Sign RCA-MA	12,0%	8.6%	6,0%
Sign RCA-MA (FHS)	9.8%	6.8%	4.8%

Note: N denotes the window size.

Table 6. Results of traditional VaR tests for forecasts carried out from rolling estimation of models using window size of 250 observations for the WIG-spozyw index

Model	LR_{POF}	LR_{IND}	LR_{TBF}	RL	FL
AR-GARCH	2.459	0.897	36.501	171.509	1386.669
AR-GARCH (FHS)	12.518***	1.291	64.588**	205.050	1259.617
RCA	2.459	0.897	49.856**	205.547	1281.037
RCA-GARCH	2.459	0.897	31.090	190.525	1411.081
RCA-GARCH (FHS)	12.518***	1.291	57.746*	219.107	1289.683
RCA (FHS)	5.317**	0.256	58.441**	215.699	1267.312
RCA-MA	1.903	4.389**	43.849*	191.254	1287.360
RCA-MA (FHS)	10.347***	0.496	25.287***	102.243	1927.830
Sign RCA	3.081	1.065	51.368***	207.307	1279.988
Sign RCA-GARCH	4.511**	0.171	39.764	200.195	1389.469
Sign RCA-GARCH (FHS)	10.194***	0.101	53.739	224.806	1289.986
Sign RCA (FHS)	5.317**	0.256	58.441	217.562	1272.351
Sign RCA-MA	0.992	3.840*	37.125	193.384	1289.115
Sign RCA-MA (FHS)	0.043	2.426	27.264	157.853	1412.718
Sym. Hist	9.1102***	0.134	56.766	235.584	1232.198

Note: *, **, *** indicate rejection of H_0 at the 10% ,5% and 1% significant level, respectively, LR_{POF} – the values of the proportion of failures test statistics, LR_{IND} – the values of the independence test statistics, LR_{TBF} – the values of the time between failures test statistics, RL – regulatory loss function, FL – firm's loss function.

For AR-GARCH, RCA-GARCH and Sign RCA-MA (with FHS method) models the null hypothesis for the traditional VaR tests is not rejected, what means that the proportion of failures is equal to the given tolerance level 5%, failures are serially independent and the time between failures is independent (see the Table 6). Slightly worse results are obtained for models: RCA (with and without FHS method), RCA-MA, Sign RCA and Sign RCA-MA.

In that case the regulatory loss function is the smallest for the Sign RCA-MA (with FHS method) model. Hence this model is preferred by the regulatory institution. However, the difference between this value of the regulatory loss function and their values calculated from other models are small, so these models are comparable. The firm's loss function takes the smallest value for AR-GARCH (with FHS method), so this model is preferred from the firm point of view, but other values of the firm's loss function are not much higher than for the best model (AR-GARCH, with FHS method).

Table 7. Values of loss function with the expected loss (LF_{ES}) for different window sizes for the WIG-spozyw index

Model	LF_{ES}		
	N = 1500	*N* = 500	*N* = 250
AR-GARCH (Dowd)	0,238	0,238	0,252
AR-GARCH (FHS)	0,451	0,451	0,385
AR-GARCH (YY)	0,235	0,235	0,266
RCA GARCH (Dowd)	0,243	0,233	0,296
RCA GARCH (FHS)	0,439	0,403	0,391
RCA GARCH (YY)	0,241	0,248	0,317
RCA (Dowd)	0,415	0,363	0,313
RCA (FHS)	0,567	0,440	0,389
RCA (YY)	0,408	0,381	0,332
RCA-MA (Dowd)	0,401	0,353	0,301
RCA-MA (FHS)	0,283	0,232	0,231
RCA-MA (YY)	0,399	0,371	0,319
Sign RCA GARCH (Dowd)	0,242	0,263	0,295
Sign RCA GARCH (FHS)	0,451	0,421	0,428
Sign RCA GARCH (YY)	0,240	0,277	0,316
Sign RCA (Dowd)	0,422	0,368	0,311
Sign RCA (FHS)	0,580	0,445	0,385
Sign RCA (YY)	0,418	0,387	0,331
Sign RCA-MA (Dowd)	0,411	0,370	0,306
Sign RCA-MA (FHS)	0,661	0,486	0,383
Sign RCA-MA (YY)	0,411	0,393	0,322

Note: N denotes the window size.

Next, the values of loss function with the expected loss (LF_{ES}) for different window size of rolling estimation for example for one of indexes presented in Table 7.

For the window size of 500 and 250 observations the smallest value of the loss function with the expected loss is obtained for the RCA-MA model (with FHS method), but for the window size of 1500 observations the best result is obtained from AR-GARCH model (with FHS method). It is seen that for the models without GARCH residuals the value of the LF_{ES} decreases as the window size decreases. Models with GARCH errors prefer bigger windows. The comparison of results of the LF_{ES} for models with and without the sign function shows that introducing the sign function into the model causes the increase of the loss function with the expected loss.

5. Summary

In this paper, the family of Sign RCA models to obtain the selected risk measures was presented. Empirical results showed that:

- Models with sign function occur seldom. It means that percentage log returns do not represent the asymmetric reaction to good or bad news coming from the market.
- Accuracy of the VaR measures for models with sign function and without GARCH errors depends on the size of window and almost all of them are underestimated (except the RCA-MA model with FHS method).
- Accuracy of the VaR forecasts for models with GARCH errors (without using FHS method) does not depend on the size of window and all of them are underestimated.
- Filtered historical simulation (FHS) (Christoffersen version) is sensitive to the size of window, i.e. for smaller window the empirically determined probability is closer to the nominal significance level for all models from the family of Sign RCA models.
- Using the Sign RCA-MA model with FHS method the empirical and nominal significance level are almost the same.
- For RCA-MA models with FHS method the forecasts of VaR are overestimated at the 5% significance level.
- The smallest values of the regulatory loss function were obtained for the VaR forecasts from RCA-MA model (FHS method).
- The smallest values of the firm's loss function were obtained for the VaR forecasts from the Sign RCA, RCA, AR-GARCH models (all models with FHS method).
- Loss function with expected loss (LF_{ES}) takes the smallest values for AR-GARCH models (Dowd and YY method) for each window size and

from the RCA-MA models (FHS method) for windows of 500 and 250 observations.

- Filtered historical simulation (Christoffersen version) generates bigger value of LF_{ES} than other analyzed method (except the RCA-MA model with FHS method).

To sum up, some models from the family of Sign RCA models can generate useful results of the VaR and ES measures only in some cases.

References

Acerbi, C., Tasche, D. (2002), Expected Shortfall: A Natural Coherent Alternative to Value at Risk, *Economic Notes*, 31, 379–388.

Angelidis, T., Degiannakis, S. (2006), Backtesting VaR Models: An Expected Shortfall Approach, Working Paper Series, http://econpapers.repec.org/paper/crtwpaper/0701.htm (2.09.2009).

Appadoo, S., Thavaneswaran, A., Singh J. (2006), RCA Models with Correlated Errors *Applied Mathematics Letters*, 19, 824–829.

Aue, A. (2004), Strong Approximation for RCA(1) Time Series with Applications, *Statistics & Probability Letters*, 68, 369–382.

Bollerslev, T. (1986), Generalized Autoregressive Conditional Heteroscedasticity, *Journal of Econometrics*, 31, 307–327.

Christoffersen, P. F. (2009), *Value-at-Risk Models*, in Andersen, T. G., Davis, R. A., Kreiss, J.-P., Mikosch, T. (ed.), *Handbook of Financial Time Series*, Springer Verlag.

Dowd, K. (2002), *Measuring Market Risk*, John Wiley & Sons, ltd.

Engle, R. F. (1982), Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation, *Econometrica*, 50, 987–1006.

Górka, J. (2008), Description the kurtosis of distributions by selected models with sing function, *Dynamic Econometric Models*, vol.8, Toruń

Nicholls, D., Quinn, B. (1982), *Random Coefficient Autoregressive Models: An Introduction*, Springer, New York.

Thavaneswaran, A., Appadoo, S. (2006), Properties of a New Family of Volatility Sing Models, *Computers and Mathematics with Applications*, 52, 809–818.

Thavaneswaran, A., Appadoo, S., Bector, C. (2006), Recent Developments in Volatility Modeling and Application, *Journal of Applied Mathematics and Decision Sciences*, 1–23.

Thavaneswaran, A., Peiris, S., Appadoo, S. (2008), Random Coefficient Volatility Models, *Statistics & Probability Letters*, 78, 582–593.

Thavaneswaran, S., Appadoo, S., i Ghahramani, M. (2009), RCA models with GARCH innovations, *Applied Mathematics Letters*, 22, 110–114.

Yamai, Y., Yoshiba, T. (2002), Comparative Analyses of Expected Shortfall and Value-at-Risk: Their Estimation Error, Decomposition and Optimization, *Monetary and Economic Studies*, 20(1), 87–121.

Zastosowanie modeli klasy Sign RCA do wyznaczenia wybranych miar zagrożenia

Z a r y s t r e ś c i. W finansach bardzo ważne jest aby dokładnie ocenić ryzyko. Istnieje wiele metod szacowania ryzyka jednak żadna z istniejących już metod nie jest najlepsza. W niniejszym artykule, do wyznaczenie takich miar ryzyka jak Value at Risk (VaR) i Expected Shortfall (ES) zastosowano modele klasy Sign RCA Obliczone zostały jednookresowe prognozy VaR oraz ES dla ostatnich 500 obserwacji z wykorzystaniem modeli oszacowanych w oknach na podstawie prób wielkości 250, 500 i 1500 obserwacji. Otrzymane wyniki zweryfikowano wykorzystując testowanie wsteczne oraz funkcje strat.

S ł o w a k l u c z o w e: Modele klasy Sign RCA Models, miary ryzyka, Value at Risk, Expected Shortfall.