On the structure of entanglement witnesses and new class of positive indecomposable maps

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Abstract

We construct a new class of positive indecomposable maps in the algebra of $d \times d$ complex matrices. Each map is uniquely characterized by a cyclic bistochastic matrix. This class generalizes a Choi map for $d = 3$. It provides a new reach family of indecomposable entanglement witnesses which define important tool for investigating quantum entanglement.

1 Introduction

One of the most important problems of quantum information theory [1] is the characterization of mixed states of composed quantum systems. In particular it is of primary importance to test whether a given quantum state exhibits quantum correlation, i.e. whether it is separable or entangled. For low dimensional systems there exists simple necessary and sufficient condition for separability. The celebrated Peres-Horodecki criterium [2, 3] states that a state of a bipartite system living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable iff its partial transpose is positive. Unfortunately, for higher-dimensional systems there is no single *universal* separability condition. A different useful separability criterion, that has been used to show entanglement of PPT states, is the range criterion [3]. It is based on the fact that for every separable state ρ there exist a set of pure product states $\psi_i \otimes \varphi_i$ that span the range of ρ while $\psi_i \otimes \overline{\varphi}_i$ span the range of its partial transposition $(\mathbb{1} \otimes \tau)\rho$. Other criteria, that are in general weaker than PPT are the reduction criterion [4] and the majorization criterion [5]. None of these criteria, nor a combination of them are sufficient to give a complete characterization of separable states.

The most general approach to separability problem is based on the following theorem [6]: a state ρ of a bipartite system living in $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable iff $\mathrm{Tr}(W\rho) \geq 0$ for any Hermitian operator W satisfying $\mathrm{Tr}(W P_A \otimes P_B) \geq 0$, where P_A and P_B are projectors acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Recall, that a Hermitian operator $W \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is an entanglement witness [6, 7] iff: i) it is not positively defined, i.e. $W \ngeq 0$, and ii) $\mathrm{Tr}(W\sigma) \geq 0$ for all separable states σ. A bipartite state ρ living in $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled iff there exists an entanglement witness W detecting ρ, i.e. such that $\mathrm{Tr}(W\rho) < 0$. It should be stressed that there is no universal W, i.e. there is no entanglement witness which detects all entangled states. Each entangled state ρ may be detected by a specific choice of W. It is clear that each W provides a new separability test and it may be interpreted as a new type of Bell inequality [8]. There is, however, no general procedure for constructing W's.

The separability problem may be equivalently formulated in terms positive maps [6]: a state ρ is separable iff $(\mathbb{1} \otimes \Lambda)\rho$ is positive for any positive map Λ which sends positive operators on $\mathcal{H}_B$ into

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positive operators on $\mathcal{H}_A$. Unfortunately, in spite of the considerable effort, the structure of positive maps is rather poorly understood [9–44].

Note, that performing a PPT test we may reduce the separability problem to PPT states. Positive maps which may be used to detect PPT entangled states define a class of so called indecomposable positive maps. There are only few examples of indecomposable positive maps known in the literature (see review in Section 4). They seem to be hard to find and no general construction method is available. Therefore, any new example provides new tools to investigate quantum entanglement. In the present paper we construct a new class of such maps. The paper is organized as follows: in the next Section we introduce a natural hierarchy of positive convex cones in the space of (unnormalized) states of bipartite $d \otimes d$ quantum systems. In Section 3 we recall basis notions from the theory of positive maps and introduce a duality between positive maps entanglement witnesses. Section 4 serves as a catalog of known indecomposable positive maps. Sections 5 and 6 introduce basic classes of positive maps which we are going to use in our search for indecomposable maps. Finally in Section 7 we show how to construct a new family of indecomposable maps within a class of positive maps discussed in previous sections. This class defines a natural generalization of Choi maps on M_3. A brief discussion is included in the last section.

2 The structure of entanglement witnesses

In the preset paper we shall consider a bipartite quantum system living in $\mathbb{C}^d \otimes \mathbb{C}^d$. Denote by M_d a set of $d \times d$ complex matrices and let M_d^+ be a convex set of semi-positive elements in M_d, that is, M_d^+ defines a space of (unnormalized) states of d-level quantum system. For any $\rho \in (M_d \otimes M_d)^+$ denote by $\mathrm{SN}(\rho)$ a Schmidt number of ρ [45]. Now, let us introduce the following family of positive cones:

$$\mathrm{V}_r = \{ \, \rho \in (M_d \otimes M_d)^+ \mid \mathrm{SN}(\rho) \leq r \, \} \ . \tag{1}$$

One has the following chain of inclusions

$$\mathrm{V}_1 \subset \ldots \subset \mathrm{V}_d \equiv (M_d \otimes M_d)^+ \ . \tag{2}$$

Clearly, V_1 is a cone of separable (unnormalized) states and $V_d \smallsetminus V_1$ stands for a set of entangled states. Note, that a partial transposition $(\mathbb{1}_d \otimes \tau)$ gives rise to another family of cones:

$$\mathrm{V}^l = (\mathbb{1}_d \otimes \tau)\mathrm{V}_l \ , \tag{3}$$

such that $\mathrm{V}^1 \subset \ldots \subset \mathrm{V}^d$. One has $\mathrm{V}_1 = \mathrm{V}^1$, together with the following hierarchy of inclusions:

$$\mathrm{V}_1 = \mathrm{V}_1 \cap \mathrm{V}^1 \subset \mathrm{V}_2 \cap \mathrm{V}^2 \subset \ldots \subset \mathrm{V}_d \cap \mathrm{V}^d \ . \tag{4}$$

Note, that $\mathrm{V}_d \cap \mathrm{V}^d$ is a convex set of PPT (unnormalized) states. Finally, $\mathrm{V}_r \cap \mathrm{V}^s$ is a convex subset of PPT states ρ such that $\mathrm{SN}(\rho) \leq r$ and $\mathrm{SN}[(\mathbb{1}_d \otimes \tau)\rho] \leq s$.

Let us denote by W a space of entanglement witnesses, i.e. a space of non-positive Hermitian operators $W \in M_d \otimes M_d$ such that $\mathrm{Tr}(W\rho) \geq 0$ for all $\rho \in \mathrm{V}_1$. Define a family of subsets $\mathrm{W}_r \subset M_d \otimes M_d$:

$$\mathrm{W}_r = \{ \, W \in M_d \otimes M_d \mid \mathrm{Tr}(W\rho) \geq 0 \ , \ \rho \in \mathrm{V}_r \, \} \ . \tag{5}$$

One has

$$(M_d \otimes M_d)^+ \equiv \mathrm{W}_d \subset \ldots \subset \mathrm{W}_1 \ . \tag{6}$$

Clearly, $\mathrm{W} = \mathrm{W}_1 \smallsetminus \mathrm{W}_d$. Moreover, for any $k > l$, entanglement witnesses from $\mathrm{W}_l \smallsetminus \mathrm{W}_k$ can detect entangled states from $\mathrm{V}_k \smallsetminus V_l$, i.e. states ρ with Schmidt number $l < \mathrm{SN}(\rho) \leq k$. In particular $W \in \mathrm{W}_k \smallsetminus \mathrm{W}_{k+1}$ can detect state ρ with $\mathrm{SN}(\rho) = k$.

Consider now the following class

$$\mathrm{W}_r^s = \mathrm{W}_r + (\mathbb{1} \otimes \tau)\mathrm{W}_s \ , \tag{7}$$

that is, $W \in \mathrm{W}_r^s$ iff

$$W = P + (\mathbb{1} \otimes \tau)Q \ , \tag{8}$$

with $P \in \mathrm{W}_r$ and $Q \in \mathrm{W}_s$. Note, that $\mathrm{Tr}(W\rho) \geq 0$ for all $\rho \in \mathrm{V}_r \cap \mathrm{V}^s$. Hence such W can detect PPT states ρ such that $\mathrm{SN}(\rho) \geq r$ or $\mathrm{SN}[(\mathbb{1}_d \otimes \tau)\rho] \geq s$. Entanglement witnesses from W_d^d are called decomposable [46]. They cannot detect PPT states. One has the following chain of inclusions:

$$\mathrm{W}_d^d \subset \ldots \subset \mathrm{W}_2^2 \subset \mathrm{W}_1^1 \equiv \mathrm{W} \ . \tag{9}$$

The 'weakest' entanglement can be detected by elements from $\mathrm{W}_1^1 \smallsetminus \mathrm{W}_2^2$. We shall call them *atomic entanglement witnesses*.

3 Positive maps and duality

It is well known that the separability problem may be reformulated in terms of positive maps [6]. Recall, that a linear map $\varphi : M_d \longrightarrow M_d$ is called positive iff $\varphi(a) \in M_d^+$ for any $a \in M_d^+$. It is well known [6] that a state $\rho \in (M_d \otimes M_d)^+$ is separable iff $(\mathbb{1}_d \otimes \varphi)\rho \geq 0$ for all positive maps $\varphi : M_d \longrightarrow M_d$ ($\mathbb{1}_d$ stands for an identity map). Hence, having a positive map φ such that $(\mathbb{1}_d \otimes \varphi)$ acting on ρ is no longer positive we are sure that ρ is entangled. However, the crucial problem with the above criterion is that the classification and characterization of positive maps is an open question.

A linear map $\varphi : M_d \longrightarrow M_d$ is called k-positive iff the extended map

$$\mathbb{1}_k \otimes \varphi : M_k \otimes M_d \longrightarrow M_k \otimes M_d \ ,$$

is positive. If φ is k-positive for all extensions, i.e. for $k = 2, 3, \ldots$, then φ is completely positive (CP). Actually, it was shown by Choi that $\varphi : M_d \longrightarrow M_d$ is CP iff it is d-positive. Note, that using the hierarchy of cones V_k we may reformulate the above definitions as follows: a linear map φ is k-positive iff

$$(\mathbb{1}_d \otimes \varphi)(\mathrm{V}_k) \subset (M_d \otimes M_d)^+ \ . \tag{10}$$

Let us denote by P_k a convex cone of k-positive maps. One has, therefore, a natural chain of inclusions

$$\mathrm{P}_d \subset \mathrm{P}_{d-1} \subset \ldots \subset \mathrm{P}_2 \subset \mathrm{P}_1 \ , \tag{11}$$

where P_d stands for CP maps. Due to the celebrated Kraus theorem any CP map can be written in the following Kraus representation

$$\varphi(a) = \sum_\alpha K_\alpha a K_\alpha^\dagger \ , \tag{12}$$

with $K_\alpha \in M_d$. Additional condition $\sum_\alpha K_\alpha^\dagger K_\alpha = I_d$ implies that $\mathrm{Tr}\,\varphi(a) = \mathrm{Tr}\,a$.

Note, that we cannot detect entangled state using CP map. Therefore, we are interested in positive maps which are not CP. It turns out that any positive map φ may be written as a difference of two CP maps, i.e.

$$\varphi(a) = \sum_\alpha K_\alpha a K_\alpha^\dagger - \sum_\beta L_\beta a L_\beta^\dagger \ , \tag{13}$$

with $K_\alpha, L_\beta \in M_d$. The most prominent example of a positive map which is not completely positive is a transposition $\tau(a) = a^T$. Composing positive maps with transposition gives rise to a new class of maps: a map $\varphi : M_d \longrightarrow M_d$ is called k-copositive iff $\varphi \circ \tau$ is k-positive. Finally, φ is completely copositive (CcP) iff $\varphi \circ \tau$ is CP. Equivalently, φ is k-copositive iff

$$(\mathbb{1}_d \otimes \varphi)(\mathrm{V}^k) \subset (M_d \otimes M_d)^+ \ . \tag{14}$$

Denoting by P^k a convex cone of k-copositive maps one has

$$\mathrm{P}^d \subset \mathrm{P}^{d-1} \subset \ldots \subset \mathrm{P}^2 \subset \mathrm{P}^1 \ , \tag{15}$$

where P^d stands for CcP maps.

A crucial role in detecting quantum entanglement is played by indecomposable maps: a positive map φ is decomposable iff it can be written as $\varphi = \varphi_1 + \varphi_2$ with φ_1 and φ_2 being CP and CcP maps, respectively. Otherwise it is called indecomposable. Note that a positive partial transpose (PPT) state can not be detected by any decomposable map. Therefore, to detect PPT entangled states one needs indecomposable maps. Having defined cones P_r and P^s let $\mathrm{P}_r + \mathrm{P}^s$ stand for a set of maps which can be written as $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in \mathrm{P}_r$ and $\varphi_2 \in \mathrm{P}^s$. Clearly, φ is indecomposable iff $\varphi \notin \mathrm{P}_d + \mathrm{P}^d$. An important subset of indecomposable maps contains so called atomic ones [28]: φ is atomic iff $\varphi \notin \mathrm{P}_2 + \mathrm{P}^2$. The importance of atomic maps follows from the fact that they may be used to detect the 'weakest' bound entanglement.

Now, $M_d \otimes M_d$ is isomorphic to the space of linear maps $\varphi : M_d \to M_d$ denoted by $\mathcal{L}(M_d, M_d)$: for any $\varphi \in \mathcal{L}(M_d, M_d)$ one defines [19]

$$\widehat{\varphi} = (\mathbb{1}_d \otimes \varphi) P^+ \in M_d \otimes M_d \ , \tag{16}$$

where P^+ stands for (unnormalized) maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$. If $e_i = |i\rangle$ $(i = 1, \ldots, d)$ is an orthonormal base in $\mathbb{C}^d$, then

$$\widehat{\varphi} = \sum_{i,j=1}^{d} e_{ij} \otimes \varphi(e_{ij}) \ , \tag{17}$$

where $e_{ij} = |i\rangle\langle j|$. Conversely, if $W \in M_d \otimes M_d$ the corresponding linear map is defined as follows

$$\varphi_W(a) = \mathrm{Tr}_2 \left[W \left(I_d \otimes a^{\mathrm{T}} \right) \right] \ . \tag{18}$$

It is clear that if φ is a positive but not CP map then the corresponding operator $\widehat{\varphi}$ is an entanglement witness.

Now, the space $\mathcal{L}(M_d, M_d)$ is endowed with a natural inner product:

$$(\varphi, \psi) = \mathrm{Tr} \Big(\sum_{\alpha=1}^{d^2} \varphi(f_\alpha)^\dagger \psi(f_\alpha) \Big) \ , \tag{19}$$

where f_α is an arbitrary orthonormal base in M_d. Taking $f_\alpha = e_{ij}$ one finds

$$\begin{aligned} (\varphi, \psi) &= \mathrm{Tr} \Big(\sum_{i,j=1}^{d} \varphi(e_{ij})^\dagger \psi(e_{ij}) \Big) \\ &= \mathrm{Tr} \Big(\sum_{i,j=1}^{d} \varphi(e_{ij}) \psi(e_{ji}) \Big) \ . \end{aligned} \tag{20}$$

This inner product is compatible with the standard Hilbert-Schmidt product in $M_d \otimes M_d$. Indeed, taking $\widehat{\varphi}$ and $\widehat{\psi}$ corresponding to φ and ψ, one has

$$(\widehat{\varphi}, \widehat{\psi})_{\rm HS} = {\rm Tr}(\widehat{\varphi}^\dagger \widehat{\psi}) \tag{21}$$

and using (17) one easily finds

$$(\varphi, \psi) = (\widehat{\varphi}, \widehat{\psi})_{\rm HS} \ , \tag{22}$$

that is, formula (17) defines an inner product isomorphism. This way one establishes the duality between maps from $\mathcal{L}(M_d, M_d)$ and operators from $M_d \otimes M_d$ [35]: for any $\rho \in M_d \otimes M_d$ and $\varphi \in \mathcal{L}(M_d, M_d)$ one defines

$$\langle \rho, \varphi \rangle = (\rho, \widehat{\varphi})_{\rm HS} \ . \tag{23}$$

In particular, if ρ is an unnormalized state and φ is a positive map, then

$$\langle \rho, \varphi \rangle = {\rm Tr}(\widehat{\varphi}\rho) = {\rm Tr}\Big(\sum_{i,j=1}^d \varphi(e_{ij})\, \rho_{ji} \Big) \ , \tag{24}$$

where

$$\rho = \sum_{i,j=1}^d e_{ij} \otimes \rho_{ij} \ , \tag{25}$$

with $\rho_{ij} \in M_d$. Formula (24) reproduces the formula for an entanglement witness $W = \widehat{\varphi}$.

This construction shows that two sets of cones — V_k and P^k — are dual to each other. It follows from (24) that

$$\rho \in \mathrm{V}_r \quad \Longleftrightarrow \quad \langle \rho, \varphi \rangle \geq 0 \quad \text{for all} \quad \varphi \in \mathrm{P}^r \ .$$

Moreover,

$$\rho \in \mathrm{V}_r \cap \mathrm{V}^s \quad \Longleftrightarrow \quad \langle \rho, \varphi \rangle \geq 0 \quad \text{for all} \quad \varphi \in \mathrm{P}^r + \mathrm{P}_s \ .$$

Conversely,

$$\varphi \in \mathrm{P}^r \quad \Longleftrightarrow \quad \langle \rho, \varphi \rangle \geq 0 \quad \text{for all} \quad \rho \in \mathrm{V}_r \ ,$$

and

$$\varphi \in \mathrm{P}^r + \mathrm{P}_s \quad \Longleftrightarrow \quad \langle \rho, \varphi \rangle \geq 0 \quad \text{for all} \quad \varphi \in \mathrm{V}_r \cap \mathrm{V}^s \ .$$

Clearly, formula (24) may be used to witness entanglement: ρ is entangled iff there exists $\varphi \in P^1$ such that $\langle \rho, \varphi \rangle < 0$. More generally, a positive operator $\rho \notin \mathrm{V}_r$ iff there exists $\varphi \in P^r$ such that $\langle \rho, \varphi \rangle < 0$, and $\rho \notin \mathrm{V}_r \cap \mathrm{V}^s$ iff there exists $\varphi \in P^r + \mathrm{P}_s$ such that $\langle \rho, \varphi \rangle < 0$.

Dually, we may use (24) to check whether a given positive map φ is indecomposable or atomic: φ is indecomposable iff there exists $\rho \in \mathrm{V}_d \cap \mathrm{V}^d$ (i.e. ρ is PPT) such that $\langle \rho, \varphi \rangle < 0$. Finally, φ is atomic iff there exists $\rho \in \mathrm{V}_2 \cap \mathrm{V}^2$ such that $\langle \rho, \varphi \rangle < 0$.

4 Indecomposable maps – review

For $d = 2$ all positive maps $\varphi : M_2 \to M_2$ are decomposable [17, 18].

4.1 Choi map for $d = 3$

The first example of an indecomposable positive linear map in M_3 was found by Choi [15]. The (normalized) Choi map reads as follows

$$\begin{aligned} \Phi_{\rm C}(e_{ii}) &= \sum_{i,j=1}^{3} a^{\rm C}_{ij} e_{jj} \ , \\ \Phi_{\rm C}(e_{ij}) &= -\frac{1}{2}\, e_{ij} \ , \qquad i \neq j \ , \end{aligned} \tag{26}$$

where $[a^{\rm C}_{ij}]$ is the following bistochatic matrix:

$$a^{\rm C}_{ij} = \frac{1}{2} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right) \ . \tag{27}$$

This map may be generalized as follows [31]: for any $a, b, c \geq 0$ let us define

$$\begin{aligned} \Phi[a,b,c](e_{ii}) &= \sum_{i,j=1}^{3} a_{ij} e_{jj} \ , \\ \Phi[a,b,c](e_{ij}) &= -\frac{1}{a+b+c}\, e_{ij} \ , \quad i \neq j \ , \end{aligned} \tag{28}$$

with

$$a_{ij} = \frac{1}{a+b+c} \left(\begin{array}{ccc} a & b & c \\ c & a & b \\ b & c & a \end{array} \right) \ . \tag{29}$$

Clearly, $\Phi^{\rm C} = \Phi[1,1,0]$. The map $\Phi[1,0,\mu]$ with $\mu \geq 1$ is the example of indecomposable map introduced by Choi [16]. Now, it was shown [31] that $\Phi[a,b,c]$ is an indecomposable positive map if and only if the following conditions are satisfied:

$$\begin{array}{ll} (i) & 0 \leq a < 2 \ , \\ (ii) & a + b + c \geq 2 \ , \\ (iii) & \left\{ \begin{array}{ll} (1-a)^2 \leq bc < (2-a)^2/4 \ , & \text{if } \ 0 \leq a \leq 1 \\ 0 \leq bc < (2-a)^2/4 \ , & \text{if } \ 1 \leq a < 2 \end{array} \right. \end{array} \ .$$

Actually, $\Phi[a,b,c]$ is indecomposable if and only if it is atomic, i.e. it cannot be decomposed into the sum of a 2-positive and 2-copositive maps.

4.2 Indecomposable maps for $d \geq 3$

For $d \geq 3$ there are three basic families of indecomposable maps:

1) A discrete family $\tau_{d,k}$, $k = 1, \ldots, d-2$ [36]. Let s be a unitary shift defined by:

$$s\, e_i = e_{i+1} \ , \qquad i = 1, \ldots, d \ ,$$

where the indices are understood mod d. The maps $\tau_{d,k}$ are defined as follows:

$$\tau_{d,k}(X) = (d-k)\, \epsilon(X) + \sum_{i=1}^{k} \epsilon(s^i\, X\, s^{*i}) - X \ , \tag{30}$$

where $\epsilon(X)$ is defined in (53). The map $\tau_{d,0}$ defined in (54) is completely positive and it is well known that the map corresponding to $k = d-1$ is completely co-positive [36].

Note that $\tau_{d,k}(I_d) = (d-1)I_d$, and $\mathrm{Tr}\,\tau_{d,k}(X) = (d-1)\mathrm{Tr}\,X$, hence the normalized maps

$$\Phi_{d,k}(X) = \frac{1}{d-1}\,\tau_{d,k}(X) \ , \tag{31}$$

are bistochastic. In particular $\Phi[1,0,1] = \Phi_{3,1}$.

2) A class of maps $\varphi_{\mathbf{p}}$ parameterized by $d+1$ parameters $\mathbf{p} = (p_0, p_1, \ldots, p_d)$:

$$\begin{aligned} \varphi_{\mathbf{p}}(e_{11}) &= p_0 e_{11} + p_d e_{dd} \ , \\ \varphi_{\mathbf{p}}(e_{22}) &= p_0 e_{22} + p_1 e_{11} \ , \\ &\vdots \\ \varphi_{\mathbf{p}}(e_{dd}) &= p_0 e_{dd} + p_{d-1} e_{d-1,d-1} \ , \\ \varphi_{\mathbf{p}}(e_{ij}) &= -e_{ij} \ , \quad i \neq j \ . \end{aligned} \tag{32}$$

It was shown [30, 37] that if

$$\begin{aligned} &a) \quad p_1, \ldots, p_d > 0 \ , \\ &b) \quad d-1 > p_0 \geq d-2 \ , \\ &c) \quad p_1 \cdot \ldots \cdot p_d \geq (d-1-p_0)^d \ , \end{aligned}$$

then $\varphi_{\mathbf{p}}$ is a positive indecomposable map. Actually, $\varphi_{\mathbf{p}}$ is atomic, i.e. it cannot be decomposed into the sum of a 2-positive and 2-copositive maps.

3) A family of maps constructed by Terhal [41] from unextendible product bases [48, 49]. Let $|\alpha_i\rangle \otimes |\beta_i\rangle$; $i = 1, \ldots, K < d^2$ be an unextendible product basis in $\mathbb{C}^d \otimes \mathbb{C}^d$. Then an unnormalised density matrix

$$\rho = I_d \otimes I_d - \sum_{i=1}^{K} |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i| \ ,$$

defines a PPT entangled state. This state my be detected by the following entanglement witness:

$$W = \sum_{i=1}^{K} |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i| - d\varepsilon |\Psi\rangle\langle\Psi| \ , \tag{33}$$

where $|\Psi\rangle$ is a maximally entangled state such that $\langle\Psi, \rho\Psi\rangle > 0$. A parameter ε is defined by

$$\varepsilon = \min_{|\phi_1\rangle\otimes|\phi_2\rangle} \sum_{i=1}^{K} |\langle\alpha_i|\phi_1\rangle|^2 \langle\beta_i|\phi_2\rangle|^2 \ , \tag{34}$$

where the minimum is taken over all pure separable states $|\phi_1\rangle \otimes |\phi_2\rangle$. It is therefore clear that the corresponding map

$$\Phi(X) = \mathrm{Tr}_2\left[W\left(I_d \otimes X^{\mathrm{T}}\right)\right] \ , \tag{35}$$

is an indecomposable positive map in M_d.

Another example of an indecomposable map (also outside the class (36)) was given by Robertson [20, 21, 22, 23]. Robertson map $\varphi_4 : M_4 \rightarrow M_4$ is defined by

$$\begin{aligned}
\varphi_4(e_{11}) &= \varphi_4(e_{22}) = \frac{1}{2}(e_{33}+e_{44}) \ , \\
\varphi_4(e_{33}) &= \varphi_4(e_{44}) = \frac{1}{2}(e_{11}+e_{22}) \ , \\
\varphi_4(e_{13}) &= \frac{1}{2}(e_{13}+e_{42}) \ , \\
\varphi_4(e_{14}) &= \frac{1}{2}(e_{14}-e_{32}) \ , \\
\varphi_4(e_{23}) &= \frac{1}{2}(e_{23}-e_{41}) \ , \\
\varphi_4(e_{24}) &= \frac{1}{2}(e_{24}+e_{31}) \ ,
\end{aligned}$$

and the remaining

$$\varphi_4(e_{12}) = \varphi_4(e_{21}) = \varphi_4(e_{34}) = \varphi_4(e_{43}) = 0 \ .$$

It satisfies $\varphi_4(I_4) = I_4$ and $\mathrm{Tr}\,\varphi_4(X) = \mathrm{Tr}\,X$, and it is known that φ_4 is atomic and hence indecomposable.

5 On certain class of positive maps

Consider the following class of linear maps $\varphi : M_d \longrightarrow M_d$:

$$\begin{aligned}
\varphi(e_{ii}) &= \sum_{j=1}^{d} a_{ij} e_{jj} \ , \\
\varphi(e_{ii}) &= -e_{ij} \ , \quad i \neq j \ ,
\end{aligned} \tag{36}$$

with $||a_{ij}||$ being a $d \times d$ real positive matrix. Let us observe that most of well known positive maps reviewed in the previous section do belong to this class (only the class based on unextendible product bases and the example constructed by Robertson do not).

Theorem 1 *A map belonging to a class (36) is positive iff*

$$\left(1 - \sum_{i=1}^{d} \frac{|x_i|^2}{B_i(x)}\right) \prod_{k=1}^{d} B_k(x) \geq 0 \ , \tag{37}$$

for all $x \in \mathbb{C}^d$ such that $|x|^2 = \sum_{i=1}^{d} |x_i|^2 = 1$, and

$$B_i(x) = |x_i|^2 + \sum_{j=1}^{d} a_{ij}|x_j|^2 \ . \tag{38}$$

If all $B_i \neq 0$, then (37) simplifies to

$$\sum_{i=1}^{d} \frac{|x_i|^2}{B_i(x)} \leq 1 \ . \tag{39}$$

Proof: φ is positive iff for any normalized $x \in \mathbb{C}^d$ one has $\varphi(P_x) \geq 0$, where $P_x = |x\rangle\langle x|$ denotes the corresponding 1-dimensional projector. Let us denote the corresponding $d \times d$ matrix $\varphi(P_x)$ by $A(x) = [A_{ij}(x)]$, that is

$$A_{ii}(x) = \sum_{j=1}^{d} a_{ij}|x_j|^2 , \tag{40}$$

$$A_{ij}(x) = -x_i\overline{x}_j , \quad i \neq j . \tag{41}$$

Positivity of φ is therefore equivalent to the positivity of $A(x)$ for any normalized $x \in \mathbb{C}^d$. Now, to check for positivity of $A(x)$ one computes the characteristic polynomial

$$\det || A_{ij}(x) - \lambda\delta_{ij} || = \sum_{k=0}^{d} (-\lambda)^{d-k} C_k(x) , \tag{42}$$

and $||A_{ij}(x)|| \geq 0$ iff $C_k(x) \geq 0$ for $k = 0, 1, \ldots, d$. The determinant of $||A_{ij}(x) - \lambda\delta_{ij}||$ is easy to calculate. Using the following formula

$$\left| \begin{array}{ccccc} \gamma_1 & \alpha_2\beta_1 & \alpha_3\beta_1 & \ldots & \alpha_n\beta_1 \\ \alpha_1\beta_2 & \gamma_2 & \alpha_3\beta_2 & \ldots & \alpha_n\beta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1\beta_n & \alpha_2\beta_n & \alpha_3\beta_n & \ldots & \gamma_n \end{array} \right| = \left(1 + \sum_{k=1}^{n} \frac{\alpha_k\beta_k}{\gamma_k - \alpha_k\beta_k} \right) \prod_{i=1}^{n} (\gamma_i - \alpha_i\beta_i) , \tag{43}$$

with $\gamma_k \neq \alpha_k\beta_k$ for $k = 1, 2, \ldots, n$, one easily finds

$$\det || A_{ij}(x) - \lambda\delta_{ij} || = \left(1 - \sum_{k=1}^{d} \frac{|x_k|^2}{B_k(x) - \lambda} \right) \prod_{i=1}^{d} (B_i(x) - \lambda) , \tag{44}$$

where $B_k(x)$ is given by (38). Now, formula (42) implies for the coefficients $C_{d-l}(x)$

$$C_{d-l}(x) = \frac{(-1)^l}{(l-1)!} \frac{d^l}{d\lambda^l} \det || A_{ij}(x) - \lambda\delta_{ij} || \Big|_{\lambda=0} , \tag{45}$$

and hence, using (44) one finds

$$C_k(x) = \left(1 - \sum_{i=1}^{d} \frac{|x_i|^2}{B_i(x)} \right) \sum_{i_1 < i_2 < \ldots < i_k} B_{i_1}(x) \ldots B_{i_k}(x) . \tag{46}$$

It is therefore clear that $C_k(x) \geq 0$ iff $C_d(x) \geq$. Hence, using (46) one obtains the following condition for the positivity of φ:

$$C_d(x) = \left(1 - \sum_{i=1}^{d} \frac{|x_i|^2}{B_i(x)} \right) \prod_{k=1}^{d} B_k(x) \geq 0 , \tag{47}$$

which finally proves (37). □

As a direct application of Theorem 1 let us observe that a celebrated Choi map in M_3 defined by the matrix

$$a_{ij} = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right) , \tag{48}$$

gives rise to

$$\begin{aligned} B_1(x) &= 2|x_1|^2 + |x_2|^2 \ , \\ B_2(x) &= 2|x_2|^2 + |x_3|^2 \ , \\ B_3(x) &= 2|x_3|^2 + |x_1|^2 \ . \end{aligned}$$

and direct calculation shows that the condition (37) is satisfied and hence φ is positive.

Let us observe that the map φ defined in (36) acting on $X \in M_d$ give

$$\begin{aligned} \varphi(X) &= \sum_{i,j=1}^{d} (a_{ij} + \delta_{ij})\, e_{ii} \langle e_j | X e_j \rangle - X \\ &= \sum_{i,j=1}^{d} (a_{ij} + \delta_{ij})\, e_{ij}\, X\, e_{ij}^* - \sum_{i,j=1}^{d} e_{ii}\, X\, e_{jj} \ . \end{aligned}$$

Now introducing $||b_{ij}||$ by

$$b_{ii} = a_{ii} \ , \quad\quad b_{ij} = -1 \ , \quad i \neq j \ , \tag{49}$$

one has

$$\varphi(X) = \sum_{i \neq j}^{d} a_{ij}\, e_{ij}\, X\, e_{ij}^* + \sum_{i \neq j}^{d} b_{ij}\, e_{ii}\, X\, e_{jj} \ . \tag{50}$$

This observation gives rise to the following

Theorem 2 *A map φ defined in (36) is completely positive iff the matrix $||b_{ij}||$ is positive.*

Let us note that if $a_{11} = \ldots = a_{dd} = a$, that is,

$$b_{ij} = \left(\begin{array}{ccccc} a & -1 & -1 & \ldots & -1 \\ -1 & a & -1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \ldots & a \end{array} \right) \ , \tag{51}$$

then $||b_{ij}|| \geq 0$ iff $a \geq d-1$. In particular if $a_{ij} = a\delta_{ij}$ then the action of φ is given by

$$\varphi(X) = (a+1)\epsilon(X) - X \ , \tag{52}$$

where $\epsilon : M_d \longrightarrow M_d$ stands for the projector onto the diagonal part:

$$\epsilon(X) = \sum_{i=1}^{d} \mathrm{Tr}[X\, e_{ii}]\, e_{ii} \ . \tag{53}$$

Hence the map (52) is completely positive for $a \geq d-1$. For $a = d-1$ one recovers a CP map

$$\tau_{d,0}(X) = d\, \epsilon(X) - X \ , \tag{54}$$

considered in [36].

6 A family of positive maps parameterized by contractions

Recently a rich family of positive maps was constructed in [43]. Let us consider the following map $\varphi : M_d \longrightarrow M_d$:

$$\varphi(X) = \frac{I_d}{d}\,\mathrm{Tr}X + \frac{1}{d-1}\sum_{\alpha,\beta=1}^{d^2-1} f_\alpha A_{\alpha\beta}\mathrm{Tr}(f_\beta X)\ , \tag{55}$$

where $A = [A_{\alpha\beta}]$ is a real matrix representing contraction in $\mathbb{R}^{d^2-1}$ and $f_\alpha \in M_d$ define the generators of $SU(d)$ such that $f_\alpha = f_\alpha^*$, $\mathrm{Tr}(f_\alpha f_\beta) = \delta_{\alpha\beta}$, and $\mathrm{Tr}\, f_\alpha = 0$ for $\alpha, \beta = 1, \ldots, d^2 - 1$. The explicit construction of f_α reads as follows:

$$(f_1, \ldots, f_{d^2-1}) = (d_l, u_{kl}, v_{kl})\ ,$$

for $l = 1, \ldots, d-1$ and $1 \le k < l \le d$, where the diagonal operators

$$d_l \;=\; \frac{1}{\sqrt{l(l+1)}}\Big(\sum_{k=1}^{l} e_{kk} - l e_{l+1,l+1}\Big)\ , \tag{56}$$

define Cartan subalgebra of $su(d)$, and off-diagonal

$$u_{kl} = \frac{1}{\sqrt{2}}(e_{kl} + e_{lk})\ , \qquad v_{kl} = \frac{-i}{\sqrt{2}}(e_{kl} - e_{lk})\ . \tag{57}$$

It was shown in [43] that this map is positive for an arbitrary contraction $A_{\alpha\beta}$. Moreover, one has

$$\varphi(I_d) = I_d \quad \text{and} \quad \mathrm{Tr}\,\varphi(X) = \mathrm{Tr}\,X\ .$$

Consider now a special case corresponding to

$$A = \left(\begin{array}{cc} \mathbf{A} & 0 \\ 0 & -I \end{array}\right)\ , \tag{58}$$

where $\mathbf{A}$ is a contraction in $\mathbb{R}^{d-1}$. Recall, that any contraction $\mathbf{A}_{\alpha\beta}$ may be represented as follows

$$\mathbf{A} = R_1 D R_2\ , \tag{59}$$

where R_i represent rotations in $\mathbb{R}^{d-1}$, i.e. $R_i \in SO(d-1)$, and D is a diagonal matrix with $|\lambda_i| = |D_{ii}| \le 1$. Let us consider the special case of (59) such that $D = \lambda I_{d-1}$ with $0 \le \lambda \le 1$, that is,

$$\mathbf{A} = \lambda\, R\ , \tag{60}$$

where $R = R_1 R_2 \in SO(d-1)$. The general case (59) produces much more complicated situation even in $d = 3$ (see Appendix). The action of φ is given by

$$\varphi(e_{ij}) \;=\; -\frac{1}{d-1}\, e_{ij}\ , \quad i \neq j\ , \tag{61}$$

$$\varphi(e_{ii}) \;=\; \frac{I_d}{d} + \frac{\lambda}{d-1}\sum_{\alpha,\beta=1}^{d-1} f_\alpha R_{\alpha\beta}(e_i, f_\beta e_i)\ . \tag{62}$$

Note that $\varphi(e_{ii})$ may be rewritten as follows

$$\varphi(e_{ii}) = \sum_{j=1}^{d} a_{ij} e_{jj} \ , \tag{63}$$

where $a_{ij} = \mathrm{Tr}[\varphi(e_{ii})e_{jj}]$ is given by the following bistochastic matrix:

$$a_{ij} = \frac{1}{d} + \frac{\lambda}{d-1} \sum_{\alpha,\beta=1}^{d-1} (e_j, f_\alpha e_j) R_{\alpha\beta} (e_i, f_\beta e_i) \ . \tag{64}$$

Therefore, up to the normalization factor $1/(d-1)$, this family belongs to our class (36) discussed in the previous section. Consider now a class of positive maps defined by

$$\begin{aligned} \varphi(e_{ii}) &= \sum_{j=1}^{d} a_{ij} e_{jj} \ , \\ \varphi(e_{ii}) &= -\frac{1}{d-1} e_{ij} \ , \qquad i \neq j \ , \end{aligned} \tag{65}$$

with bistochastic $||a_{ij}||$.

Theorem 3 *A bistochastic matrix $||a_{ij}||$ corresponds to contraction λR with $R \in SO(d-1)$ and $\lambda \leq 1$, that is $||a_{ij}||$ is given by (64), iff*

$$\sum_{k=1}^{d} a_{ik} a_{jk} \ = \ \frac{1}{(d-1)^2} \left(\lambda^2 \delta_{ij} + d - 2 + \frac{1-\lambda^2}{d} \right) \ . \tag{66}$$

Proof: To prove (66) define a new map $\Phi : M_d \longrightarrow M_d$

$$\Phi(e_{ii}) = (d-1) \left(\varphi(e_{ii}) - \frac{I_d}{d} \right) = \lambda \sum_{\alpha,\beta=1}^{d-1} f_\alpha R_{\alpha\beta} (e_i, f_\beta e_i) \ , \tag{67}$$

together with a dual map

$$\widetilde{\Phi}(e_{ii}) = (d-1) \left(\widetilde{\varphi}(e_{ii}) - \frac{I_d}{d} \right) = \lambda \sum_{\alpha,\beta=1}^{d-1} (e_i, f_\alpha e_i) R_{\alpha\beta} f_\beta \ . \tag{68}$$

One has $\Phi(I_d) = \widetilde{\Phi}(I_d) = 0$. Now, let us compute $\widetilde{\Phi}[\Phi(e_{ii})]$:

$$\widetilde{\Phi}[\Phi(e_{ii})] = \lambda \sum_{\alpha,\beta=1}^{d-1} \widetilde{\Phi}(f_\alpha) R_{\alpha\beta} (e_i, f_\beta e_i) = \lambda \sum_{\alpha,\beta=1}^{d-1} \sum_{j=1}^{d} \widetilde{\Phi}(e_{jj}) (e_j, f_\alpha e_j) R_{\alpha\beta} (e_i, f_\beta e_i) \ , \tag{69}$$

and hence using (68)

$$\widetilde{\Phi}[\Phi(e_{ii})] \ = \ \lambda^2 \sum_{\alpha,\beta=1}^{d-1} \sum_{\mu,\nu=1}^{d-1} \sum_{j=1}^{d} (e_j, f_\alpha e_j) R_{\alpha\beta} (e_i, f_\beta e_i) (e_j, f_\mu e_j) R_{\mu\nu} f_\nu \ . \tag{70}$$

Taking into account that

$$\sum_{j=1}^{d}(e_j, f_\alpha e_j)(e_j, f_\mu e_j) = \delta_{\alpha\mu} \ ,$$

one obtains

$$\begin{aligned}\widetilde{\Phi}[\Phi(e_{ii})] &= \lambda^2 \sum_{\alpha,\beta,\nu=1}^{d-1} R_{\alpha\beta} R_{\alpha\nu} (e_i, f_\beta e_i) f_\nu = \lambda^2 \sum_{\beta=1}^{d-1} (e_i, f_\beta e_i) f_\beta \\ &= \lambda^2 \left(\sum_{\beta=0}^{d-1} (e_i, f_\beta e_i) f_\beta - (e_i, f_0 e_i) f_0 \right) \ , \end{aligned} \tag{71}$$

where $f_0 = I_d/\sqrt{d}$. This leads to the following formula

$$\widetilde{\Phi}[\Phi(e_{ii})] = \lambda^2 \left(e_{ii} - \frac{I_d}{d} \right) \ . \tag{72}$$

Now, using (67)

$$\begin{aligned}\widetilde{\Phi}[\Phi(e_{ii})] &= (d-1)\widetilde{\Phi}\left[\varphi(e_{ii}) - \frac{I_d}{d}\right] = (d-1)\sum_{j=1}^{d} a_{ij}\widetilde{\Phi}(e_{jj}) \\ &= (d-1)^2 \sum_{j=1}^{d} a_{ij}\left[\widetilde{\varphi}(e_{jj}) - \frac{I_d}{d}\right] = (d-1)^2 \left[\sum_{j=1}^{d} a_{ij}\, \widetilde{\varphi}(e_{jj}) - \frac{I_d}{d}\right] \ ,\end{aligned}$$

where we have used $\sum_{j=1}^{d} a_{ij} = 1$. Finally, taking into account the definition of the dual map

$$\widetilde{\varphi}(e_{jj}) = \sum_{k=1}^{d} a_{kj} e_{kk} \ ,$$

one gets

$$\widetilde{\Phi}[\Phi(e_{ii})] = (d-1)^2 \left[\sum_{j,k=1}^{d} a_{ij}\, a_{kj} e_{kk} - \frac{I_d}{d} \right] \ , \tag{73}$$

and comparing formulae (72) and (73)

$$(d-1)^2 \left(\sum_{j,k=1}^{d} a_{ij}\, a_{kj} e_{kk} - \frac{I_d}{d} \right) = \lambda^2 \left(e_{ii} - \frac{I_d}{d} \right) \ ,$$

one shows (66). □

7 Main result

Now we show that for certain class of bistochastic satisfying (66) matrices the corresponding positive map (65) is indecomposable.

Theorem 4 *Let $\varphi : M_d \longrightarrow M_d$ be a positive map defined by (65) with a bistochastic matrix $||a_{ij}||$ satisfying (66). Suppose that a matrix $||a_{ij}||$ is cyclic, i.e.*

$$a_{ij} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_{d-1} \\ \alpha_{d-1} & \alpha_0 & \alpha_1 & \dots & \alpha_{d-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_0 \end{pmatrix} , \tag{74}$$

with $\alpha_i \geq 0$, and $\alpha_0 + \alpha_1 + \ldots + \alpha_{d-1} = 1$. Then φ is indecomposable if :
i) for $d = 2k + 1$ one of the following two conditions is satisfied

$$1) \quad \begin{cases} \alpha_1 + \ldots + \alpha_k > 0 \\ \alpha_1 + \ldots + \alpha_k \neq \alpha_{k+1} + \ldots + \alpha_{2k} \end{cases} ,$$

$$2) \quad \begin{cases} \alpha_1 + \ldots + \alpha_k = 0 \\ 1 > \alpha_0 > 0 \end{cases} ,$$

ii) for $d = 2k$ one of the following two conditions is satisfied

$$1) \quad \begin{cases} \alpha_1 + \ldots + \alpha_{k-1} > 0 \\ \alpha_1 + \ldots + \alpha_{k-1} \neq \alpha_{k+1} + \ldots + \alpha_{2k-1} \end{cases} ,$$

$$2) \quad \begin{cases} \alpha_1 + \ldots + \alpha_{k-1} = 0 \\ 1 > \alpha_0 + \alpha_k > 0 \end{cases} .$$

Proof. — To show that a positive map φ is indecomposable we use the duality formula (24), i.e. we construct a PPT matrix $\rho \in (M_d \otimes M_d)^+$ such that $\langle \rho, \varphi \rangle < 0$. Consider the following matrix

$$\rho = \sum_{i,j=1}^{d} A_{ij}\, e_{ij} \otimes e_{ij} + \sum_{i \neq j} D_{ij}\, e_{ii} \otimes e_{jj} \ . \tag{75}$$

It is positive iff the Hermitian matrix $[A_{ij}] \geq 0$ and all coefficients $D_{ij} \geq 0$. It was shown in [50] that ρ is PPT if

$$D_{ij} D_{ji} - |A_{ij}| \geq 0 \ , \quad i \neq j \ . \tag{76}$$

Let us consider two separate cases:

i) If $d = 2k + 1$, let us take $A_{ij} = a > 0$, and

$$\begin{aligned} D_{i,i+1} &= D_{i,i+2} = \ldots = D_{i,i+k} = a^2 \ , \\ D_{i,i+k+1} &= D_{i,i+k+2} = \ldots = D_{i,i+2k} = 1 \ , \end{aligned}$$

where the addition is mod d. Clearly, the condition (76) is satisfied and hence the corresponding ρ is PPT. Note, that $\langle \rho, \varphi \rangle = dF(a)$ with

$$F(a) = -a(1 - \alpha_0) + a^2(\alpha_1 + \ldots + \alpha_k) + (\alpha_{k+1} + \ldots + \alpha_{2k}) \ .$$

Note, that if $\alpha_1 + \ldots + \alpha_k > 0$ the function $F = F(a)$ attains its minimum for

$$a = a_0 = \frac{1 - \alpha_0}{2(\alpha_1 + \ldots + \alpha_k)} \ ,$$

and

$$F(a_0) = -\frac{[(\alpha_1 + \ldots + \alpha_k) - (\alpha_{k+1} + \ldots + \alpha_{2k})]^2}{4(\alpha_1 + \ldots + \alpha_k)} \ ,$$

which, for $\alpha_1 + \ldots + \alpha_k \neq \alpha_{k+1} + \ldots + \alpha_{2k}$, implies that $\langle \rho, \varphi \rangle < 0$. Now, if $\alpha_1 + \ldots + \alpha_k = 0$, then $\alpha_{k+1} + \ldots + \alpha_{2k} = 1 - \alpha_0$ and

$$F(a) = (1 - a)(1 - \alpha_0) \ .$$

Hence, $F(a) < 0$ iff $a > 1$ and $1 > \alpha_0 > 0$.

ii) If $d = 2k$, let us take $A_{ij} = a > 0$, and

$$\begin{aligned} D_{i,i+1} &= D_{i,i+2} = \ldots = D_{i,i+k-1} = a^2 \ , \\ D_{i,i+k} &= a \ , \\ D_{i,i+k+1} &= D_{i,i+k+2} = \ldots = D_{i,i+2k-1} = 1 \ . \end{aligned}$$

Clearly, the condition (76) is satisfied and hence the corresponding ρ is PPT. Note, that $\langle \rho, \varphi \rangle = dG(a)$ with

$$\begin{aligned} G(a) &= -a(1 - \alpha_0 - \alpha_k) + a^2(\alpha_1 + \ldots + \alpha_{k-1}) \\ &+ (\alpha_{k+1} + \ldots + \alpha_{2k-1}) \ . \end{aligned}$$

Now, if $\alpha_1 + \ldots + \alpha_{k-1} > 0$ the function $G = G(a)$ attains its minimum for

$$a = a_0' = \frac{1 - \alpha_0 - \alpha_k}{2(\alpha_1 + \ldots + \alpha_{k-1})} \ ,$$

and

$$G(a_0') = -\frac{[(\alpha_1 + \ldots + \alpha_{k-1}) - (\alpha_{k+1} + \ldots + \alpha_{2k-1})]^2}{4(\alpha_1 + \ldots + \alpha_{k-1})}$$

which, for $\alpha_1 + \ldots + \alpha_{k-1} \neq \alpha_{k+1} + \ldots + \alpha_{2k-1}$, implies that $\langle \rho, \varphi \rangle < 0$. If $\alpha_1 + \ldots + \alpha_{k-1} = 0$, then $\alpha_{k+1} + \ldots + \alpha_{2k} = 1 - \alpha_0 - \alpha_k$ and

$$G(a) = (1 - a)(1 - \alpha_0 - \alpha_k) \ .$$

Hence, $G(a) < 0$ iff $a > 1$ and $1 > \alpha_0 + \alpha_k > 0$.

8 Conclusions

We have constructed a new class of positive indecomposable maps $\varphi : M_d \longrightarrow M_d$ which generalizes a Choi map on M_3 [15]. Each such map is characterized by a cyclic bistochastic $d \times d$ matrix $||a_{ij}||$ satisfying conditions of Theorem 1. Now, any indecomposable map provides a new tool for investigation of quantum entanglement: a PPT state ρ is entangled iff there exists an indecomposable map φ such that $(\mathbb{1}_d \otimes \varphi)\rho \not\geq 0$, i.e. $(\mathbb{1}_d \otimes \varphi)\rho$ has at least one negative eigenvalue. Recall that a characteristic feature of transposition τ is that τ and τ_U defined by

$$\tau_U(X) = U\, X^T\, U^\dagger \ ,$$

for $U \in U(d)$, are equivalent, i.e. $(\mathbb{1}_d \otimes \tau)\rho$ and $(\mathbb{1}_d \otimes \tau_U)\rho$ have the same eigenvalues [2]. This property is no longer true for other positive maps. In general φ and φ_U:

$$\varphi_U(X) = U\, \varphi(X)\, U^\dagger \ ,$$

are not equivalent, that is, even if $(\mathbb{1}_d \otimes \varphi)\rho \geq 0$ there may still exist $U \in U(d)$ such that $(\mathbb{1}_d \otimes \varphi_U)\rho \ngeq 0$.

Therefore any indecomposable map φ defined by (65) gives rise to the whole class of indecomposable maps φ_U:

$$\begin{aligned} \varphi_U(e_{ii}) &= \sum_{j,k,l=1}^{d} a_{ij} U_{jk} \overline{U}_{jl} e_{kl} \ , \\ \varphi_U(e_{ij}) &= -\frac{1}{d-1} \sum_{k,l=1}^{d} U_{ik} \overline{U}_{jl}\, e_{kl} \ , \quad i \neq j \ , \end{aligned} \tag{77}$$

with $U_{ik} = \mathrm{Tr}(Ue_{ki})$. This construction leads to a new family of indecomposable entanglement witnesses

$$\widehat{\varphi}_U = (\mathbb{1}_d \otimes \varphi_U)P^+ = \sum_{i,j=1}^{d} e_{ij} \otimes U\varphi(e_{ij})\, U^\dagger \ .$$

As a byproduct we showed that this family of indecomposable entanglement witnesses detect quantum entanglement within a large class of PPT states proposed recently in [50].

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Appendix

Consider $d = 3$. Any contraction in $d-1 = 2$ dimensions is represented by $A = R_1 D R_2$, where D is a diagonal matrix with $D_{ii} = \lambda_i$ such that $|\lambda_i| \leq 1$ and R_k are orthogonal 2×2 matrices. Hence, R_k may be parameterized as follows:

$$R_k = \begin{pmatrix} \cos\phi_k & -\sin\phi_k \\ \sin\phi_k & \cos\phi_k \end{pmatrix} \ . \tag{A.1}$$

Now, the corresponding bistochastic 3×3 matrix $a = ||a_{ij}||$ reads as follows:

$$a = P_0 + P_1 + P_2 \ , \tag{A.2}$$

where

$$P_0 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \ , \tag{A.3}$$

$$P_1 = \frac{\lambda_+}{12} \begin{pmatrix} 2\cos\phi_+ & -\cos\phi_+ - \sqrt{3}\sin\phi_+ & -\cos\phi_+ + \sqrt{3}\sin\phi_+ \\ -\cos\phi_+ + \sqrt{3}\sin\phi_+ & 2\cos\phi_+ & -\cos\phi_+ - \sqrt{3}\sin\phi_+ \\ -\cos\phi_+ - \sqrt{3}\sin\phi_+ & -\cos\phi_+ + \sqrt{3}\sin\phi_+ & 2\cos\phi_+ \end{pmatrix} \ , \tag{A.4}$$

and

$$P_2 = \frac{\lambda_-}{12} \begin{pmatrix} \cos\phi_- + \sqrt{3}\sin\phi_- & -2\cos\phi_- & \cos\phi_- - \sqrt{3}\sin\phi_- \\ -2\cos\phi_- & \cos\phi_- - \sqrt{3}\sin\phi_- & \cos\phi_- + \sqrt{3}\sin\phi_- \\ \cos\phi_- - \sqrt{3}\sin\phi_- & \cos\phi_- + \sqrt{3}\sin\phi_- & -2\cos\phi_- \end{pmatrix} \ , \tag{A.5}$$

with $\phi_\pm = \phi_1 \pm \phi_2$ and $\lambda_\pm = \lambda_1 \pm \lambda_2$. Note that for $\lambda_1 = \lambda_2 = 1$ and $\phi_1 = 0$, $\phi_2 = -\pi/3$ one recovers (27).

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