On multipartite invariant states II. Orthogonal symmetry.

Dariusz Chruściński and Andrzej Kossakowski
Institute of Physics, Nicolaus Copernicus University, Grudziądzka 5/7, 87–100 Toruń, Poland

We construct a new class of multipartite states possessing orthogonal symmetry. This new class defines a convex hull of multipartite states which are invariant under the action of local unitary operations introduced in our previous paper *On multipartite invariant states I. Unitary symmetry.* We study basic properties of multipartite symmetric states: separability criteria and multi-PPT conditions.

PACS numbers: 03.65.Ud, 03.67.-a

I. INTRODUCTION

In a recent paper [1] we analyzed multipartite states invariant under local unitary operations. For bipartite systems one has two classes of unitary invariant states: Werner states [2] invariant under

$$\rho \longrightarrow U\otimes U\, \rho\, (U\otimes U)^\dagger \ , \tag{1}$$

for any $U \in U(d)$, where $U(d)$ denotes the group of unitary $d\times d$ matrices, and isotropic states [3] which are invariant under

$$\rho \longrightarrow U\otimes \overline{U}\, \rho\, (U\otimes \overline{U})^\dagger \ , \tag{2}$$

where $\overline{U}$ is the complex conjugate of U in some basis. In [1] we proposed a natural generalization of bipartite symmetric states to multipartite systems consisting of an arbitrary even number of d-dimensional subsystems (qudits).

In the present paper we introduce a new class of states which combines above symmetries (1) and (2), i.e. it contains states which are both $U\otimes U$ and $U\otimes\overline{U}$–invariant, that is, states invariant under all unitary operations U such that $U=\overline{U}$:

$$\rho \longrightarrow O\otimes O\, \rho\, (O\otimes O)^T \ , \tag{3}$$

with $O\in O(d)\subset U(d)$, where $O(d)$ denotes the set of $d\times d$ orthogonal matrices. Such states were first considered in [4] (see also [5]). In a slightly different context symmetric states were studied also in [6]. Recently [7] unitary invariant 3-partite states were used to test multipartite separability criteria.

Here we present a general construction of $O\otimes O$–invariant states for multipartite systems consisting of an arbitrary even number of d-dimensional subsystems. It turns out that orthogonally invariant states of $2K$–partite system (with K being a positive integer) define (3^K-1)–invariant simplex. We analyze (multi)separability criteria and the hierarchy of multi-PPT conditions [1, 8, 9]. It is hoped that these new family would serve as a useful laboratory to study multipartite entanglement [10, 11, 12, 13, 14, 15].

II. BIPARTITE STATES

A. Simplex of orthogonally invariant states

Let us consider a bipartite Alice–Bob system living in $\mathcal{H}_{AB}=\mathcal{H}_A\otimes\mathcal{H}_B=(\mathbb{C}^d)^{\otimes 2}$. Recall that the space of $U\otimes U$–invariant hermitian operators in $\mathcal{H}_{AB}$ is spanned by two orthogonal projectors

$$Q^0=\frac12(I^{\otimes 2}+\mathbf{F})\ ,\quad Q^1=\frac12(I^{\otimes 2}-\mathbf{F})\ , \tag{4}$$

where $\mathbf{F}$ is a flip operator, i.e. $\mathbf{F}(\psi\otimes\varphi)=\varphi\otimes\psi$, defined by

$$\mathbf{F}=\sum_{i,j=1}^d |ij\rangle\langle ji|\ . \tag{5}$$

In particular this 2-dimensional space contains a line of normalized (i.e. with unit trace) operators:

$$L:\quad (1-q)\,\widetilde{Q}^0+q\,\widetilde{Q}^1\ , \tag{6}$$

with $q\in\mathbb{R}$, and throughout the paper $\widetilde{A}$ stands for $A/\mathrm{Tr}A$. A segment of L with vertices $\widetilde{Q}^0$ and $\widetilde{Q}^1$ defines a family of bipartite Werner states:

$$\mathcal{W}_{\mathbf{q}}=q_0\,\widetilde{Q}^0+q_1\,\widetilde{Q}^1\ , \tag{7}$$

with $q_\alpha\geq 0$, and $q_0+q_1=1$.

Now, a partial transposition $\mathbb{1}\otimes\tau$ sends points of L into another line $L_\tau=(\mathbb{1}\otimes\tau)L$:

$$L_\tau:\quad (1-p)\,\widetilde{P}^0+p\,\widetilde{P}^1\ , \tag{8}$$

with $p\in\mathbb{R}$, and P^α denote the following orthogonal projectors:

$$P^1=P^+_d\ ,\quad P^0=I^{\otimes 2}-P^1\ , \tag{9}$$

with P_d^+ being a 1-dimensional projector corresponding to a canonical maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$:

$$P_d^+ = \frac{1}{d}\,(\mathbb{1}\otimes\tau)\mathbf{F} = \frac{1}{d}\sum_{i,j=1}^{d}|ii\rangle\langle jj| \ . \qquad (10)$$

A segment of L_τ with vertices $\widetilde{P}^0$ and $\widetilde{P}^1$ defines a family of bipartite isotropic states:

$$\mathcal{I}_{\mathbf{p}} = p_0\,\widetilde{P}^0 + p_1\,\widetilde{P}^1 \ , \qquad (11)$$

with $p_\alpha \geq 0$, and $p_0 + p_1 = 1$.

Let us introduce a new class Σ_1 of bipartite states which are both $U\otimes U$ and $U\otimes\overline{U}$–invariant for all $U \in U(d)$ such that $U = \overline{U}$. Such U's represent real orthogonal matrices in $O(d)$. Hence, Σ_1 defines a new family of symmetric $O\otimes O$–invariant states:

$$\rho \ \longrightarrow\ O\otimes O\,\rho\,(O\otimes O)^T \ , \qquad (12)$$

with $O \in O(d) \subset U(d)$. Clearly Σ_1 contains both Werner and isotropic states and, therefore, it contains a convex hull of $\widetilde{Q}^\alpha$ and $\widetilde{P}^\alpha$:

$$\Sigma_1 \supset \mathrm{conv}\,\{\widetilde{Q}^0,\widetilde{Q}^1,\widetilde{P}^0,\widetilde{P}^1\} \ . \qquad (13)$$

It is easy to see that these four states are co-planar, i.e. they belong to a common 2-dimensional plane in d^2-dimensional space of hermitian operators in $\mathbb{C}^d\otimes\mathbb{C}^d$. Indeed, one shows that

$$\det\left[\begin{array}{c|c}\mathrm{Tr}(\widetilde{Q}^\alpha\widetilde{Q}^\beta) & \mathrm{Tr}(\widetilde{Q}^\alpha\widetilde{P}^\beta)\\ \hline \mathrm{Tr}(\widetilde{P}^\alpha\widetilde{Q}^\beta) & \mathrm{Tr}(\widetilde{P}^\alpha\widetilde{P}^\beta)\end{array}\right] = 0 \ , \qquad (14)$$

and hence Σ_1 is 2-dimensional. Therefore the two lines L and L_τ intersect and the point $L\cap L_\tau$ is described by

$$q = \frac{1}{2} - \frac{1}{d(d+1)} \ , \qquad (15)$$

and

$$p = \frac{2}{d(d+1)}\left[\frac{1}{2} + \frac{1}{d(d+1)}\right] \ . \qquad (16)$$

Note that $q,p\in[0,1]$ and hence the intersection point $L\cap L_\tau \in \Sigma_1$ defines a state which is both Werner and isotropic. Moreover, since $q < 1/2$ (and $p < 1/d$) this state is separable.

Now, it turns out that Σ_1 defines a simplex with vertices $\widetilde{\Pi}^\alpha$; $\alpha = 0,1,2$, where

$$\begin{aligned}\Pi^0 &= Q^0 - P^1 \ ,\\ \Pi^1 &= Q^1 \ ,\\ \Pi^2 &= P^1 \ .\end{aligned} \qquad (17)$$

One may call it a 'minimal simplex' containing $\mathrm{conv}\,\{\widetilde{Q}^0,\widetilde{Q}^1,\widetilde{P}^0,\widetilde{P}^1\}$. In particular

$$\widetilde{Q}^0 = \frac{1}{d(d+1)}\left[(d-1)(d+2)\,\widetilde{\Pi}^0 + 2\widetilde{\Pi}^2\right] \ , \qquad (18)$$

and

$$\widetilde{P}^0 = \frac{1}{2(d+1)}\left[(d+2)\,\widetilde{\Pi}^0 + d\widetilde{\Pi}^1\right] \ . \qquad (19)$$

Note, that the family Π^k gives rise to the orthogonal resolution of identity in $\mathcal{H}_A\otimes\mathcal{H}_B$:

$$\Pi^i\Pi^j = \delta_{ij}\Pi^j \ , \qquad (20)$$

and

$$\Pi^0 + \Pi^1 + \Pi^2 = I^{\otimes 2} \ . \qquad (21)$$

Any state ρ in Σ_1 may be written as follows

$$\rho = \sum_{k=0}^{2}\pi_k\,\widetilde{\Pi}^k \ , \qquad (22)$$

where $\widetilde{\Pi}^k = \Pi^k/\mathrm{Tr}\Pi^k$, and the corresponding 'fidelities'

$$\pi_k = \mathrm{Tr}(\rho\Pi^k) \ , \qquad (23)$$

satisfy $\pi_k \geq 0$ together with $\sum_k \pi_k = 1$. It is evident that an arbitrary bipartite state ρ may be projected onto the $O\otimes O$–invariant subspace by the following projection operation $\mathbf{P}^{(1)} : \mathcal{P} \longrightarrow \Sigma_1$:

$$\mathbf{P}^{(1)}\rho = \sum_{k=0}^{2}\mathrm{Tr}(\rho\boldsymbol{\Pi}^k)\,\widetilde{\Pi}^k \ . \qquad (24)$$

B. Separability and PPT condition

Consider a separable state $\sigma = P_\psi\otimes P_\varphi$, where $P_x = |x\rangle\langle x|$, and ψ,φ are normalized vectors in $\mathbb{C}^d$. One easily finds for fidelities $\mathrm{Tr}(\sigma\Pi^k)$:

$$\begin{aligned}\pi_0 &= \frac{1}{2}(1+\alpha) - \frac{\beta}{d} \ ,\\ \pi_1 &= \frac{1}{2}(1-\alpha) \ ,\\ \pi_2 &= \frac{\beta}{d} \ ,\end{aligned} \qquad (25)$$

where

$$\alpha = |\langle\psi|\varphi\rangle|^2 \ , \qquad \beta = |\langle\psi|\overline{\varphi}\rangle|^2 \ . \qquad (26)$$

Now, an arbitrary separable state is a convex combination of the extremal product states $P_\psi \otimes P_\varphi$. Noting that $0 \leq \alpha, \beta \leq 1$, the separable $O\otimes O$–invariant states satisfy

$$\pi_1 \leq \frac{1}{2}\ , \quad \pi_2 \leq \frac{1}{d}\ , \tag{27}$$

i.e. they combine separability conditions for Werner states $\pi_1 \leq 1/2$ and isotropic states $\pi_2 \leq 1/d$.

Now, applying a partial transposition $(\mathbb{1}\otimes\tau)$ to (22) one finds

$$(\mathbb{1}\otimes\tau)\rho = \sum_{\alpha=0}^{2} \pi'_\alpha\, \widetilde{\Pi}^k\ , \tag{28}$$

where

$$\pi'_\alpha = \sum_{\beta=0}^{2} \pi_\beta\, \mathbf{C}^{\beta\alpha}\ , \tag{29}$$

and $\mathbf{C}$ denotes the following 3×3 matrix:

$$\mathbf{C} = \frac{1}{2d}\begin{pmatrix} d-2 & d & 2 \\ d+2 & d & -2 \\ (d-1)(d+2) & -d(d-1) & 2 \end{pmatrix}\ . \tag{30}$$

Observe that

$$\sum_{\beta=0}^{2} \mathbf{C}^{\beta\alpha} = 1\ , \tag{31}$$

but $\mathbf{C}^{\beta\alpha}$ contains negative elements and hence it is not a stochastic matrix. The Peres-Horodecki condition [8, 9] implies $\pi'_\alpha \geq 0$ and hence

$$\pi_0 + \pi_1 - (d-1)\pi_2 \ \geq\ 0\ , \tag{32}$$

$$\pi_0 - \pi_1 + \pi_2 \ \geq\ 0\ , \tag{33}$$

which is equivalent to $\pi_1 \leq 1/2$ and $\pi_2 \leq 1/d$. This shows that bipartite $O\otimes O$–invariant state is separable iff it is PPT.

III. 2×2–PARTITE STATES

A. Construction

Consider now a 4-partite system living in $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_3\otimes\mathcal{H}_4$ with $\mathcal{H}_k = \mathbb{C}^d$. Following [1] we may introduce two Alices A_k and two Bobs B_k: A_k lives in $\mathcal{H}_k$ and B_k lives in $\mathcal{H}_{2+k}$ (for $k = 1, 2$).

Let $\boldsymbol{\alpha}$ be a trinary 2-dimensional vector, i.e. $\boldsymbol{\sigma} = (\alpha_1, \alpha_2)$ with $\alpha_j \in \{0, 1, 2\}$. Following [1] we define a family of 4–partite projectors

$$\mathbf{\Pi}^{\boldsymbol{\alpha}} = \Pi^{\alpha_1}_{1|3} \otimes \Pi^{\alpha_2}_{2|4}\ , \tag{34}$$

where $L_{i|j}$ denotes a bipartite operator acting on $\mathcal{H}_i\otimes\mathcal{H}_j$, and Π^α are defined in (17). One easily shows that 9 projectors (34) satisfy

1. $\mathbf{\Pi}^{\boldsymbol{\alpha}}$ are $\mathbf{O}\otimes\mathbf{O}$–invariant, i.e.

$$\mathbf{O}\otimes\mathbf{O}\,\mathbf{\Pi}^{\boldsymbol{\alpha}} = \mathbf{\Pi}^{\boldsymbol{\alpha}}\mathbf{O}\otimes\mathbf{O}\ , \tag{35}$$

with $\mathbf{O} = (O_1, O_2)$, and

$$\mathbf{O}\otimes\mathbf{O} = O_1\otimes O_2\otimes O_1\otimes O_2\ .$$

2. $\mathbf{\Pi}^{\boldsymbol{\alpha}}\cdot\mathbf{\Pi}^{\boldsymbol{\beta}} = \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}\,\mathbf{\Pi}^{\boldsymbol{\beta}}$,
3. $\sum_{\boldsymbol{\alpha}} \mathbf{\Pi}^{\boldsymbol{\alpha}} = \mathbb{1}^{\otimes 4}$,

that is, $\mathbf{\Pi}^{\boldsymbol{\alpha}}$ define spectral resolution of identity in $(\mathbb{C}^d)^{\otimes 4}$. Hence, any 4-partite $\mathbf{O}\otimes\mathbf{O}$–invariant state may be uniquely represented by

$$\rho = \sum_{\boldsymbol{\alpha}} \pi_{\boldsymbol{\alpha}}\, \widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}\ , \tag{36}$$

where the corresponding 'fidelities' $\pi_{\boldsymbol{\alpha}} = \mathrm{Tr}(\rho\,\mathbf{\Pi}^{\boldsymbol{\alpha}})$ satisfy $\pi_{\boldsymbol{\alpha}} \geq 0$ together with $\sum_{\boldsymbol{\alpha}} \pi_{\boldsymbol{\alpha}} = 1$. The above construction gives rise to 8-dimensional simplex Σ_2 with vertices $\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}$. Note, that Σ_2 contains a convex hull of 4 classes of 4-partite invariant states introduced in [1]:

$$\Sigma_2 \supset \mathrm{conv}\left\{\Sigma_2^{(00)}, \Sigma_2^{(01)}, \Sigma_2^{(10)}, \Sigma_2^{(11)}\right\}\ , \tag{37}$$

where

$$\Sigma_2^{(00)} \ =\ \mathrm{conv}\,\{\widetilde{Q}^i_{1|3}\otimes\widetilde{Q}^j_{2|4}\}\ , \tag{38}$$

$$\Sigma_2^{(01)} \ =\ \mathrm{conv}\,\{\widetilde{Q}^i_{1|3}\otimes\widetilde{P}^j_{2|4}\}\ , \tag{39}$$

$$\Sigma_2^{(10)} \ =\ \mathrm{conv}\,\{\widetilde{P}^i_{1|3}\otimes\widetilde{Q}^j_{2|4}\}\ , \tag{40}$$

$$\Sigma_2^{(11)} \ =\ \mathrm{conv}\,\{\widetilde{P}^i_{1|3}\otimes\widetilde{P}^j_{2|4}\}\ , \tag{41}$$

with $i, j \in \{0, 1\}$. A 3-dimensional simplex $\Sigma_2^{\boldsymbol{a}}$, where $\boldsymbol{a} = (a_1, a_2)$ denotes 2-dimensional binary vector, defines a set of $\boldsymbol{a}$-invariant states. Recall that a 4-partite state ρ is $\boldsymbol{a}$–invariant iff $\tau_{\boldsymbol{a}}\rho$, with

$$\tau_{\boldsymbol{a}} = \mathbb{1}\otimes\mathbb{1}\otimes\tau^{a_1}\otimes\tau^{a_2}\ , \tag{42}$$

is $\mathbf{U}\otimes\mathbf{U}$–invariant. In particular $\Sigma_2^{(00)}$ and $\Sigma_2^{(11)}$ denote 4-partite Werner and isotropic states, respectively.

B. Separability

To find the corresponding separability criteria note that a general 4-partite $O\otimes O$–invariant state ρ is 4-separable iff there exists a 4-separable state σ such that $\mathbf{P}^{(2)}\rho = \sigma$, where

$$\mathbf{P}^{(2)} : \mathcal{P} \longrightarrow \Sigma_2\ , \tag{43}$$

defines a projection onto 4-partite $O\otimes O$–invariant states. Consider an extremal product state $\sigma = P_{\psi_1}\otimes P_{\psi_2}\otimes P_{\varphi_1}\otimes P_{\varphi_2}$, where ψ_i,φ_j are normalized vectors in $\mathbb{C}^d$. One easily finds for fidelities $\mathrm{Tr}(\sigma\Pi^{\boldsymbol{\sigma}})$:

$$\begin{aligned}\pi_{\boldsymbol{\sigma}} &= \mathrm{Tr}(P_{\psi_1}\otimes P_{\varphi_1}\cdot\Pi^{\sigma_1}_{1|3})\,\mathrm{Tr}(P_{\psi_2}\otimes P_{\varphi_2}\cdot\Pi^{\sigma_2}_{2|4}) \\ &= u_1\cdot u_2\ ,\end{aligned} \tag{44}$$

with

$$u_i = \begin{cases} (1+\alpha_i)/2-\beta_i/d & ,\quad \sigma_i=0 \\ (1-\alpha_i)/2 & ,\quad \sigma_i=1 \\ \beta_i/d & ,\quad \sigma_i=2\end{cases}\ , \tag{45}$$

where

$$\alpha_i = |\langle\psi_i|\varphi_i\rangle|^2\ ,\qquad \beta_i = |\langle\psi_i|\overline{\varphi}_i\rangle|^2\ . \tag{46}$$

Now, since $\alpha_i,\beta_i\le 1$, the projection $\mathbf{P}^{(2)}$ of the convex hull of extremal separable states gives the subset of separable $O\otimes O$–invariant states defined by the following relations:

$$\pi_{\boldsymbol{\sigma}} \le \frac{1}{f_{\sigma_1}f_{\sigma_2}}\ , \tag{47}$$

where

$$f_\sigma = \begin{cases} 1 & ,\ \sigma=0 \\ 2 & ,\ \sigma=1 \\ d & ,\ \sigma=2\end{cases}\ . \tag{48}$$

It is evident, that (47) generalize formulae (27). Clearly, separable states in Σ_2 contain a convex hull of separable states in each $\boldsymbol{a}$–invariant simplex $\Sigma_2^{\boldsymbol{a}}$:

$$\mathrm{Sep}(\Sigma_2) \supset \mathrm{conv}\bigcup_{\boldsymbol{a}}\mathrm{Sep}(\Sigma_2^{\boldsymbol{a}})\ . \tag{49}$$

Is 4-separability equivalent to PPT condition? Note, that one may perform 3 different partial transpositions (42):

$$\begin{aligned}\tau_{(01)} &= \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\tau\ ,\\ \tau_{(10)} &= \mathbb{1}\otimes\mathbb{1}\otimes\tau\otimes\mathbb{1}\ ,\\ \tau_{(11)} &= \mathbb{1}\otimes\mathbb{1}\otimes\tau\otimes\tau\ .\end{aligned} \tag{50}$$

It is easy to see that

$$\tau_{(01)}\,\rho = \sum_{\boldsymbol{\alpha}}\pi'_{\boldsymbol{\alpha}}\,\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}\ , \tag{51}$$

$$\tau_{(10)}\,\rho = \sum_{\boldsymbol{\alpha}}\pi''_{\boldsymbol{\alpha}}\,\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}\ , \tag{52}$$

$$\tau_{(11)}\,\rho = \sum_{\boldsymbol{\alpha}}\pi'''_{\boldsymbol{\alpha}}\,\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}}\ , \tag{53}$$

with

$$\pi'_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}}\pi_{\boldsymbol{\beta}}\,(\mathbf{I}\otimes\mathbf{C})^{\boldsymbol{\beta\alpha}}\ , \tag{54}$$

$$\pi''_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}}\pi_{\boldsymbol{\beta}}\,(\mathbf{C}\otimes\mathbf{I})^{\boldsymbol{\beta\alpha}}\ , \tag{55}$$

$$\pi'''_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}}\pi_{\boldsymbol{\beta}}\,(\mathbf{C}\otimes\mathbf{C})^{\boldsymbol{\beta\alpha}}\ , \tag{56}$$

where $\mathbf{I}$ denotes 3×3 identity matrix and $\mathbf{C}$ is defined in (30). For example one finds that a state $\rho\in\Sigma_2$ is (01)–PPT, i.e. $\tau_{01}\rho\ge 0$ iff

$$\begin{aligned}\pi_{00}+\pi_{01}-(d-1)\pi_{02} &\ge 0\ ,\\ \pi_{00}-\pi_{01}+\pi_{02} &\ge 0\ ,\\ \pi_{10}+\pi_{11}-(d-1)\pi_{12} &\ge 0\ ,\\ \pi_{10}-\pi_{11}+\pi_{12} &\ge 0\ ,\\ \pi_{20}+\pi_{21}-(d-1)\pi_{22} &\ge 0\ ,\\ \pi_{20}-\pi_{21}+\pi_{22} &\ge 0\ .\end{aligned} \tag{57}$$

Similarly, it is (10)–PPT iff

$$\begin{aligned}\pi_{00}+\pi_{10}-(d-1)\pi_{20} &\ge 0\ ,\\ \pi_{00}-\pi_{10}+\pi_{20} &\ge 0\ ,\\ \pi_{01}+\pi_{11}-(d-1)\pi_{21} &\ge 0\ ,\\ \pi_{01}-\pi_{11}+\pi_{21} &\ge 0\ ,\\ \pi_{02}+\pi_{12}-(d-1)\pi_{22} &\ge 0\ ,\\ \pi_{02}-\pi_{12}+\pi_{22} &\ge 0\ .\end{aligned} \tag{58}$$

Now, it was proved in [1] that any 4-partite $\mathbf{U}\otimes\mathbf{U}$–invariant state is 4-separable iff it is (01)- (10)- and (11)-PPT. Moreover, any symmetric state is $A|B$ bi-separable iff it is (11)–PPT. We conjecture that the same property holds for $\mathbf{O}\otimes\mathbf{O}$–invariant states. To prove it one has to apply the same techniques as in [1]. To investigate all PPT conditions one needs together with (57) and (58) a highly complicated (11)–PPT condition which we shall not consider here.

IV. $2K$–PARTITE STATES

A. General contruction

Generalization to $2K$–partite system is straightforward. Following [1] we introduce $2K$ qudits with the total space $\mathcal{H} = \mathcal{H}_1\otimes\ldots\otimes\mathcal{H}_{2K} = (\mathbb{C}^d)^{\otimes 2K}$. We may still interpret the total system as a bi-partite one with $\mathcal{H}_A = \mathcal{H}_1\otimes\ldots\otimes\mathcal{H}_K$ and $\mathcal{H}_B = \mathcal{H}_{K+1}\otimes\ldots\otimes\mathcal{H}_{2K}$. Equivalently, we may introduce K Alices and K Bobs with $\mathcal{H}_{A_i} = \mathcal{H}_i$ and $\mathcal{H}_{B_i} = \mathcal{H}_{K+i}$, respectively. Then $\mathcal{H}_A$ and $\mathcal{H}_B$ stand for the composite K Alices' and Bobs' spaces.

Now, let $\boldsymbol{\alpha}$ be a trinary K-dimensional vector, i.e. $\boldsymbol{\sigma} = (\alpha_1, \ldots, \alpha_K)$ with $\alpha_j \in \{0, 1, 2\}$. In analogy to (34) let us define a family of $2K$–partite projectors

$$\mathbf{\Pi}^{\boldsymbol{\alpha}} = \Pi^{\alpha_1}_{1|K+1} \otimes \ldots \otimes \Pi^{\alpha_K}_{K|2K} \ . \tag{59}$$

One easily shows that

1. $\mathbf{\Pi}^{\boldsymbol{\alpha}}$ are $\mathbf{O} \otimes \mathbf{O}$–invariant, i.e.
$$\mathbf{O} \otimes \mathbf{O}\, \mathbf{\Pi}^{\boldsymbol{\alpha}} = \mathbf{\Pi}^{\boldsymbol{\alpha}} \mathbf{O} \otimes \mathbf{O} \ , \tag{60}$$
with $\mathbf{O} = (O_1, \ldots, O_K)$, and
$$\mathbf{O} \otimes \mathbf{O} = O_1 \otimes \ldots \otimes O_K \otimes O_1 \otimes \ldots \otimes O_K \ .$$
2. $\mathbf{\Pi}^{\boldsymbol{\alpha}} \cdot \mathbf{\Pi}^{\boldsymbol{\beta}} = \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} \mathbf{\Pi}^{\boldsymbol{\beta}}$,
3. $\sum_{\boldsymbol{\alpha}} \mathbf{\Pi}^{\boldsymbol{\alpha}} = \mathbb{1}^{\otimes 2K}$.

Therefore, $2K$–partite $\mathbf{O} \otimes \mathbf{O}$–invariant states define a $(3^K - 1)$–dimensional simplex Σ_K:

$$\rho = \sum_{\boldsymbol{\alpha}} \pi_{\boldsymbol{\alpha}} \widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}} \ , \tag{61}$$

where

$$\widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}} = \widetilde{\Pi}^{\alpha_1}_{1|K+1} \otimes \ldots \otimes \widetilde{\Pi}^{\alpha_K}_{K|2K} \ , \tag{62}$$

and the corresponding 'fidelities'

$$\pi_{\boldsymbol{\alpha}} = \mathrm{Tr}(\rho \Pi^{\boldsymbol{\alpha}}) \ , \tag{63}$$

satisfy $\pi_{\boldsymbol{\alpha}} \geq 0$ together with $\sum_{\boldsymbol{\alpha}} \pi_{\boldsymbol{\alpha}} = 1$.

Denote by $\Sigma^{\boldsymbol{a}}_K$ a $(2^K - 1)$–dimensional simplex of $\boldsymbol{a}$–invariant states, where $\boldsymbol{a} = (a_1, \ldots, a_K)$ denotes a binary K-vector. Recall that a $2K$-partite state ρ is $\boldsymbol{a}$–invariant iff $\tau_{\boldsymbol{a}}\rho$, with

$$\tau_{\boldsymbol{a}} = \mathbb{1}^{\otimes K} \otimes \tau^{a_1} \otimes \ldots \otimes \tau^{a_K} \ , \tag{64}$$

is $\mathbf{U} \otimes \mathbf{U}$–invariant. In particular $\Sigma^{(0\ldots0)}_K$ and $\Sigma^{(1\ldots1)}_K$ denote the simplex of $2K$-partite Werner and isotropic states, respectively (see [1]). It is therefore clear that Σ_K contains a convex hull of 2^K single $\boldsymbol{a}$-invariant simplexes $\Sigma^{\boldsymbol{a}}_K$:

$$\Sigma_K \supset \mathrm{conv} \bigcup_{\boldsymbol{a}} \Sigma^{\boldsymbol{a}}_K \ . \tag{65}$$

B. Separability and multi–PPT conditions

To find separability conditions for $2K$-partite $\mathbf{O} \otimes \mathbf{O}$–invariant states consider a separable state

$$\sigma = P_{\psi_1} \otimes \ldots \otimes P_{\psi_K} \otimes P_{\varphi_1} \otimes \ldots \otimes P_{\varphi_K} \ ,$$

where ψ_i, φ_j are normalized vectors in $\mathbb{C}^d$. One easily finds for fidelities $\mathrm{Tr}(\sigma \Pi^{\boldsymbol{\sigma}})$:

$$\begin{aligned} \pi_{\boldsymbol{\sigma}} &= \prod_{i=1}^{K} \mathrm{Tr}(P_{\psi_i} \otimes P_{\varphi_i} \cdot \Pi^{\sigma_i}_{i|K+i}) \\ &= u_1 \ldots u_K \ , \end{aligned} \tag{66}$$

where u_i are defined in (45). The projection $\mathbf{P}^{(K)}$ of the convex hull of extremal separable states gives the subset of separable $O \otimes O$–invariant states defined by the following relations:

$$\pi_{\boldsymbol{\sigma}} \leq \frac{1}{f_{\sigma_1} \ldots f_{\sigma_K}} \ , \tag{67}$$

where f's are defined in (48). Clearly, a set of separable states in Σ_K contains a convex hull of separable states in each $\boldsymbol{a}$–invariant simplex $\Sigma^{\boldsymbol{a}}_K$:

$$\mathrm{Sep}(\Sigma_K) \supset \mathrm{conv} \bigcup_{\boldsymbol{a}} \mathrm{Sep}(\Sigma^{\boldsymbol{a}}_K) \ . \tag{68}$$

For $2K$–partite state one may look for $2^K - 1$ partial transpositions

$$\tau_{\boldsymbol{a}} = \mathbb{1}^{\otimes K} \otimes \tau^{a_1} \otimes \ldots \otimes \tau^{a_K} \ . \tag{69}$$

Note, that

$$\tau_{\boldsymbol{a}}\rho = \sum_{\boldsymbol{\alpha}} \pi'_{\boldsymbol{\alpha}} \widetilde{\mathbf{\Pi}}^{\boldsymbol{\alpha}} \ , \tag{70}$$

with

$$\pi'_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} \pi_{\boldsymbol{\beta}} \left(\mathbf{C}^{a_1} \otimes \ldots \otimes \mathbf{C}^{a_K}\right)^{\boldsymbol{\beta}\boldsymbol{\alpha}} \ , \tag{71}$$

where

$$\mathbf{C}^a = \begin{cases} \mathbf{I} & , \ a = 0 \\ \mathbf{C} & , \ a = 1 \end{cases} \ . \tag{72}$$

In analogy to 4-partite symmetric states we conjecture that a $2K$-partite state in Σ_K is $2K$-separable iff it is $\boldsymbol{b}$-PPT for all binary 2-vectors $\boldsymbol{b}$. Moreover, a state in Σ_K is $A|B$ bi-separable iff it is $(1\ldots1)$–PPT.

C. Reductions

It is evident that reducing the $2K$ partite state $\rho \in \Sigma_K$ with respect to $A_i \otimes B_i$ pair one obtains $2(K-1)$–partite state $\rho' \in \Sigma_{K-1}$ living in

$$\mathcal{H}_1 \otimes \ldots \check{\mathcal{H}}_i \otimes \ldots \otimes \check{\mathcal{H}}_{i+K} \otimes \ldots \otimes \mathcal{H}_{2K} \ , \tag{73}$$

where $\check{\mathcal{H}}_i$ denotes the omitting of $\mathcal{H}_i$. The corresponding fidelities are given by

$$\pi'_{(\alpha_1\ldots\alpha_{K-1})} \;=\; \sum_\beta \pi_{(\alpha_1\ldots\alpha_{i-1}\beta\alpha_i\ldots\alpha_{K-1})}\ . \qquad (74)$$

Note, that reduction with respect to a 'mixed' pair, say $A_i \otimes B_j$ with $i \neq j$, is equivalent to two 'natural' reductions with respect to $A_i \otimes B_i$ and $A_j \otimes B_j$ and hence it gives rise to $2(K-2)$–partite invariant state. This procedure establishes a natural hierarchy of multipartite $\mathbf{O} \otimes \mathbf{O}$–invariant states.

Acknowledgments

This work was partially supported by the Polish State Committee for Scientific Research Grant *Informatyka i inżynieria kwantowa* No PBZ-Min-008/P03/03.

[1] D. Chruściński and A. Kossakowski, *On multipartite invariant states I. Unitary symmetry*, e-print quant-ph/0601027

[2] R.F. Werner, Phys. Rev. A **40**, 4277 (1989).

[3] M. Horodecki and P. Horodecki, Phys. Rev. **A** 59, 4206 (1999).

[4] K.G.H. Vollbrecht and R.F. Werner, Phys. Rev. A **64**, 062307 (2001).

[5] T. Eggeling and R.F. Werner, Phys. Rev. A **63**, 042111 (2001).

[6] S. Virmani and M.B. Plenio, Phys. Rev. A **67** 062308 (2003).

[7] W. Hall, *Multipartite reduction criteria for separability*, e-print quant-ph/0504154

[8] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[9] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **223**, 1 (1996); P. Horodecki, Phys. Lett. A **232**, 333 (1997).

[10] A. Miyake and H-J. Briegel, Phys. Rev. Lett. **95**, 220501 (2005).

[11] A.C. Doherty, P.A. Parrilo and F.M. Spedalieri, Phys. Rev. A, Vol. **71**, 032333 (2005).

[12] G. Toth and O. Guehne, Phys. Rev. Lett. **94**, 060501 (2005).

[13] M. Bourennane, M. Eibl, Ch. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Guehne, P. Hyllus, D. Bruss, M. Lewenstein and A. Sanpera, Phys. Rev. Lett. **92**, 087902 (2004).

[14] A. Acin, Phys. Rev. Lett. **88**, 027901 (2002)

[15] W. Dür, J. I. Cirac and R. Tarrach, Phys. Rev. Lett. **83**, 3562 (1999); W. Dur and J.I. Cirac, Phys. Rev. A **61**, 042314 (2000).