On partially entanglement breaking channels

Dariusz Chruściński and Andrzej Kossakowski
Institute of Physics, Nicolaus Copernicus University,
Grudziądzka 5/7, 87–100 Toruń, Poland

Abstract

Using well known duality between quantum maps and states of composite systems we introduce the notion of Schmidt number of a quantum channel. It enables one to define classes of quantum channels which partially break quantum entanglement. These classes generalize the well known class of entanglement breaking channels.

1 Introduction

In quantum information theory [1] a quantum channel is represented by a completely positive trace preserving map (CPT) between states of two quantum systems living in $\mathcal{H}_A$ and $\mathcal{H}_B$. Consider $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^d$. Then the states of both systems are defined by semi-positive elements from $M_d \cong \mathbb{C}^d \otimes \mathbb{C}^d$. Due to the Kraus-Choi representation theorem [2] any CPT map

$$\Phi \ : \ M_d \ \longrightarrow \ M_d \ , \tag{1}$$

may be represented by

$$\Phi(\rho) = \sum_\alpha K_\alpha \, \rho \, K_\alpha^* \ , \tag{2}$$

where the Kraus operators $K_\alpha \in M_d$ satisfies trace-preserving condition $\sum_\alpha K_\alpha^* K_\alpha = I_d$. It is, therefore, clear that all the properties of Φ are encoded into the family K_α. In the present paper we show how the structure of Φ depends upon the rank of Kraus operators. In particular it is well known [3, 4] that if all K_α are rank one then Φ defines so called entanglement breaking channel (EBT), that is, for any state ρ from $M_d \otimes M_d$, $(\mathrm{id}_d \otimes \Phi)\rho$ is separable in $M_d \otimes M_d$.

Definition 1 *We call a channel (1) an r–partially entanglement breaking channel (r–PEBT) iff for an arbitrary ρ*

$$SN[(id_d \otimes \Phi)\rho] \leq r \ , \tag{3}$$

where $SN(\sigma)$ denotes the Schmidt number of σ.

Clearly, EBT channels are 1–PEBT. Let us recall [5] that

$$\mathrm{SN}(\sigma) = \min_{p_k, \psi_k} \left\{ \max_k \ \mathrm{SR}(\psi_k) \right\} \ , \tag{4}$$

where the minimum is taken over all possible pure states decompositions

$$\sigma = \sum_k p_k \, |\psi_k\rangle\langle\psi_k| \ ,$$

with $p_k \geq 0$, $\sum_k p_k = 1$ and ψ_k are normalized vectors in $\mathbb{C}^d \otimes \mathbb{C}^d$. The Schmidt rank $\mathrm{SR}(\psi)$ denotes the number of non-vanishing Schmidt coefficients in the Schmidt decomposition of ψ. This number characterizes the minimum Schmidt rank of the pure states that are needed to construct such density matrix. It is evident that $1 \leq \mathrm{SN}(\rho) \leq d$ and ρ is separable iff $\mathrm{SN}(\rho) = 1$. Moreover, it was proved [5] that the Schmidt number is non-increasing under local operations and classical communication.

Let us denote by S_k the set of density matrices on $\mathbb{C}^d \otimes \mathbb{C}^d$ that have Schmidt number at most k. One has $\mathcal{S} = S_1 \subset S_2 \subset \ldots \subset S_d = \mathcal{P}$ with $\mathcal{S}$ and $\mathcal{P}$ being the sets of separable and all density matrices, respectively. Recall, that a positive map $\Lambda : M_d \longrightarrow M_d$ is k-positive, if $(\mathrm{id}_k \otimes \Lambda)$ is positive on $M_k \otimes M_d$. Due to Choi [6] Λ is completely positive iff it is d-positive. Now, Λ is k-positive iff $(\mathrm{id}_d \otimes \Lambda)$ is positive on S_k. The set of k-positive maps which are not $(k+1)$-positive may be used to construct a Schmidt number witness operator W which is non-negative on all states in S_{k-1}, but detects at least one state ρ belonging to S_k [7, 8] (see also [9]), i.e.

$$\mathrm{Tr}\,(W\sigma) \geq 0 \ , \qquad \sigma \in S_{k-1} \ , \tag{5}$$

and there is a $\rho \in S_k$ such that $\mathrm{Tr}\,(W\rho) < 0$.

In the next section we investigate basic properties of PEBT channels. Then in section 4 we generalize the discussion to multipartite entangled states.

2 Properties of PEBT channels

Using well know duality between quantum CPT maps (1) and states of the composite quantum system living in $\mathbb{C}^d \otimes \mathbb{C}^d$ [10, 11] we may assign a Schmidt number to any CPT map. Take any CPT map Φ and define a state [12]

$$\rho_\Phi = (\mathrm{id}_d \otimes \Phi)\, P^+_d \ , \tag{6}$$

where $P^+_d = |\psi^+_d\rangle\langle\psi^+_d|$ with $\psi^+_d = d^{-1/2} \sum_k e_k \otimes e_k$ being a maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$ ($e_k\,;\ k = 1, 2, \ldots, d$ denote the orthonormal base in $\mathbb{C}^d$).

Definition 2 *A Schmidt number of Φ is defined by*

$$SN(\Phi) = SN(\rho_\Phi) \ , \tag{7}$$

where ρ_Φ stands for the 'dual' state defined in (6).

Actually, in [11] a CPT map $\Phi : M_d \longrightarrow M_d$ was called an r–CPT iff $\mathrm{SN}(\Phi) \leq r$. We show that r–PEBT channels are represented by r–CPT maps.

Note, that using Kraus decomposition (2) we may express the Schmidt number of Φ in analogy to (4) as follows:

$$\mathrm{SN}(\Phi) = \min_{K_\alpha} \left\{ \max_\alpha \mathrm{rank}\, K_\alpha \right\} \ . \tag{8}$$

The analogy between (4) and (8) is even more visible if we make the following observation: any vector $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$ may be written as $\psi = \sum_{i,j=1}^d x_{ij} e_i \otimes e_j$ and hence, introducing a ψ-dependent operator $F \in M_d$ such that $x_{ij} = \langle j|F|i\rangle$, one has

$$\psi = \sum_{i=1}^d e_i \otimes F e_i \ . \tag{9}$$

Using the maximally entangled state ψ_d^+ it may be rewritten in perfect analogy to (6):

$$\psi = \sqrt{d}\,(\mathrm{id}_d \otimes F)\psi_d^+ \ . \tag{10}$$

Clearly, the above formula realizes an isomorphism between $\mathbb{C}^d \otimes \mathbb{C}^d$ and M_d. Note, that the normalization condition $\langle\psi|\psi\rangle = 1$ implies $\mathrm{Tr}(F^*F) = 1$. Moreover, two vectors ψ_1 and ψ_2 are orthogonal iff the corresponding operators F_1 and F_2 are trace-orthogonal, i.e. $\mathrm{Tr}(F_1^\dagger F_2) = 0$. It is evident that $\mathrm{SR}(\psi) = \mathrm{rank}\, F$. Moreover, the singular values of F are nothing but the Schmidt coefficients of ψ. Hence, the separable pure states from $\mathbb{C}^d \otimes \mathbb{C}^d$ correspond to rank one operators from M_d.

Consider now the corresponding one-dimensional projector $|\psi\rangle\langle\psi|$. It may be written as

$$|\psi\rangle\langle\psi| = \sum_{i,j=1}^d e_{ij} \otimes F e_{ij} F^* \ , \tag{11}$$

with $\mathrm{Tr}(F^\dagger F) = 1$. In (11) a rank one operator $e_{ij} \in M_d$ equals to $|i\rangle\langle j|$ in Dirac notation. Hence the Schmidt class S_k may be defined as follows: $\rho \in S_k$ iff

$$\rho = \sum_\alpha p_\alpha P_\alpha \ , \tag{12}$$

with $p_\alpha \geq 0$, $\sum_\alpha p_\alpha = 1$ and

$$P_\alpha = \sum_{i,j=1}^d e_{ij} \otimes F_\alpha e_{ij} F_\alpha^* \ , \tag{13}$$

with $\mathrm{rank}\, F_\alpha \leq k$, and $\mathrm{Tr}(F_\alpha F_\alpha^*) = 1$. That is, S_k is a convex combination of one dimensional projectors corresponding to F's of rank at most k.

Theorem 1 *A quantum channel $\Phi \in r$–PEBT iff $SN(\Phi) \leq r$.*

Proof. Note, that $\mathrm{SN}(\Phi) \leq r$ iff there exists a Kraus decomposition such that all Kraus operators K_α satisfy $\mathrm{rank}\, K_\alpha \leq r$. Indeed, using (2) and (13) one has

$$(\mathrm{id}_d \otimes \Phi)\, P_d^+ = \sum_{i,j=1}^d e_{ij} \otimes \Phi(e_{ij}) = \sum_\alpha p_\alpha P_\alpha \ ,$$

with

$$p_\alpha = \frac{1}{d}\mathrm{Tr}(K_\alpha^\dagger K_\alpha) \ , \qquad F_\alpha = \frac{1}{\sqrt{dp_\alpha}}\, K_\alpha \ .$$

The above relations simply translate the isomorphism between states and CPT maps in terms of operators K_α and F_α. Suppose now that Φ is r-PEBT and let ρ be an arbitrary state in M_d

$$\rho = \sum_\beta p_\beta \sum_{i,j=1}^d e_{ij} \otimes F_\beta \, e_{ij} \, F_\beta^* \ ,$$

with arbitrary $F_\alpha \in M_d$ such that $\mathrm{Tr}(F_\alpha F_\alpha^*) = 1$. One has

$$(\mathrm{id}_d \otimes \Phi)\rho \ = \ \sum_{\alpha,\beta} p_{\alpha\beta} \sum_{i,j=1}^d e_{ij} \otimes \widetilde{F}_{\alpha\beta} e_{ij} \widetilde{F}_{\alpha\beta}^* \ , \tag{14}$$

with

$$p_{\alpha\beta} \ = \ \frac{1}{d} \mathrm{Tr}(K_\alpha K_\alpha^*) \, p_\beta \ , \qquad \widetilde{F}_{\alpha\beta} \ = \ \sqrt{\frac{dp_\beta}{p_{\alpha\beta}}} \, K_\alpha F_\beta \ ,$$

where K_α are Kraus operators representing an r–CPT map Φ satisfying $\mathrm{rank} K_\alpha \leq r$. Now,

$$\mathrm{rank}\,(K_\alpha F_\beta) \leq \min\{\mathrm{rank}\, K_\alpha \, , \mathrm{rank}\, F_\beta\} \leq r \ ,$$

and hence $(\mathrm{id}_d \otimes \Phi)\, \rho \in S_r$. The converse follows immediately. $\Box$
As a corollary note that since $\mathrm{rank}\,(K_\alpha F_\beta) \leq \mathrm{rank}\, F_\beta$ one finds

$$\mathrm{SN}((\mathrm{id}_d \otimes \Phi)\, \rho) \leq \mathrm{SN}(\rho) \ , \tag{15}$$

which shows that indeed SN does not increase under a local operation defined by $\mathrm{id}_d \otimes \Phi$.

Theorem 2 *A map Φ is r-CPT iff $\Lambda \circ \Phi$ is CPT for any r-positive map Λ.*

Proof. Suppose that Φ is r-CPT and take an arbitrary k-positive Λ:

$$(\mathrm{id}_d \otimes \Lambda \circ \Phi)\, P_d^+ = (\mathrm{id}_d \otimes \Lambda) \left[(\mathrm{id}_d \otimes \Phi)\, P_d^+ \right] \geq 0 \ ,$$

since $(\mathrm{id}_d \otimes \Phi)\, P_d^+ \in S_r$. Conversely, let $\Lambda \circ \Phi$ be CPT for any r-positive Λ, then $(\mathrm{id}_d \otimes \Lambda \circ \Phi)\, P_d^+ \geq 0$ implies that $(\mathrm{id}_d \otimes \Phi)\, P_d^+ \in S_r$ and hence Φ is r-CPT. Actually, the same is true for $\Phi \circ \Lambda$. $\Box$

To introduce another class of quantum operations let us recall the notion of co-positivity: a map Λ is r–co-positive iff $\tau \circ \Lambda$ is r-positive, where τ denotes transposition in M_d. In the same way Φ is completely co-positive (CcP) iff $\tau \circ \Phi$ is CP. Let us define the following convex subsets in $M_d \otimes M_d$: $S^r = (\mathrm{id}_d \otimes \tau)\, S_r$. One obviously has: $S^1 \subset S^2 \subset \ldots \subset S^n$. Note, that $S^1 = S_1 = \mathcal{S}$ and $S_n \cap S^n$ is a set of all PPT states.

Now, following [11] we call a CcPT map Φ an (r,s)-CPT if

$$(\mathrm{id}_d \otimes \Phi)\, P_d^+ \in S_r \cap S^s \ , \tag{16}$$

that is

$$\rho_\Phi \in S_r \quad \text{and} \quad (\mathrm{id}_d \otimes \tau)\rho_\Phi \in S_s \ .$$

Hence, if ρ_ϕ is a PPT state, then Φ is (r,s)-CPT for some r and s. In general there is no relation between (r,s)-CPT and (k,l)-CPT for arbitrary r,s and k,l. However, one has

$$(1,1)\text{-CPT} \subset (2,2)\text{-CPT} \subset \ldots \subset (n,n)\text{-CPT} \ ,$$

and (n,n)-CPT $\equiv$ CPT $\cap$ CcPT.
Theorem 3: A map Φ is (r,s)-CPT iff for any r-positive map Λ_1 and s–co-positive map Λ_2 the composite map $\Lambda_1 \circ \Lambda_2 \circ \Phi$ is CPT.

3 Examples

Example 1: Let us consider so called isotropic state in d dimensions

$$\mathcal{I}_\lambda = \frac{1-\lambda}{d^2} I_d \otimes I_d + \lambda P^+_d \ , \tag{17}$$

with $-1/(d^2-1) \leq \lambda \leq 1$. It is well known [13] that $\mathcal{I}_\lambda$ is separable iff $\lambda \leq 1/(d+1)$. Now, let $\Psi : M_d \longrightarrow M_d$ be an arbitrary positive trace preserving map and define a CPT map Φ_λ by

$$(\mathrm{id}_d \otimes \Phi_\lambda) P^+_d = (\mathrm{id}_d \otimes \Psi) \mathcal{I}_\lambda \ . \tag{18}$$

One easily finds

$$\Phi_\lambda(\rho) = \frac{1-\lambda}{d} \mathrm{Tr}\rho \, I_d + \lambda \Psi(\rho) \ . \tag{19}$$

Clearly, for $\lambda \leq 1/(d+1)$ (i.e. when $\mathcal{I}_\lambda$ is separable) Φ_λ is $(1,1)$-CPT, i.e. both Φ_λ and $\tau \circ \Phi_\lambda$ are EBT.
Example 2: Let us rewrite an isotropic state $\mathcal{I}_\lambda$ in terms of fidelity $f = \mathrm{Tr}(\mathcal{I}_\lambda \, P^+_d)$:

$$I_f = \frac{1-f}{d^2-1}(I_d \otimes I_d - P^+_d) + f P^+_d \ . \tag{20}$$

It was shown in [5] that $\mathrm{SN}(\mathcal{I}_f) = k$ iff

$$\frac{k-1}{d} < f \leq \frac{k}{d} \ . \tag{21}$$

Defining a CPT map Φ_f

$$(\mathrm{id}_d \otimes \Phi_f) P^+_d = \mathcal{I}_f \ , \tag{22}$$

one finds

$$\Phi_f(\rho) = \frac{1-f}{d^2-1} \mathrm{Tr}\rho \, I_d + \frac{d^2 f - 1}{d^2-1} \, \rho \ . \tag{23}$$

This map is k–CPT iff f satisfies (21) and hence it represents an r–PEBT channel.
Example 3: Consider

$$\rho = \sum_{\alpha=1}^{d^2} p_\alpha \sum_{i,j=1}^{d} e_{ij} \otimes F_\alpha \, e_{ij} \, F^*_\alpha \ , \tag{24}$$

where

$$p_\alpha \geq 0 \ , \qquad \sum_{\alpha=1}^{d^2} p_\alpha = 1 \ , \qquad F_\alpha = \frac{U_\alpha}{\sqrt{d}} \ , \tag{25}$$

and U_α defines a family of unitary operators from $U(d)$ such that

$$\mathrm{Tr}(U_\alpha\, U_\beta^*) = \delta_{\alpha\beta} \ , \qquad \alpha, \beta = 1, 2, \ldots, d^2 \ . \tag{26}$$

The corresponding 'dual' quantum channel Φ is given by

$$\Phi(\sigma) = \sum_{\alpha=1}^{d^2} K_\alpha\, \sigma\, K_\alpha^* \ , \tag{27}$$

with $K_\alpha = \sqrt{p_\alpha}\, U_\alpha$. Note, that for $p_\alpha = 1/d^2$ one obtains a completely depolarizing channel, i.e.

$$\frac{1}{d^2} \sum_{\alpha=1}^{d^2} U_\alpha\, e_{ij}\, U_\alpha^* \ = \delta_{ij} \ . \tag{28}$$

Now, following [14] consider a map

$$\Lambda_\mu(\sigma) = I_d\, \mathrm{Tr}\, \sigma - \mu\sigma \ , \tag{29}$$

which is k (but not $(k+1)$)–positive for

$$\frac{1}{k+1} \leq \mu \leq \frac{1}{k} \ . \tag{30}$$

One has

$$\begin{aligned} (\mathrm{id}_d \otimes \Lambda_\mu)\rho &= \sum_{\alpha=1}^{d^2} p_\alpha \sum_{i,j=1}^{d} e_{ij} \otimes [I_d\, \mathrm{Tr}(F_\alpha\, e_{ij}\, F_\alpha^*) - \mu\, F_\alpha\, e_{ij}\, F_\alpha^*] \\ &= \frac{1}{d} I_d \otimes I_d - \sum_{\alpha=1}^{d^2} \mu p_\alpha \sum_{i,j=1}^{d} e_{ij} \otimes F_\alpha\, e_{ij}\, F_\alpha^* \\ &= \frac{1}{d} \sum_{\alpha=1}^{d^2} (1 - d\mu p_\alpha) \sum_{i,j=1}^{d} e_{ij} \otimes F_\alpha\, e_{ij}\, F_\alpha^* \ , \end{aligned} \tag{31}$$

where we have used (28). It is therefore clear that if for some $1 \leq \alpha \leq d^2,\ p_\alpha > 1/(d\mu)$ and μ satisfies (30), then $\mathrm{SN}(\rho) \geq k+1$. Equivalently, a 'dual' quantum channel (27) belongs to $\{\, d$–PEBT $-$ k–PEBT$\}$.

4 PEBT channels and multipartite entanglement

Consider now a multipartite entangled state living in $\mathcal{H} = (\mathbb{C}^d)^{\otimes N}$ for some $N \geq 2$. Any $\psi \in \mathcal{H}$ may be written as follows:

$$\psi = \sum_{i_1,\ldots,i_K=1}^{d} e_{i_1} \otimes \ldots \otimes e_{i_K} \otimes F(e_{i_1} \otimes \ldots \otimes e_{i_K}) \ , \tag{32}$$

where F is an operator

$$F \;:\; (\mathbb{C}^d)^{\otimes K} \;\longrightarrow\; (\mathbb{C}^d)^{\otimes N-K} \ ,$$

and $1 \leq K \leq N-1$. Again, normalization of ψ implies $\mathrm{Tr}(F^*F) = 1$. Clearly, such representation of ψ is highly non-unique. One may freely choose K and take K copies of $\mathbb{C}^d$ out of $(\mathbb{C}^d)^{\otimes N}$. Any specific choice of representation depends merely on a specific question we would like to ask. For example (32) gives rise to the following reduced density matrices:

$$\rho_B = \mathrm{Tr}_A\, |\psi\rangle\langle\psi| = \mathrm{Tr}_{12\ldots K}\, |\psi\rangle\langle\psi| \;=\; FF^* \;\in\; M_d^{\otimes N-K} \ , \tag{33}$$

and

$$\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi| = \mathrm{Tr}_{K+1\ldots N}\, |\psi\rangle\langle\psi| \;=\; F^*F \;\in\; M_d^{\otimes K} \ . \tag{34}$$

A slightly different way to represent ψ reads as follows

$$\psi = \sum_{i_1,\ldots,i_{N-1}=1}^{d} e_{i_1} \otimes \ldots \otimes e_{i_{N-2}} \otimes e_{i_{N-1}} \otimes F_{i_1\ldots i_{N-2}} e_{i_{N-1}} \ , \tag{35}$$

where

$$F_{i_1\ldots i_{N-2}} \;:\; \mathbb{C}^d \;\longrightarrow\; \mathbb{C}^d \ ,$$

for any $i_1,\ldots,i_{N-2} = 1,2,\ldots,d$. Now, normalization of ψ implies

$$\sum_{i_1,\ldots,i_{N-2}=1}^{d} \mathrm{Tr}\left(F^*_{i_1\ldots i_{N-2}} F_{i_1\ldots i_{N-2}} \right) = 1 \ . \tag{36}$$

One has the following relation between different representations:

$$\langle e_{i_N} | F_{i_1\ldots i_{N-2}} | e_{i_{N-1}} \rangle \;=\; \langle e_{i_1} \otimes \ldots \otimes e_{i_{N-1}} | F | e_{i_N} \rangle \ . \tag{37}$$

Example 4. For $N = 3$ we have basically three representations:

$$\psi = \sum_{i=1}^{d} e_i \otimes F e_i \ , \tag{38}$$

$$\psi = \sum_{i,j=1}^{d} e_i \otimes e_j \otimes F'(e_i \otimes e_j) \ , \tag{39}$$

and

$$\psi = \sum_{i,j=1}^{d} e_i \otimes e_j \otimes F_i\, e_j \ , \tag{40}$$

with

$$F \;:\; \mathbb{C}^d \;\longrightarrow\; (\mathbb{C}^d)^{\otimes 2} \ , \quad F' = F^T \;:\; (\mathbb{C}^d)^{\otimes 2} \;\longrightarrow\; \mathbb{C}^d \ , \quad F_i \;:\; \mathbb{C}^d \;\longrightarrow\; \mathbb{C}^d \ .$$

As an example take $d = 2$ and let us consider two well known 3-qubit states [15]:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle \right) \ , \tag{41}$$

and

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right) \ . \tag{42}$$

One finds for GHZ–state:

$$F' = (F_1, F_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = F^T \ , \tag{43}$$

and for W–state:

$$\widetilde{F}' = (\widetilde{F}_1, \widetilde{F}_2) = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \widetilde{F}^T \ . \tag{44}$$

Note, that for both states $\mathrm{rank}(F) = \mathrm{rank}(\widetilde{F}) = 2$. There is, however, crucial difference between F_i and $\widetilde{F}_i$: $\mathrm{rank}(F_i) = 1$, whereas $\mathrm{rank}(\widetilde{F}_1) = 2$. Both states possess genuine 3–qubit entanglement. The difference consists in the fact that GHZ–state is 2–qubit separable whereas W–state is 2–qubit entangled [16]:

$$\rho_{23}^{\mathrm{GHZ}} = \mathrm{Tr}_1|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| = \sum_{k=0}^{1}\sum_{i,j=0}^{1} e_{ij}\otimes F_k\, e_{ij}\, F_k^* \ , \tag{45}$$

with $\mathrm{SN}(\,\rho_{23}^{\mathrm{GHZ}}\,) = 1$,

and

$$\rho_{23}^{\mathrm{W}} = \mathrm{Tr}_1|\mathrm{W}\rangle\langle\mathrm{W}| = \sum_{k=0}^{1}\sum_{i,j=0}^{1} e_{ij}\otimes \widetilde{F}_k\, e_{ij}\, \widetilde{F}_k^* \ , \tag{46}$$

with $\mathrm{SN}(\,\rho_{23}^{\mathrm{W}}\,) = 2$.

If $N = 2K$ any state vector $\psi \in (\mathbb{C}^d)^{\otimes N} = (\mathbb{C}^d)^{\otimes K} \otimes (\mathbb{C}^d)^{\otimes K}$ may be represented by (32) with

$$F \ : \ (\mathbb{C}^d)^{\otimes K} \ \longrightarrow \ (\mathbb{C}^d)^{\otimes K} \ . \tag{47}$$

Hence, an arbitrary state ρ from $M_d^{\otimes K} \otimes M_d^{\otimes K}$ reads as follows

$$\rho = \sum_\alpha p_\alpha \sum_{i_1,\ldots,i_K=1}^{d}\ \sum_{j_1,\ldots,j_K=1}^{d} e_{i_1j_1}\otimes \ldots \otimes e_{i_Kj_K}\otimes F_\alpha(e_{i_1j_1}\otimes \ldots \otimes e_{i_Kj_K})F_\alpha^* \ . \tag{48}$$

Clearly, $\mathrm{SN}(\rho) \leq r$ iff $\mathrm{rank}(F_\alpha) \leq r$ for all F_α appearing in (48). Then the corresponding quantum channel

$$\Phi \ : \ M_d^{\otimes K} \ \longrightarrow \ M_d^{\otimes K} \ , \tag{49}$$

possesses Kraus decomposition with $K_\alpha = \sqrt{d^K p_\alpha}\, F_\alpha$ and hence is r–PEBT. For other aspects of multipartite entanglement se e.g. [17].

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References

[1] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge, 2000

[2] K. Kraus, *States, Effects and Operations: Fundamental Notions of Quantum Theory*, Springer-Verlag, 1983

[3] M. Horodecki, P. Shor and M.B. Ruskai, Rev. Math. Phys **15**, 629 (2003)

[4] M.B. Ruskai, Rev. Math. Phys. **15**, 643 (2003)

[5] B. Terhal and P. Horodecki, Phys. Rev. A **61**, 040301 (2000)

[6] M.-D. Choi, Lin. Alg. Appl. **10**, 285 (1975)

[7] A. Sanpera, D. Bruss and M. Lewenstein, Phys. Rev. A **63**, 050301(R) (2001)

[8] F. Hulpke, D. Bruss, M. Lewenstein and A. Sanpera, Quant. Inf. Comp. **4**, 207 (2004)

[9] J. Eisert and H.J. Briegel, Phys. Rev. A **64** (2001) 022306

[10] K. Życzkowski and I. Bengtsson, Open Syst. Inf. Dyn. **11**, 3 (2004)

[11] M. Asorey, A. Kossakowski, G. Marmo and E.C.G. Sudarshan, Open Syst. Inf. Dyn. **12**, 319 (2005)

[12] A. Jamiołkowski, Rep. Math. Phys. **3**, (1972)

[13] M. Horodecki and P. Horodecki Phys. Rev. **A** 59, 4206 (1999)

[14] J. Tomiyama, Lin. Alg. Appl. **69** (1985) 169

T. Takasaki and J. Tomiyama, Math. Z. **184** (1983) 101

[15] D.M. Greenberger, M. Horne and A. Zelinger, in *Bell's theorem , Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, The Netherlands, 1989), pp. 69.

[16] J.K. Korbicz, J.I. Cirac and M. Lewenstein, Phys. Rev. Lett. **95** (2005) 120502

[17] W. Dür, J.I. Cirac and R. Tarrach, Phys. Rev. Lett. **83** (1999) 3562;

W. Dür and J.I. Cirac, Phys. Rev. A **61** (2000) 042314;

A. V. Thapliyal, Phys Rev. A **59** (1999) 3336;

H.A. Carteret, A. Higuchi and A. Sudbery, J. Math. Phy. **41** (2000) 7932