On circulant states with positive partial transpose

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We construct a large class of quantum $d \otimes d$ states which are positive under partial transposition (so called PPT states). The construction is based on certain direct sum decomposition of the total Hilbert space displaying characteristic circular structure — that is way we call them circulant states. It turns out that partial transposition maps any such decomposition into another one and hence both original density matrix and its partially transposed partner share similar cyclic properties. This class contains many well known examples of PPT states from the literature and gives rise to a huge family of completely new states.

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I. INTRODUCTION

The interest on quantum entanglement has dramatically increased during the last two decades due to the emerging field of quantum information theory [1]. It turns out that quantum entanglement may be used as basic resources in quantum information processing and communication. The prominent examples are quantum cryptography, quantum teleportation, quantum error correction codes and quantum computation.

It is well known that it is extremely hard to check whether a given density matrix describing a quantum state of the composite system is separable or entangled. There are several operational criteria which enable one to detect quantum entanglement (see e.g. [2] for the recent review). The most famous Peres-Horodecki criterion [3, 4] is based on the partial transposition: if a state ρ is separable then its partial transposition $(\mathbb{1} \otimes \tau)\rho$ is positive. States which are positive under partial transposition are called PPT states. Clearly each separable state is necessarily PPT but the converse is not true. It was shown by Horodecki et al. [5] that PPT condition is both necessary and sufficient for separability for $2 \otimes 2$ and $2 \otimes 3$ systems.

Now, since all separable states belong to a set of PPT states, the structure of this set is of primary importance in quantum information theory. Unfortunately, this structure is still unknown, that is, one may check whether a given state is PPT but we do not know how to construct a general quantum state with PPT property. There are only several examples of PPT states which do not show any systematic methods of constructing them (with one exception, i.e. a class of PPT entangled states which is based on a concept of unextendible product bases [6] (see also [7]). Other examples of PPT entangled states were constructed in [4, 8–15] and the extreme points of the set of PPT states were recently analyzed in [19]. PPT states play also a crucial role in mathematical theory of positive maps and, as is well know, these maps are very important in the study of quantum entanglement. The mathematical structure of quantum entangled states with positive partial transposition were studied in [16–18].

Recently in [20] we proposed a class of PPT states in $d \otimes d$ which are invariant under the maximal commutative subgroup of $U(d)$, i.e. d-dimensional torus $U(1) \times \ldots \times U(1)$. In the present paper we propose another class which defines considerable generalization of [20]. The construction of this new class is based on certain decomposition of the total Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$ into direct sum of d-dimensional subspaces. This decomposition is controlled by some cyclic property, that is, knowing one subspace, say Σ_0, the remaining subspaces $\Sigma_1, \ldots, \Sigma_{d-1}$ are uniquely determined by applying a cyclic shift to elements from Σ_0. Now, we call a density matrix ρ a *circulant state* if ρ is a convex combination of density matrices supported on Σ_α. The crucial observation is that a partial transposition of the circulant state has again a circular structure corresponding to another direct sum decomposition $\widetilde{\Sigma}_0 \oplus \ldots \oplus \widetilde{\Sigma}_{d-1}$.

The paper is organized as follows: for pedagogical reason we first illustrate our general method for $d = 2$ in Section II and for $d = 3$ in Section III. Interestingly, there is only one circular decomposition for $d = 2$ and exactly two different decompositions for $d = 3$. In general case presented in Section IV there are $(d-1)!$ decompositions labeled by permutations from the symmetric group S_{d-1}. Section V presents several known examples of PPT states that do belong to our class. Final conclusions are collected in the last section.

II. TWO QUBITS

A. An instructive example

Consider a density matrix living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ which has the following form:

$$\rho = \left(\begin{array}{cc|cc} a_{00} & \cdot & \cdot & a_{01} \\ \cdot & b_{00} & b_{01} & \cdot \\ \hline \cdot & b_{10} & b_{11} & \cdot \\ a_{10} & \cdot & \cdot & a_{11} \end{array} \right) . \tag{1}$$

In order to have more transparent pictures we replaced all vanishing matrix elements by dots (we use this convention through out this paper). It is clear that (1) defines a positive operator iff the following 2×2 matrices

$$a = \left(\begin{array}{cc} a_{00} & a_{01} \\ a_{10} & a_{11} \end{array} \right) , \quad b = \left(\begin{array}{cc} b_{00} & b_{11} \\ b_{10} & b_{11} \end{array} \right) , \tag{2}$$

are positive. Normalization adds additional condition

$$\mathrm{Tr}\, a + \mathrm{Tr}\, b = 1 \ .$$

Now, the crucial observation is that partially transposed matrix $\rho^\tau = (\mathbb{1} \otimes \tau)\rho$ belongs to the same class as original ρ

$$\rho^\tau = \left(\begin{array}{cc|cc} \widetilde{a}_{00} & \cdot & \cdot & \widetilde{a}_{01} \\ \cdot & \widetilde{b}_{00} & \widetilde{b}_{01} & \cdot \\ \hline \cdot & \widetilde{b}_{10} & \widetilde{b}_{11} & \cdot \\ \widetilde{a}_{10} & \cdot & \cdot & \widetilde{a}_{11} \end{array} \right) , \tag{3}$$

where the matrices $\widetilde{a} = [\widetilde{a}_{ij}]$ and $\widetilde{b} = [\widetilde{b}_{ij}]$ read as follows

$$\widetilde{a} = \left(\begin{array}{cc} a_{00} & b_{01} \\ b_{10} & a_{11} \end{array} \right) , \quad \widetilde{b} = \left(\begin{array}{cc} b_{00} & a_{01} \\ a_{10} & b_{11} \end{array} \right) . \tag{4}$$

Hence, ρ defined in (1) is PPT iff

$$\widetilde{a} \geq 0 \quad \text{and} \quad \widetilde{b} \geq 0 \ .$$

The above conditions together with $a \geq 0$ and $b \geq 0$ may be equivalently rewritten as follows

$$\begin{aligned} a_{00}a_{11} &\geq |a_{01}|^2 \ , \\ a_{00}a_{11} &\geq |b_{01}|^2 \ , \end{aligned}$$

and

$$\begin{aligned} b_{00}b_{11} &\geq |a_{01}|^2 \ , \\ b_{00}b_{00} &\geq |b_{01}|^2 \ , \end{aligned}$$

which presents the full characterization of PPT states within a class (1). We stress that for $b_{01} = b_{10} = 0$ the above class reduces to the family of PPT states considered in [20].

B. Cyclic structure

In order to generalize the above example to higher dimensional cases let us observe that there is an interesting property of cyclicity which governs the structure of (1). For this reason we call (1) circulant state. Note that ρ may be written as a direct sum

$$\rho = \rho_0 + \rho_1 \ , \tag{5}$$

where ρ_0 and ρ_1 are supported on two orthogonal subspaces

$$\begin{aligned} \Sigma_0 &= \mathrm{span}\, \{e_0 \otimes e_0 \, , e_1 \otimes e_1\} \ , \\ \Sigma_1 &= \mathrm{span}\, \{e_0 \otimes e_1 \, , e_1 \otimes e_0\} \ , \end{aligned} \tag{6}$$

where $\{e_0, e_1\}$ is a computational base in $\mathbb{C}^2$, and clearly

$$\Sigma_0 \oplus \Sigma_1 = \mathbb{C}^2 \otimes \mathbb{C}^2 \ .$$

One has

$$\rho_0 = \sum_{i,j=0}^{1} a_{ij}\, e_{ij} \otimes e_{ij} \ , \tag{7}$$

$$\rho_1 = \sum_{i,j=0}^{1} b_{ij}\, e_{ij} \otimes e_{i+1,j+1} \ , \tag{8}$$

where

$$e_{ij} = |e_i\rangle\langle e_j| \ , \tag{9}$$

and one adds mod 2. Now, let us introduce the shift operator $S : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined by

$$S\, e_i = e_{i+1} \ , \qquad (\mathrm{mod}\ 2) \ . \tag{10}$$

It is clear that matrix elements S_{ij} define the following circulant matrix [21]

$$S = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \ . \tag{11}$$

One finds that

$$\Sigma_1 = (\mathbb{1} \otimes S)\, \Sigma_0 \ . \tag{12}$$

Moreover, introducing two orthogonal projectors P_0 and $P_1 = (\mathbb{1} \otimes S) P_0 (\mathbb{1} \otimes S)^*$ projecting onto Σ_0 and Σ_1, respectively

$$P_0 \ = \ \sum_{i=0}^{1} e_{ii} \otimes e_{ii} \ , \tag{13}$$

$$P_1 \ = \ \sum_{i=0}^{1} e_{ii} \otimes e_{i+1,i+1} \ , \tag{14}$$

one finds

$$\rho_i = P_i\, \rho\, P_i \ , \tag{15}$$

and hence

$$\rho = P_0\, \rho\, P_0 + P_1\, \rho\, P_1 \ . \tag{16}$$

Now, it turns out that (1) may be nicely rewritten in terms of S. Introducing the following diagonal matrices

$$x_{ij} = \begin{pmatrix} a_{ij} & \cdot \\ \cdot & b_{ij} \end{pmatrix} \ , \tag{17}$$

one may rewrite (1) in the following form

$$\rho = \left(\begin{array}{c|c} S^0\, x_{00}\, S^0 & S^0\, x_{01}\, S^1 \\ \hline S^1\, x_{10}\, S^0 & S^1\, x_{11}\, S^1 \end{array} \right) \ . \tag{18}$$

It is therefore clear that partially transposed matrix ρ^τ also possesses a cyclic structure

$$\rho^\tau = \left(\begin{array}{c|c} S^0\, x_{00}\, S^0 & S^1\, x_{01}\, S^0 \\ \hline S^0\, x_{10}\, S^1 & S^1\, x_{11}\, S^1 \end{array} \right) \ , \tag{19}$$

and may be decomposed as the following direct sum

$$\rho^\tau = \widetilde{\rho}_0 + \widetilde{\rho}_1 \ , \tag{20}$$

with

$$\widetilde{\rho}_0 \ = \ \sum_{i,j=0}^{1} \widetilde{a}_{ij}\, e_{ij} \otimes e_{ij} \ , \tag{21}$$

$$\widetilde{\rho}_1 \ = \ \sum_{i,j=0}^{1} \widetilde{b}_{ij}\, e_{ij} \otimes e_{i+1,j+1} \ . \tag{22}$$

In analogy with (23) and (24) one has

$$\widetilde{\rho}_i = P_i\, \rho^\tau\, P_i \ , \tag{23}$$

and

$$\rho^\tau = P_0\, \rho^\tau\, P_0 + P_1\, \rho^\tau\, P_1 \ . \tag{24}$$

Note, that partial transposition $\rho \longrightarrow \rho^\tau$ reduces to the following operations on the level on 2×2 matrices:

$$a \ \longrightarrow \ \widetilde{a} \quad \text{and} \quad b \longrightarrow \ \widetilde{b} \ .$$

Again these operations are fully controlled by the circulant matrix S

$$\widetilde{a} = a \circ \mathbb{I} + b \circ S \ , \tag{25}$$

and similarly

$$\widetilde{b} = b \circ \mathbb{I} + a \circ S \ , \tag{26}$$

where $x \circ y$ denotes the Hadamard product of two matrices x and y [22].

III. TWO QUTRITS

A similar construction may be performed in $\mathbb{C}^3 \otimes \mathbb{C}^3$. The basic idea is to decompose the total Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$ into a direct sum of three orthogonal subspaces Σ_i related by a certain cyclic property. In analogy to (10) let us define a shift operator $S : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ via

$$S\, e_i = e_{i+1} \ , \qquad (\text{mod } 3) \ . \tag{27}$$

It is clear that matrix elements S_{ij} define the following 3×3 circulant matrix

$$S = \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \ . \tag{28}$$

Now, let us define three orthogonal 3-dimensional subspaces in $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\Sigma_0 = \text{span}\, \{ e_0 \otimes e_0 \, , e_1 \otimes e_1 \, , e_2 \otimes e_2 \} \ , \tag{29}$$

and

$$\Sigma_1 = (\mathbb{1} \otimes S) \Sigma_0 \ , \qquad \Sigma_2 = (\mathbb{1} \otimes S^2) \Sigma_0 \ . \tag{30}$$

One easily finds

$$\begin{aligned} \Sigma_1 \ &= \ \text{span}\, \{ e_0 \otimes e_1 \, , e_1 \otimes e_2 \, , e_2 \otimes e_0 \} \ , \\ \Sigma_2 \ &= \ \text{span}\, \{ e_0 \otimes e_2 \, , e_1 \otimes e_0 \, , e_2 \otimes e_1 \} \ , \end{aligned} \tag{31}$$

together with

$$\Sigma_0 \oplus \Sigma_1 \oplus \Sigma_2 \ = \ \mathbb{C}^3 \otimes \mathbb{C}^3 \ .$$

The construction of a circulant state in $\mathbb{C}^3 \otimes \mathbb{C}^3$ goes as follows: define three positive operators ρ_i which are supported on Σ_i $(i = 0, 1, 2)$:

$$\rho_0 \ = \ \sum_{i,j=0}^{2} a_{ij}\, e_{ij} \otimes e_{ij} \ , \tag{32}$$

$$\rho_1 \ = \ \sum_{i,j=0}^{2} b_{ij}\, e_{ij} \otimes S\, e_{ij}\, S^* \ ,$$

$$\ = \ \sum_{i,j=0}^{2} b_{ij}\, e_{ij} \otimes e_{i+1,j+1} \tag{33}$$

$$\rho_2 \ = \ \sum_{i,j=0}^{2} c_{ij}\, e_{ij} \otimes S^2\, e_{ij}\, S^{*2} \ ,$$

$$\ = \ \sum_{i,j=0}^{2} c_{ij}\, e_{ij} \otimes e_{i+2,j+2} \ , \tag{34}$$

where a_{ij}, b_{ij} and c_{ij} give rise to the following 3×3 matrices:

$$a \ = \ \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \ , \quad b \ = \ \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} \ ,$$

$$c \ = \ \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix} \ .$$

Positivity of ρ_i is guaranteed by positivity of a, b and c. Finally, define a circulant $3 \otimes 3$ state by

$$\rho = \rho_0 + \rho_1 + \rho_2 \ . \tag{35}$$

It is clear that

$$\rho_\alpha = P_\alpha\, \rho\, P_\alpha \ , \qquad \alpha = 0, 1, 2 \ ,$$

where P_α denotes orthogonal projector onto Σ_α:

$$P_0 = \sum_{i=0}^{2} e_{ii} \otimes e_{ii} \ ,$$

and

$$P_\alpha = (\mathbb{1} \otimes S^\alpha)\, P_0\, (\mathbb{1} \otimes S^*)^\alpha \ ,$$

with

$$P_0 + P_1 + P_2 = \mathbb{I} \otimes \mathbb{I} \ .$$

Using definitions of ρ_i one easily finds

$$\rho = \left(\begin{array}{ccc|ccc|ccc} a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\ \cdot & b_{00} & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot \\ \cdot & \cdot & c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{02} & \cdot \\ \hline \cdot & \cdot & c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{12} & \cdot \\ a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\ \cdot & b_{10} & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\ \hline \cdot & b_{20} & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\ \cdot & \cdot & c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{22} & \cdot \\ a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22} \end{array} \right) . \tag{36}$$

Normalization of ρ implies

$$\mathrm{Tr}\Big(a + b + c\Big) = 1 \ .$$

It turns out that (36) may be nicely rewritten in terms of S. Introducing the following diagonal matrices

$$x_{ij} = \begin{pmatrix} a_{ij} & \cdot & \cdot \\ \cdot & b_{ij} & \cdot \\ \cdot & \cdot & c_{ij} \end{pmatrix} \ , \tag{37}$$

one may rewrite (36) in the following block form

$$\rho = \left(\begin{array}{c|c|c} S^0\, x_{00}\, S^{*0} & S^0\, x_{01}\, S^{*1} & S^0\, x_{02}\, S^{*2} \\ \hline S^1\, x_{10}\, S^{*0} & S^1\, x_{11}\, S^{*1} & S^1\, x_{12}\, S^{*2} \\ \hline S^2\, x_{20}\, S^{*0} & S^2\, x_{21}\, S^{*1} & S^2\, x_{22}\, S^{*2} \end{array} \right) \ . \tag{38}$$

Partially transposed ρ^τ has the following form

$$\rho^\tau = \left(\begin{array}{ccc|ccc|ccc} \widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \widetilde{a}_{02} & \cdot \\ \cdot & \widetilde{b}_{00} & \cdot & \widetilde{b}_{01} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{02} \\ \cdot & \cdot & \widetilde{c}_{00} & \cdot & \widetilde{c}_{01} & \cdot & \widetilde{c}_{02} & \cdot & \cdot \\ \hline \cdot & \widetilde{b}_{10} & \cdot & \widetilde{b}_{11} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{12} \\ \cdot & \cdot & \widetilde{c}_{10} & \cdot & \widetilde{c}_{11} & \cdot & \widetilde{c}_{12} & \cdot & \cdot \\ \widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \widetilde{a}_{12} & \cdot \\ \hline \cdot & \cdot & \widetilde{c}_{20} & \cdot & \widetilde{c}_{21} & \cdot & \widetilde{c}_{22} & \cdot & \cdot \\ \widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \widetilde{a}_{22} & \cdot \\ \cdot & \widetilde{b}_{20} & \cdot & \widetilde{b}_{21} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{22} \end{array} \right) \ , \tag{39}$$

where the matrices $\widetilde{a} = [\widetilde{a}_{ij}]$, $\widetilde{b} = [\widetilde{b}_{ij}]$ and $\widetilde{c} = [\widetilde{c}_{ij}]$ read as follows

$$\widetilde{a} = \begin{pmatrix} a_{00} & c_{01} & b_{02} \\ c_{10} & b_{11} & a_{12} \\ b_{20} & a_{21} & c_{22} \end{pmatrix} , \quad \widetilde{b} = \begin{pmatrix} b_{00} & a_{01} & c_{02} \\ a_{10} & c_{11} & b_{12} \\ c_{20} & b_{21} & a_{22} \end{pmatrix} ,$$

$$\widetilde{c} = \begin{pmatrix} c_{00} & b_{01} & a_{02} \\ b_{10} & a_{11} & c_{12} \\ a_{20} & c_{21} & b_{22} \end{pmatrix} .$$

Note, that

$$\rho^\tau = \widetilde{\rho}_0 + \widetilde{\rho}_1 + \widetilde{\rho}_2 , \qquad (40)$$

where $\widetilde{\rho}_k$ are supported on three orthogonal subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^3$:

$$\begin{aligned} \widetilde{\Sigma}_0 &= \mathrm{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_2\,, e_2 \otimes e_1\} , \\ \widetilde{\Sigma}_1 &= \mathrm{span}\,\{e_0 \otimes e_1\,, e_1 \otimes e_0\,, e_2 \otimes e_2\} , \qquad (41) \\ \widetilde{\Sigma}_2 &= \mathrm{span}\,\{e_0 \otimes e_2\,, e_1 \otimes e_1\,, e_2 \otimes e_0\} . \end{aligned}$$

One has therefore

Theorem 1 *A circulant $3 \otimes 3$ state ρ is* PPT *iff the matrices $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ are positive.*

Note, that

$$\widetilde{\Sigma}_0 = (\mathbb{1} \otimes \widetilde{\Pi})\, \Sigma_0 ,$$

where $\widetilde{\Pi}$ is the following permutation matrix

$$\widetilde{\Pi} = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix} . \qquad (42)$$

Again, one has a cyclic structure

$$\widetilde{\Sigma}_i = (\mathbb{1} \otimes S) \widetilde{\Sigma}_0 . \qquad (43)$$

Moreover, it is clear that

$$\widetilde{\rho}_\alpha = \widetilde{P}_\alpha\, \rho^\tau\, \widetilde{P}_\alpha , \qquad \alpha = 0, 1, 2 ,$$

where $\widetilde{P}_\alpha$ denotes orthogonal projector onto $\widetilde{\Sigma}_\alpha$:

$$\widetilde{P}_0 = (\mathbb{1} \otimes \Pi)\, P_0\, (\mathbb{1} \otimes \Pi^*) ,$$

and

$$\widetilde{P}_\alpha = (\mathbb{1} \otimes S^\alpha)\, \widetilde{P}_0\, (\mathbb{1} \otimes S^*)^\alpha ,$$

with

$$\widetilde{P}_0 + \widetilde{P}_1 + \widetilde{P}_2 = \mathbb{I} \otimes \mathbb{I} .$$

It is therefore clear that ρ^τ is again a circulant operator and its circular structure is governed by

$$\rho^\tau = \left(\begin{array}{c|c|c} S^0\, \widetilde{x}_{00}\, S^{*0} & S^0\, \widetilde{x}_{01}\, S^{*2} & S^0\, \widetilde{x}_{02}\, S^{*1} \\ \hline S^2\, \widetilde{x}_{10}\, S^{*0} & S^2\, \widetilde{x}_{11}\, S^{*2} & S^2\, \widetilde{x}_{12}\, S^{*1} \\ \hline S^1\, \widetilde{x}_{20}\, S^{*0} & S^1\, \widetilde{x}_{21}\, S^{*2} & S^1\, \widetilde{x}_{22}\, S^{*1} \end{array} \right) , \qquad (44)$$

where

$$\widetilde{x}_{ij} = \begin{pmatrix} \widetilde{a}_{ij} & \cdot & \cdot \\ \cdot & \widetilde{b}_{ij} & \cdot \\ \cdot & \cdot & \widetilde{c}_{ij} \end{pmatrix} . \qquad (45)$$

Interestingly, matrices $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ may be nicely defined in terms of $\widetilde{\Pi}$ and S. It is not difficult to show that

$$\begin{aligned} \widetilde{a} &= a \circ \widetilde{\Pi} + b \circ (\widetilde{\Pi}\, S) + c \circ (\widetilde{\Pi}\, S^2) , \\ \widetilde{b} &= b \circ \widetilde{\Pi} + c \circ (\widetilde{\Pi}\, S) + a \circ (\widetilde{\Pi}\, S^2) , \qquad (46) \\ \widetilde{c} &= c \circ \widetilde{\Pi} + a \circ (\widetilde{\Pi}\, S) + b \circ (\widetilde{\Pi}\, S^2) , \end{aligned}$$

where "$\circ$" denotes the Hadamard product.

Let us stress that this class in a significant way enlarges the class considered in [20]. One reconstruct [20] by taking as b and c diagonal matrices:

$$b = \begin{pmatrix} b_{00} & \cdot & \cdot \\ \cdot & b_{11} & \cdot \\ \cdot & \cdot & b_{22} \end{pmatrix} , \quad c = \begin{pmatrix} c_{00} & \cdot & \cdot \\ \cdot & c_{11} & \cdot \\ \cdot & \cdot & c_{22} \end{pmatrix} .$$

Then one finds for partially transposed matrix:

$$\widetilde{a} = \begin{pmatrix} a_{00} & \cdot & \cdot \\ \cdot & b_{11} & a_{12} \\ \cdot & a_{21} & c_{22} \end{pmatrix} , \quad \widetilde{b} = \begin{pmatrix} b_{00} & a_{01} & \cdot \\ a_{10} & c_{11} & \cdot \\ \cdot & \cdot & a_{22} \end{pmatrix} ,$$

and

$$\widetilde{c} = \begin{pmatrix} c_{00} & \cdot & a_{02} \\ \cdot & a_{11} & \cdot \\ a_{20} & \cdot & b_{22} \end{pmatrix} .$$

IV. GENERAL $d \otimes d$ CASE

Now we are ready to construct circular states in $d \otimes d$. The basic idea is to decompose the total Hilbert space

$\mathbb{C}^d \otimes \mathbb{C}^d$ into a direct sum of d orthogonal d-dimensional subspaces related by a certain cyclic property. It turns out that there are $(d-1)!$ different cyclic decompositions and it is therefore clear that they may be labeled by permutations from the symmetric group S_{d-1}. For $d = 2$ one has only one decomposition

$$\begin{aligned} \Sigma_0 &= \mathrm{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_1\}\ , \\ \Sigma_1 &= \mathrm{span}\,\{e_0 \otimes e_1\,, e_1 \otimes e_0\}\ , \end{aligned} \tag{47}$$

whereas for $d = 3$ we have found 2 different cyclic decompositions

$$\begin{aligned} \Sigma_0 &= \mathrm{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_1\,, e_2 \otimes e_2\}\ , \\ \Sigma_1 &= \mathrm{span}\,\{e_0 \otimes e_1\,, e_1 \otimes e_2\,, e_2 \otimes e_0\}\ , \\ \Sigma_2 &= \mathrm{span}\,\{e_0 \otimes e_2\,, e_1 \otimes e_0\,, e_2 \otimes e_1\}\ , \end{aligned} \tag{48}$$

and

$$\begin{aligned} \widetilde{\Sigma}_0 &= \mathrm{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_2\,, e_2 \otimes e_1\}\ , \\ \widetilde{\Sigma}_1 &= \mathrm{span}\,\{e_0 \otimes e_1\,, e_1 \otimes e_0\,, e_2 \otimes e_2\}\ , \\ \widetilde{\Sigma}_2 &= \mathrm{span}\,\{e_0 \otimes e_2\,, e_1 \otimes e_1\,, e_2 \otimes e_0\}\ . \end{aligned} \tag{49}$$

Let us introduce a basic d-dimensional subspace

$$\Sigma_0 = \mathrm{span}\,\{e_0 \otimes e_0\,, e_1 \otimes e_1\,, \ldots\,, e_{d-1} \otimes e_{d-1}\}\ . \tag{50}$$

Now, for any permutation $\pi \in S$ let us define Σ_0^π which is spanned by

$$e_0 \otimes e_{\pi(0)}\,, e_1 \otimes e_{\pi(1)}\,, \ldots\,, e_{d-1} \otimes e_{\pi(d-1)}\ . \tag{51}$$

Note, that introducing a permutation matrix Π corresponding to π one has

$$\Sigma_0^\pi = (\mathbb{1} \otimes \Pi)\Sigma_0\ . \tag{52}$$

Actually, it is enough to consider only a subset of permutations such that $\pi(0) = 0$, it means that vector $e_0 \otimes e_0$ always belongs to the subspace number '0' in each decomposition. Finally, the remaining $(d-1)$ subspaces in the decomposition labeled by π are defined via

$$\begin{aligned} \Sigma_\alpha^\pi &= (\mathbb{1} \otimes S^\alpha)\Sigma_0^\pi\ , \\ &= (\mathbb{1} \otimes S^\alpha\,\Pi)\Sigma_0\ , \end{aligned} \tag{53}$$

where S is a circulant matrix corresponding to shift in $\mathbb{C}^d$:

$$S = \begin{pmatrix} \cdot & \cdot & \ldots & \cdot & 1 \\ 1 & \cdot & \ldots & \cdot & \cdot \\ \cdot & 1 & \ldots & \cdot & \cdot \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \ldots & 1 & \cdot \end{pmatrix}\ . \tag{54}$$

One easily check

$$\Sigma_0^\pi \oplus \Sigma_1^\pi \oplus \ldots \oplus \Sigma_{d-1}^\pi = \mathbb{C}^d \otimes \mathbb{C}^d\ .$$

To construct a circulant state corresponding to this decomposition let us introduce d positive $d \times d$ matrices $a^{(\alpha)} = [a_{ij}^{(\alpha)}]$; $\alpha = 0, 1, \ldots, d-1$. Now, define d positive operators ρ_α^π supported on Σ_α^π via

$$\begin{aligned} \rho_\alpha^\pi &= \sum_{i,j=0}^{d-1} a_{ij}^{(\alpha)}\, e_{ij} \otimes S^\alpha\, e_{\pi(i),\pi(j)}\, S^{*\alpha} \\ &= \sum_{i,j=0}^{d-1} a_{ij}^{(\alpha)}\, e_{ij} \otimes e_{\pi(i)+\alpha,\pi(j)+\alpha}\ . \end{aligned} \tag{55}$$

Finally,

$$\rho_\pi = \rho_0^\pi + \rho_1^\pi + \ldots + \rho_{d-1}^\pi\ , \tag{56}$$

defines circulant state (corresponding to π). Normalization of ρ_π is equivalent to the following condition for matrices $a^{(\alpha)}$

$$\mathrm{Tr}\,\left(a^{(0)} + a^{(1)} + \ldots + a^{(d-1)}\right) = 1\ .$$

Interestingly, ρ_π has the following transparent block form: introduce a set of d^2 diagonal matrices

$$x_{ij} = \begin{pmatrix} a_{ij}^{(0)} & \cdot & \ldots & \cdot \\ \cdot & a_{ij}^{(1)} & \ldots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \ldots & a_{ij}^{(d-1)} \end{pmatrix}\ , \tag{57}$$

then one finds

$$
\rho_\pi = \left(\begin{array}{c|c|c|c}
S^{\pi(0)} x_{00} S^{\pi(0)*} & S^{\pi(0)} x_{01} S^{\pi(1)*} & \dots & S^{\pi(0)} x_{0,d-1} S^{\pi(d-1)*} \\ \hline
S^{\pi(1)} x_{10} S^{\pi(0)*} & S^{\pi(1)} x_{11} S^{\pi(1)*} & \dots & S^{\pi(1)} x_{1,d-1} S^{\pi(d-1)*} \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
S^{\pi(d-1)} x_{d-1,0} S^{\pi(0)*} & S^{\pi(d-1)} x_{d-1,1} S^{\pi(1)*} & \dots & S^{\pi(d-1)} x_{d-1,d-1} S^{\pi(d-1)*}
\end{array} \right) . \qquad (58)
$$

Having defined a circulant state ρ_π let us look for a partially transposed matrix ρ_π^τ. Now comes the crucial observation

Theorem 2 *If ρ_π is a circulant state corresponding to permutation π such that $\pi(0) = 0$, then its partial transposition ρ_π^τ is also circulant with respect to another decomposition corresponding to permutation $\widetilde{\pi}$ such that*

$$
\pi(i) + \widetilde{\pi}(i) = d \ , \qquad (59)
$$

for $i = 1, 2, \dots, d-1$, and $\widetilde{\pi}(0) = 0$.

Note, that for $d = 2$ there is only one (trivial) permutation $(\pi(0) = 0, \pi(1) = 1)$ and hence $\widetilde{\pi} = \pi$, that is both π and $\widetilde{\pi}$ define the same decomposition (47). For $d = 3$ one has two different permutations in S_2: the trivial one $(\pi(0) = 0, \pi(1) = 1, \pi(2) = 2)$ which corresponds to (48) and "true" permutation $(\widetilde{\pi}(0) = 0, \widetilde{\pi}(1) = 2, \widetilde{\pi}(2) = 1)$ which corresponds to (49).

Hence, ρ_π^τ may be decomposed as follows

$$
\rho_\pi^\tau = \widetilde{\rho}_0^\pi + \widetilde{\rho}_1^\pi + \dots + \widetilde{\rho}_{d-1}^\pi \ , \qquad (60)
$$

and $\widetilde{\rho}_\alpha^\pi$ are defined by

$$
\begin{aligned}
\widetilde{\rho}_\alpha^\pi &= \sum_{i,j=0}^{d-1} \widetilde{a}_{ij}^{(\alpha)} \, e_{ij} \otimes S^\alpha \, e_{\widetilde{\pi}(i),\widetilde{\pi}(j)} \, S^{*\alpha} \\
&= \sum_{i,j=0}^{d-1} \widetilde{a}_{ij}^{(\alpha)} \, e_{ij} \otimes e_{\widetilde{\pi}(i)+\alpha,\widetilde{\pi}(j)+\alpha} \ , \qquad (61)
\end{aligned}
$$

where again we trivially extended $\widetilde{\pi}$ from S_{d-1} to S_d by $\widetilde{\pi}(0) \equiv 0$.

In analogy to (58) one finds the following block form of ρ_π^τ:

$$
\rho_\pi^\tau = \left(\begin{array}{c|c|c|c}
S^{\widetilde{\pi}(0)} x_{00} S^{\widetilde{\pi}(0)*} & S^{\widetilde{\pi}(0)} x_{01} S^{\widetilde{\pi}(1)*} & \dots & S^{\widetilde{\pi}(0)} x_{0,d-1} S_\pi^{\widetilde{\pi}(d-1)*} \\ \hline
S^{\widetilde{\pi}(1)} x_{10} S^{\widetilde{\pi}(0)*} & S^{\widetilde{\pi}(1)} x_{11} S^{\widetilde{\pi}(1)*} & \dots & S^{\widetilde{\pi}(1)} x_{1,d-1} S^{\widetilde{\pi}(d-1)*} \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
S^{\widetilde{\pi}(d-1)} x_{d-1,0} S^{\widetilde{\pi}(0)*} & S^{\widetilde{\pi}(d-1)} x_{d-1,1} S^{\widetilde{\pi}(1)*} & \dots & S^{\widetilde{\pi}(d-1)} x_{d-1,d-1} S^{\widetilde{\pi}(d-1)*}
\end{array} \right) , \qquad (62)
$$

where

$$
\widetilde{x}_{ij} = \begin{pmatrix}
\widetilde{a}_{ij}^{(0)} & \cdot & \dots & \cdot \\
\cdot & \widetilde{a}_{ij}^{(1)} & \dots & \cdot \\
\vdots & \vdots & \ddots & \vdots \\
\cdot & \cdot & \dots & \widetilde{a}_{ij}^{(d-1)}
\end{pmatrix} . \qquad (63)
$$

Hence a partial transposition applied to a circulant state ρ_π reduces to

1. introducing "complementary" permutation $\widetilde{\pi}$, and
2. defining a new set of $d \times d$ matrices $\widetilde{a}^{(\alpha)} = [\widetilde{a}_{ij}^{(\alpha)}]$.

Now, "complementary" permutation $\widetilde{\pi}$ is fully characterized by (59). Finally, one finds the following intricate formula for $\widetilde{a}^{(\alpha)}$:

$$
\widetilde{a}^{(\alpha)} = \sum_{\beta=0}^{d-1} a^{(\alpha+\beta)} \circ \left(\widetilde{\Pi} \, S_\pi^\beta \right) \ , \qquad (\mathrm{mod}\ d) \ , \quad (64)
$$

where "$\circ$" denotes the Hadamard product, and

$$
S_\pi = \Pi^* \, S \, \Pi \ . \qquad (65)
$$

Therefore, we arrive at our main result

Theorem 3 *A circulant state ρ_π is* PPT *iff the matrices $\widetilde{a}^{(\alpha)}$ defined in (64) are positive.*

V. EXAMPLES

A. PPT class from [20]

One reconstructs a class of PPT states from [20] taking the circular decomposition corresponding to trivial permutation with arbitrary (but positive) $a^{(0)}$ and positive diagonal $a^{(k)}$ $(k = 1, \ldots, d-1)$. Note, however, that there are new classes defined by the same matrices $a^{(\alpha)}$ but corresponding to different permutations, that is, apart from the state defined by

$$\rho = \sum_{i,j=0}^{d-1} a_{ij}^{(0)}\, e_{ij} \otimes e_{ij} + \sum_{k=1}^{d-1}\sum_{i=0}^{d-1} a_{ii}^{(k)}\, e_{ii} \otimes e_{i+k,i+k} \ ,$$

one has its π partner

$$\begin{aligned} \rho_\pi &= (\mathbb{1}\otimes \Pi)\rho(\mathbb{1}\otimes\Pi^*) \\ &= \sum_{i,j=0}^{d-1} a_{ij}^{(0)}\, e_{ij}\otimes e_{\pi(i),\pi(j)} \\ &+ \sum_{k=1}^{d-1}\sum_{i=0}^{d-1} a_{ii}^{(k)}\, e_{ii}\otimes e_{\pi(i)+k,\pi(i)+k} \ . \end{aligned}$$

It is, therefore, clear that all examples discussed in [20] (together with the corresponding "π–partners") belong to our new class.

B. π–Isotropic state

The standard isotropic state [23] in $d\otimes d$

$$\mathcal{I} \ = \frac{1-\lambda}{d^2}\,\mathbb{I}\otimes\mathbb{I} + \frac{\lambda}{d}\sum_{i,j=0}^{d-1} e_{ij}\otimes e_{ij} \ , \qquad (66)$$

corresponds to trivial permutation and it is defined by the following set of $d\times d$ positive matrices:

$$a_{ij}^{(0)} = \begin{cases} \lambda/d & , \ i\neq j \\ \lambda/d + (1-\lambda)/d^2 & , \ i = j \end{cases} \ ,$$

and diagonal

$$a_{ij}^{(k)} = \begin{cases} 0 & , \ i\neq j \\ \lambda/d + (1-\lambda)/d^2 & , \ i = j \end{cases} \ ,$$

for $k = 1, \ldots, d-1$. Again, for each permutation π we may define π–isotropic state

$$\begin{aligned} \mathcal{I} &= (\mathbb{1}\otimes\Pi)\mathcal{I}(\mathbb{1}\otimes\Pi^*) \\ &= \frac{1-\lambda}{d^2}\,\mathbb{I}\otimes\mathbb{I} + \frac{\lambda}{d}\sum_{i,j=0}^{d-1} e_{ij}\otimes e_{\pi(i),\pi(j)} \ , \end{aligned}$$

which is defined by the same set of matrices $a^{(\alpha)}$ but corresponds to π–decomposition.

C. π–Werner state

The celebrated Werner state [24] is defined by the following well known formula

$$\mathcal{W} = (1-p)\,Q^+ + p\,Q^- \ , \qquad (67)$$

where

$$Q^\pm = \frac{1}{d(d\pm 1)}\Big(\mathbb{I}\otimes\mathbb{I} \pm \mathbb{F}\Big) \ ,$$

and $\mathbb{F}$ denotes a flip operator defined by

$$\mathbb{F} = \sum_{i,j=0}^{d-1} e_{ij}\otimes e_{ji} \ .$$

It is clear that $\mathcal{W}$ belongs to a class of bipartite operators obtained from the class of isotropic states by applying a partial transposition. One easily finds

$$\widetilde{a}_{ij}^{(0)} = \begin{cases} x_- & , \ i\neq j \\ x_- + x_+ & , \ i = j \end{cases} \ ,$$

and

$$\widetilde{a}^{(k)} \ = \ x_+\,\mathbb{I} \ , \qquad k = 1, \ldots, d-1 \ .$$

where

$$x_\pm = \frac{1-p}{d^2+d} \pm \frac{p}{d^2-d} \ .$$

It is clear that for any permutation π one may define π–Werner state

$$\begin{aligned} \mathcal{W}_\pi &= (\mathbb{1}\otimes\Pi)\,\mathcal{W}\,(\mathbb{1}\otimes\Pi^*) \\ &= (1-p)\,Q_\pi^+ + p\,Q_\pi^- \ , \ , \end{aligned}$$

where

$$Q_\pi^\pm = \frac{1}{d(d\pm 1)}\Big(\mathbb{I}\otimes\mathbb{I} \pm \mathbb{F}_\pi\Big) \ ,$$

and $\mathbb{F}_\pi$ denotes a "π–flip operator" defined by

$$\mathbb{F}_\pi = (\mathbb{1}\otimes\Pi)\,\mathbb{F}\,(\mathbb{1}\otimes\Pi^*) = \sum_{i,j=0}^{d-1} e_{ij}\otimes e_{\pi(j),\pi(j)} \ .$$

D. Ha example in $4 \otimes 4$

Ha [16] constructed a $4 \otimes 4$ PPT state which was used to check that the seminal Robertson positive map $\Lambda : M_4(\mathbb{C}) \longrightarrow M_4(\mathbb{C})$ [25] is indecomposable. Ha's state belongs to our class labeled by a trivial permutation π_1 (see Appendix) with four positive matrices defined as follows:

$$a = \begin{pmatrix} 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} , \quad b = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} ,$$

$$c = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} , \quad d = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} ,$$

The partially transposed state defines a circulant operator corresponding to decomposition labeled by $\widetilde{\pi}_1$ and defined by:

$$\widetilde{a} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} , \quad \widetilde{b} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} ,$$

$$\widetilde{c} = \begin{pmatrix} 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} , \quad \widetilde{d} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} .$$

Evidently $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \geq 0$. Interestingly, as was shown by Ha [16] both ρ and ρ^τ are of Schmidt rank two (it proves that Robertson map is not only indecomposable but even atomic, i.e. it can not be written as a sum of 2-positive and 2-copositive maps). We stress that this example does not belong to the previous class defined in [20].

E. Fei et. al. bound entangled state in $4 \otimes 4$

Fei et. al. [27] constructed $4 \otimes 4$ bound entangled state which correspond to $\widetilde{\pi}_1$–decomposition (see Appendix). It is defined by the following set of $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$:

$$\begin{pmatrix} x_1 & \cdot & \cdot & \cdot \\ \cdot & x_5 & \cdot & -x_5 \\ \cdot & \cdot & x_1 & \cdot \\ \cdot & -x_5 & \cdot & x_5 \end{pmatrix} , \quad \begin{pmatrix} x_3 & -x_3 & \cdot & \cdot \\ -x_3 & x_3 & \cdot & \cdot \\ \cdot & \cdot & x_4 & -x_4 \\ \cdot & \cdot & -x_4 & x_4 \end{pmatrix} ,$$

$$\begin{pmatrix} x_2 & \cdot & -x_2 & \cdot \\ \cdot & x_1 & \cdot & \cdot \\ -x_2 & \cdot & x_2 & \cdot \\ \cdot & \cdot & \cdot & x_1 \end{pmatrix} , \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} .$$

Evidently $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \geq 0$ for $x_i \geq 0$. Now, partially transposed state is circular with respect π_1–decomposition (see Appendix) and it is defined by the following set of 4×4 matrices a, b, c, d:

$$\begin{pmatrix} x_1 & -x_3 & -x_2 & \cdot \\ -x_3 & x_1 & \cdot & -x_5 \\ -x_2 & \cdot & x_1 & -x_4 \\ \cdot & -x_5 & -x_4 & x_1 \end{pmatrix} , \quad \begin{pmatrix} x_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x_4 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} ,$$

$$\begin{pmatrix} x_2 & \cdot & \cdot & \cdot \\ \cdot & x_5 & \cdot & \cdot \\ \cdot & \cdot & x_2 & \cdot \\ \cdot & \cdot & \cdot & x_5 \end{pmatrix} , \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_4 \end{pmatrix} .$$

It is clear that in general a is not a positive matrix. However, for $x_1 = (1-\varepsilon)/4$ and $x_2 = x_3 = x_4 = x_5 = \varepsilon/8$ it has three different eigenvalues

$$1/4 \ , \quad (1-2\varepsilon)/4 \ , \quad (1-\varepsilon)/4 \ ,$$

and hence ρ is PPT for $0 \leq \varepsilon \leq 1/2$. It was shown [27] that ρ being PPT is entangled.

VI. CONCLUSIONS

We have constructed a large class of PPT states in $d \otimes d$ which correspond to circular decompositions of $\mathbb{C}^d \otimes \mathbb{C}^d$ into direct sums of d-dimensional subspaces. This class significantly enlarges the previous class defined in [20]. It contains several known examples from the literature and produces a highly nontrivial family of new states.

There are many open problems: the basic question is how to detect entanglement within this class of PPT states. One may expect that there is special class of entanglement witnesses which are sensitive to entanglement encoded into circular decompositions. The related mathematical problem is the construction of linear indecomposable positive maps $\Lambda : M_d(\mathbb{C}) \longrightarrow M_d(\mathbb{C})$ satisfying

$$(\mathbb{1} \otimes \Lambda)\, \rho \ngeq 0 \ ,$$

for some circulant PPT state ρ. A corresponding class of such maps correlated with the previous class of PPT states [20] was recently proposed in [26]. It would be interesting to establish a structure of edge states [28, 29] within circulant PPT states since the knowledge of edge states is sufficient to characterize all PPT states. Finally, it is interesting to explore the possibility of other decompositions leading to new classes of PPT states. We stress that the seminal Horodecki $3 \otimes 3$ entangled PPT state [4] does not belong to our class. In a forthcoming paper we show that this state belongs to a new class of PPT states which is governed by another type of decompositions of $\mathbb{C}^d \otimes \mathbb{C}^d$.

Appendix

For $d = 4$ one has 6 different decompositions of $\mathbb{C}^4 \otimes \mathbb{C}^4$ into the direst sum of four 4-dimensional subspaces. These are labeled by permutations from the symmetric group S_3. One finds

$$\begin{aligned}
&(\pi_1(0) = 0\, , \pi_1(1) = 1\, , \pi_1(2) = 2\, , \pi_1(3) = 3) \ ,\\
&(\widetilde{\pi}_1(0) = 0\, , \widetilde{\pi}_1(1) = 3\, , \widetilde{\pi}_1(2) = 2\, , \widetilde{\pi}_1(3) = 1) \ ,\\
&(\pi_2(0) = 0\, , \pi_2(1) = 2\, , \pi_2(2) = 3\, , \pi_3(3) = 1) \ ,\\
&(\widetilde{\pi}_2(0) = 0\, , \widetilde{\pi}_2(1) = 2\, , \widetilde{\pi}_2(2) = 1\, , \widetilde{\pi}_2(3) = 3) \ ,\\
&(\pi_3(0) = 0\, , \pi_3(1) = 3\, , \pi_3(2) = 1\, , \pi_3(3) = 2) \ ,\\
&(\widetilde{\pi}_3(0) = 0\, , \widetilde{\pi}_3(1) = 1\, , \widetilde{\pi}_3(2) = 3\, , \widetilde{\pi}_3(3) = 2) \ .
\end{aligned}$$

The corresponding permutation matrices read as follows

$$\Pi_1 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \ , \quad \widetilde{\Pi}_1 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \ , \tag{A.1}$$

$$\Pi_2 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \ , \quad \widetilde{\Pi}_2 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \ , \tag{A.2}$$

$$\Pi_3 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \ , \quad \widetilde{\Pi}_3 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \ . \tag{A.3}$$

Moreover, one finds for

- $\widetilde{\Pi}_1 S_{\pi_1}$, $\widetilde{\Pi}_1 S^2_{\pi_1}$ and $\widetilde{\Pi}_1 S^3_{\pi_1}$:

$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \ , \quad \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \ , \quad \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \ ,$$

- $\widetilde{\Pi}_2 S_{\pi_2}$, $\widetilde{\Pi}_2 S^2_{\pi_2}$ and $\widetilde{\Pi}_2 S^3_{\pi_2}$:

$$\begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \ , \quad \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \ , \quad \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \ ,$$

- and for $\widetilde{\Pi}_3 S_{\pi_3}$, $\widetilde{\Pi}_3 S^2_{\pi_3}$ and $\widetilde{\Pi}_3 S^3_{\pi_3}$:

$$\begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \ , \quad \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \ , \quad \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \ ,$$

respectively.

A. π_1 and $\widetilde{\pi}_1$ circulant states

$$\begin{aligned}
\Sigma^{\pi_1}_0 &= \mathrm{span}\,\{e_0 \otimes e_0\, , e_1 \otimes e_1\, , e_2 \otimes e_2\, , e_3 \otimes e_3\} \ ,\\
\Sigma^{\pi_1}_1 &= \mathrm{span}\,\{e_0 \otimes e_1\, , e_1 \otimes e_2\, , e_2 \otimes e_3\, , e_3 \otimes e_0\} \ ,\\
\Sigma^{\pi_1}_2 &= \mathrm{span}\,\{e_0 \otimes e_2\, , e_1 \otimes e_3\, , e_2 \otimes e_1\, , e_3 \otimes e_2\} \ ,\\
\Sigma^{\pi_1}_3 &= \mathrm{span}\,\{e_0 \otimes e_3\, , e_1 \otimes e_0\, , e_2 \otimes e_1\, , e_3 \otimes e_2\} \ ,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\Sigma}^{\pi_1}_0 &= \mathrm{span}\,\{e_0 \otimes e_0\, , e_1 \otimes e_3\, , e_2 \otimes e_2\, , e_3 \otimes e_1\} \ ,\\
\widetilde{\Sigma}^{\pi_1}_1 &= \mathrm{span}\,\{e_0 \otimes e_1\, , e_1 \otimes e_0\, , e_2 \otimes e_3\, , e_3 \otimes e_2\} \ ,\\
\widetilde{\Sigma}^{\pi_1}_2 &= \mathrm{span}\,\{e_0 \otimes e_2\, , e_1 \otimes e_1\, , e_2 \otimes e_0\, , e_3 \otimes e_3\} \ ,\\
\widetilde{\Sigma}^{\pi_1}_3 &= \mathrm{span}\,\{e_0 \otimes e_3\, , e_1 \otimes e_2\, , e_2 \otimes e_1\, , e_3 \otimes e_0\} \ .
\end{aligned}$$

$$
\rho_{\pi_1} \;=\; \left(\begin{array}{cccc|cccc|cccc|cccc}
a_{00} & \cdot & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & \cdot & a_{02} & \cdot & \cdot & \cdot & \cdot & a_{03} \\
\cdot & b_{00} & \cdot & \cdot & \cdot & \cdot & b_{01} & \cdot & \cdot & \cdot & \cdot & b_{02} & b_{03} & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_{00} & \cdot & \cdot & \cdot & \cdot & c_{01} & c_{02} & \cdot & \cdot & \cdot & \cdot & c_{03} & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{00} & d_{01} & \cdot & \cdot & \cdot & \cdot & d_{02} & \cdot & \cdot & \cdot & \cdot & d_{03} & \cdot \\
\hline
\cdot & \cdot & \cdot & d_{10} & d_{11} & \cdot & \cdot & \cdot & \cdot & d_{12} & \cdot & \cdot & \cdot & \cdot & d_{13} & \cdot \\
a_{10} & \cdot & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & \cdot & a_{12} & \cdot & \cdot & \cdot & \cdot & a_{13} \\
\cdot & b_{10} & \cdot & \cdot & \cdot & \cdot & b_{11} & \cdot & \cdot & \cdot & \cdot & b_{12} & b_{13} & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_{10} & \cdot & \cdot & \cdot & \cdot & c_{11} & c_{12} & \cdot & \cdot & \cdot & \cdot & c_{13} & \cdot & \cdot \\
\hline
\cdot & \cdot & c_{20} & \cdot & \cdot & \cdot & \cdot & c_{21} & c_{22} & \cdot & \cdot & \cdot & \cdot & c_{23} & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{20} & d_{21} & \cdot & \cdot & \cdot & \cdot & d_{22} & \cdot & \cdot & \cdot & \cdot & d_{23} & \cdot \\
a_{20} & \cdot & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & \cdot & a_{22} & \cdot & \cdot & \cdot & \cdot & a_{23} \\
\cdot & b_{20} & \cdot & \cdot & \cdot & \cdot & b_{21} & \cdot & \cdot & \cdot & \cdot & b_{22} & b_{23} & \cdot & \cdot & \cdot \\
\hline
\cdot & b_{30} & \cdot & \cdot & \cdot & \cdot & b_{31} & \cdot & \cdot & \cdot & \cdot & b_{32} & b_{33} & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_{30} & \cdot & \cdot & \cdot & \cdot & c_{31} & c_{32} & \cdot & \cdot & \cdot & \cdot & c_{33} & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{30} & d_{31} & \cdot & \cdot & \cdot & \cdot & d_{32} & \cdot & \cdot & \cdot & \cdot & d_{33} & \cdot \\
a_{30} & \cdot & \cdot & \cdot & \cdot & a_{31} & \cdot & \cdot & \cdot & \cdot & a_{32} & \cdot & \cdot & \cdot & \cdot & a_{33}
\end{array}\right) . \tag{A.4}
$$

$$
\rho_{\pi_1}^{\tau} \;=\; \left(\begin{array}{cccc|cccc|cccc|cccc}
\widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \cdot & \widetilde{a}_{02} & \cdot & \cdot & \widetilde{a}_{03} & \cdot & \cdot \\
\cdot & \widetilde{b}_{00} & \cdot & \cdot & \widetilde{b}_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{02} & \cdot & \cdot & \widetilde{b}_{03} & \cdot \\
\cdot & \cdot & \widetilde{c}_{00} & \cdot & \cdot & \widetilde{c}_{01} & \cdot & \cdot & \widetilde{c}_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{03} \\
\cdot & \cdot & \cdot & \widetilde{d}_{00} & \cdot & \cdot & \widetilde{d}_{01} & \cdot & \cdot & \widetilde{d}_{02} & \cdot & \cdot & \widetilde{d}_{03} & \cdot & \cdot & \cdot \\
\hline
\cdot & \widetilde{b}_{10} & \cdot & \cdot & \widetilde{b}_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{12} & \cdot & \cdot & \widetilde{b}_{13} & \cdot \\
\cdot & \cdot & \widetilde{c}_{10} & \cdot & \cdot & \widetilde{c}_{11} & \cdot & \cdot & \widetilde{c}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{13} \\
\cdot & \cdot & \cdot & \widetilde{d}_{10} & \cdot & \cdot & \widetilde{d}_{11} & \cdot & \cdot & \widetilde{d}_{12} & \cdot & \cdot & \widetilde{d}_{13} & \cdot & \cdot & \cdot \\
\widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \cdot & \widetilde{a}_{12} & \cdot & \cdot & \widetilde{a}_{13} & \cdot & \cdot \\
\hline
\cdot & \cdot & \widetilde{c}_{20} & \cdot & \cdot & \widetilde{c}_{21} & \cdot & \cdot & \widetilde{c}_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{23} \\
\cdot & \cdot & \cdot & \widetilde{d}_{20} & \cdot & \cdot & \widetilde{d}_{21} & \cdot & \cdot & \widetilde{d}_{22} & \cdot & \cdot & \widetilde{d}_{23} & \cdot & \cdot & \cdot \\
\widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \cdot & \widetilde{a}_{22} & \cdot & \cdot & \widetilde{a}_{23} & \cdot & \cdot \\
\cdot & \widetilde{b}_{20} & \cdot & \cdot & \widetilde{b}_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{22} & \cdot & \cdot & \widetilde{b}_{23} & \cdot \\
\hline
\cdot & \cdot & \cdot & \widetilde{d}_{30} & \cdot & \cdot & \widetilde{d}_{31} & \cdot & \cdot & \widetilde{d}_{32} & \cdot & \cdot & \widetilde{d}_{33} & \cdot & \cdot & \cdot \\
\widetilde{a}_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{31} & \cdot & \cdot & \widetilde{a}_{32} & \cdot & \cdot & \widetilde{a}_{33} & \cdot & \cdot \\
\cdot & \widetilde{b}_{30} & \cdot & \cdot & \widetilde{b}_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{32} & \cdot & \cdot & \widetilde{b}_{33} & \cdot \\
\cdot & \cdot & \widetilde{c}_{30} & \cdot & \cdot & \widetilde{c}_{31} & \cdot & \cdot & \widetilde{c}_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{33}
\end{array}\right) . \tag{A.5}
$$

where the matrices $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ are given by

$$
\widetilde{a} = \begin{pmatrix} a_{00} & d_{01} & c_{02} & b_{03} \\ d_{10} & c_{11} & b_{12} & a_{13} \\ c_{20} & b_{21} & a_{22} & d_{23} \\ b_{30} & a_{31} & d_{32} & c_{33} \end{pmatrix} , \;
\widetilde{b} = \begin{pmatrix} b_{00} & a_{01} & d_{02} & c_{03} \\ a_{10} & d_{11} & c_{12} & b_{13} \\ d_{20} & c_{21} & b_{22} & a_{23} \\ c_{30} & b_{31} & a_{32} & d_{33} \end{pmatrix} , \;
\widetilde{c} = \begin{pmatrix} c_{00} & b_{01} & a_{02} & d_{03} \\ b_{10} & a_{11} & d_{12} & c_{13} \\ a_{20} & d_{21} & c_{22} & b_{23} \\ d_{30} & c_{31} & b_{32} & a_{33} \end{pmatrix} , \;
\widetilde{d} = \begin{pmatrix} d_{00} & c_{01} & b_{02} & a_{03} \\ c_{10} & b_{11} & a_{12} & d_{13} \\ b_{20} & a_{21} & d_{22} & c_{23} \\ a_{30} & d_{31} & c_{32} & b_{33} \end{pmatrix} .
$$

B. π_2 and $\widetilde{\pi}_2$ circulant states

$$
\begin{aligned}
\Sigma_0^{\pi_2} &= \mathrm{span}\{e_0\otimes e_0\,, e_1\otimes e_2\,, e_2\otimes e_3\,, e_3\otimes e_1\}\ , \\
\Sigma_1^{\pi_2} &= \mathrm{span}\{e_0\otimes e_1\,, e_1\otimes e_3\,, e_2\otimes e_0\,, e_3\otimes e_2\}\ , \\
\Sigma_2^{\pi_2} &= \mathrm{span}\{e_0\otimes e_2\,, e_1\otimes e_0\,, e_2\otimes e_1\,, e_3\otimes e_3\}\ , \\
\Sigma_3^{\pi_2} &= \mathrm{span}\{e_0\otimes e_3\,, e_1\otimes e_1\,, e_2\otimes e_2\,, e_3\otimes e_0\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Sigma}_0^{\pi_2} &= \mathrm{span}\{e_0\otimes e_0\,, e_1\otimes e_2\,, e_2\otimes e_1\,, e_3\otimes e_3\}\ , \\
\widetilde{\Sigma}_1^{\pi_2} &= \mathrm{span}\{e_0\otimes e_1\,, e_1\otimes e_3\,, e_2\otimes e_2\,, e_3\otimes e_0\}\ , \\
\widetilde{\Sigma}_2^{\pi_2} &= \mathrm{span}\{e_0\otimes e_2\,, e_1\otimes e_0\,, e_2\otimes e_3\,, e_3\otimes e_1\}\ , \\
\widetilde{\Sigma}_3^{\pi_2} &= \mathrm{span}\{e_0\otimes e_3\,, e_1\otimes e_1\,, e_2\otimes e_0\,, e_3\otimes e_2\}\ .
\end{aligned}
$$

$$
\rho_{\pi_2} = \left(\begin{array}{cccc|cccc|cccc|cccc}
a_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & \cdot & a_{02} & \cdot & a_{03} & \cdot & \cdot \\
\cdot & b_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{03} & \cdot \\
\cdot & \cdot & c_{00} & \cdot & c_{01} & \cdot & \cdot & \cdot & \cdot & c_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{03} \\
\cdot & \cdot & \cdot & d_{00} & \cdot & d_{01} & \cdot & \cdot & \cdot & \cdot & d_{02} & \cdot & d_{03} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & c_{10} & \cdot & c_{11} & \cdot & \cdot & \cdot & \cdot & c_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{13} \\
\cdot & \cdot & \cdot & d_{10} & \cdot & d_{11} & \cdot & \cdot & \cdot & \cdot & d_{12} & \cdot & d_{13} & \cdot & \cdot & \cdot \\
a_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & \cdot & a_{12} & \cdot & a_{13} & \cdot & \cdot \\
\cdot & b_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{13} & \cdot \\
\hline
\cdot & b_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{23} & \cdot \\
\cdot & \cdot & c_{20} & \cdot & c_{21} & \cdot & \cdot & \cdot & \cdot & c_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{23} \\
\cdot & \cdot & \cdot & d_{20} & \cdot & d_{21} & \cdot & \cdot & \cdot & \cdot & d_{22} & \cdot & d_{23} & \cdot & \cdot & \cdot \\
a_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & \cdot & a_{22} & \cdot & a_{23} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & d_{30} & \cdot & d_{31} & \cdot & \cdot & \cdot & \cdot & d_{32} & \cdot & d_{33} & \cdot & \cdot & \cdot \\
a_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{31} & \cdot & \cdot & \cdot & \cdot & a_{32} & \cdot & a_{33} & \cdot & \cdot \\
\cdot & b_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{31} & b_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{33} & \cdot \\
\cdot & \cdot & c_{30} & \cdot & c_{31} & \cdot & \cdot & \cdot & \cdot & c_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{33}
\end{array}\right) . \tag{A.6}
$$

$$
\rho_{\pi_2}^{\tau} = \left(\begin{array}{cccc|cccc|cccc|cccc}
\widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \cdot & \widetilde{a}_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{03} \\
\cdot & \widetilde{b}_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{01} & \cdot & \cdot & \widetilde{b}_{02} & \cdot & \widetilde{b}_{03} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \widetilde{c}_{00} & \cdot & \widetilde{c}_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{02} & \cdot & \widetilde{c}_{03} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \widetilde{d}_{00} & \cdot & \widetilde{d}_{01} & \cdot & \cdot & \widetilde{d}_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{d}_{03} & \cdot \\
\hline
\cdot & \cdot & \widetilde{c}_{10} & \cdot & \widetilde{c}_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{12} & \cdot & \widetilde{c}_{13} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \widetilde{d}_{10} & \cdot & \widetilde{d}_{11} & \cdot & \cdot & \widetilde{d}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{d}_{13} & \cdot \\
\widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \cdot & \widetilde{a}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{13} \\
\cdot & \widetilde{b}_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{11} & \cdot & \cdot & \widetilde{b}_{12} & \cdot & \widetilde{b}_{13} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \widetilde{d}_{20} & \cdot & \widetilde{d}_{21} & \cdot & \cdot & \widetilde{d}_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{d}_{23} & \cdot \\
\widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \cdot & \widetilde{a}_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{23} \\
\cdot & \widetilde{b}_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{21} & \cdot & \cdot & \widetilde{b}_{22} & \cdot & \widetilde{b}_{23} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \widetilde{c}_{20} & \cdot & \widetilde{c}_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{22} & \cdot & \widetilde{c}_{23} & \cdot & \cdot \\
\hline
\cdot & \widetilde{b}_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{31} & \cdot & \cdot & \widetilde{b}_{32} & \cdot & \widetilde{b}_{33} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \widetilde{c}_{30} & \cdot & \widetilde{c}_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{32} & \cdot & \widetilde{c}_{33} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \widetilde{d}_{30} & \cdot & \widetilde{d}_{31} & \cdot & \cdot & \widetilde{d}_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{d}_{33} & \cdot \\
\widetilde{a}_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{31} & \cdot & \cdot & \widetilde{a}_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{33}
\end{array}\right) . \tag{A.7}
$$

where the matrices $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ are given by

$$\widetilde{a} \;=\; \begin{pmatrix} a_{00} & c_{01} & b_{02} & d_{03} \\ c_{10} & a_{11} & d_{12} & b_{13} \\ b_{20} & d_{21} & c_{22} & a_{23} \\ d_{30} & b_{31} & a_{32} & c_{33} \end{pmatrix} \,,\; \widetilde{b} \;=\; \begin{pmatrix} b_{00} & d_{01} & c_{02} & a_{03} \\ d_{10} & b_{11} & a_{12} & c_{13} \\ c_{20} & a_{21} & d_{22} & b_{23} \\ a_{30} & c_{31} & b_{32} & d_{33} \end{pmatrix} \,,\; \widetilde{c} \;=\; \begin{pmatrix} c_{00} & a_{01} & d_{02} & b_{03} \\ a_{10} & c_{11} & b_{12} & d_{13} \\ d_{20} & b_{21} & a_{22} & c_{23} \\ b_{30} & d_{31} & c_{32} & a_{33} \end{pmatrix} \,,\; \widetilde{d} \;=\; \begin{pmatrix} d_{00} & b_{01} & a_{02} & c_{03} \\ b_{10} & d_{11} & c_{12} & a_{13} \\ a_{20} & c_{21} & b_{22} & d_{23} \\ c_{30} & a_{31} & d_{32} & b_{33} \end{pmatrix} \,.$$

C. π_3 and $\widetilde{\pi}_3$ circulant states

$$\begin{aligned} \Sigma_0^{\pi_3} &= \mathrm{span}\,\{e_0\otimes e_0\,, e_1\otimes e_3\,, e_2\otimes e_1\,, e_3\otimes e_2\} \;, \\ \Sigma_1^{\pi_3} &= \mathrm{span}\,\{e_0\otimes e_1\,, e_1\otimes e_0\,, e_2\otimes e_2\,, e_3\otimes e_3\} \;, \\ \Sigma_2^{\pi_3} &= \mathrm{span}\,\{e_0\otimes e_2\,, e_1\otimes e_1\,, e_2\otimes e_3\,, e_3\otimes e_0\} \;, \\ \Sigma_3^{\pi_3} &= \mathrm{span}\,\{e_0\otimes e_3\,, e_1\otimes e_2\,, e_2\otimes e_0\,, e_3\otimes e_1\} \;, \end{aligned}$$

$$\begin{aligned} \widetilde{\Sigma}_0^{\pi_3} &= \mathrm{span}\,\{e_0\otimes e_0\,, e_1\otimes e_1\,, e_2\otimes e_3\,, e_3\otimes e_2\} \;, \\ \widetilde{\Sigma}_1^{\pi_3} &= \mathrm{span}\,\{e_0\otimes e_1\,, e_1\otimes e_2\,, e_2\otimes e_0\,, e_3\otimes e_3\} \;, \\ \widetilde{\Sigma}_2^{\pi_3} &= \mathrm{span}\,\{e_0\otimes e_2\,, e_1\otimes e_3\,, e_2\otimes e_1\,, e_3\otimes e_0\} \;, \\ \widetilde{\Sigma}_3^{\pi_3} &= \mathrm{span}\,\{e_0\otimes e_3\,, e_1\otimes e_0\,, e_2\otimes e_2\,, e_3\otimes e_1\} \;. \end{aligned}$$

$$\rho_{\pi_3} \;=\; \left(\begin{array}{cccc|cccc|cccc|cccc}
a_{00} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{01} & \cdot & a_{02} & \cdot & \cdot & \cdot & \cdot & a_{03} & \cdot \\
\cdot & b_{00} & \cdot & \cdot & b_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{02} & \cdot & \cdot & \cdot & \cdot & b_{03} \\
\cdot & \cdot & c_{00} & \cdot & \cdot & c_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{02} & c_{03} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{00} & \cdot & \cdot & d_{01} & \cdot & d_{02} & \cdot & \cdot & \cdot & \cdot & d_{03} & \cdot & \cdot \\
\hline
\cdot & b_{10} & \cdot & \cdot & b_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{12} & \cdot & \cdot & \cdot & \cdot & b_{13} \\
\cdot & \cdot & c_{10} & \cdot & \cdot & c_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{12} & c_{13} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{10} & \cdot & \cdot & d_{11} & \cdot & d_{12} & \cdot & \cdot & \cdot & \cdot & d_{13} & \cdot & \cdot \\
a_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{11} & \cdot & a_{12} & \cdot & \cdot & \cdot & \cdot & a_{13} & \cdot \\
\hline
\cdot & \cdot & \cdot & d_{20} & \cdot & \cdot & d_{21} & \cdot & d_{22} & \cdot & \cdot & \cdot & \cdot & d_{23} & \cdot & \cdot \\
a_{20} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{21} & \cdot & a_{22} & \cdot & \cdot & \cdot & \cdot & a_{23} & \cdot \\
\cdot & b_{20} & \cdot & \cdot & b_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{22} & \cdot & \cdot & \cdot & \cdot & b_{23} \\
\cdot & \cdot & c_{20} & \cdot & \cdot & c_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{22} & c_{23} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & c_{30} & \cdot & \cdot & c_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & c_{32} & c_{33} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & d_{30} & \cdot & \cdot & d_{31} & \cdot & d_{32} & \cdot & \cdot & \cdot & \cdot & d_{33} & \cdot & \cdot \\
a_{30} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{31} & \cdot & a_{32} & \cdot & \cdot & \cdot & \cdot & a_{33} & \cdot \\
\cdot & b_{30} & \cdot & \cdot & b_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & b_{32} & \cdot & \cdot & \cdot & \cdot & b_{33}
\end{array}\right) \,. \tag{A.8}$$

$$
\rho_{\pi_3}^{\tau} = \left(\begin{array}{cccc|cccc|cccc|cccc}
\widetilde{a}_{00} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{02} & \cdot & \cdot & \widetilde{a}_{03} & \cdot \\
\cdot & \widetilde{b}_{00} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{01} & \cdot & \widetilde{b}_{02} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{03} \\
\cdot & \cdot & \widetilde{c}_{00} & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{01} & \cdot & \widetilde{c}_{02} & \cdot & \cdot & \widetilde{c}_{03} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \widetilde{d}_{00} & \widetilde{d}_{01} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{d}_{02} & \cdot & \cdot & \widetilde{d}_{02} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \widetilde{d}_{10} & \widetilde{d}_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{d}_{12} & \cdot & \cdot & \widetilde{d}_{13} & \cdot & \cdot \\
\widetilde{a}_{10} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{12} & \cdot & \cdot & \widetilde{a}_{13} & \cdot \\
\cdot & \widetilde{b}_{10} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{11} & \cdot & \widetilde{b}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{13} \\
\cdot & \cdot & \widetilde{c}_{10} & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{11} & \cdot & \widetilde{c}_{12} & \cdot & \cdot & \widetilde{c}_{13} & \cdot & \cdot & \cdot \\
\hline
\cdot & \widetilde{b}_{20} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{21} & \cdot & \widetilde{b}_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{23} \\
\cdot & \cdot & \widetilde{c}_{20} & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{21} & \cdot & \widetilde{c}_{22} & \cdot & \cdot & \widetilde{c}_{23} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \widetilde{d}_{20} & \widetilde{d}_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{d}_{22} & \cdot & \cdot & \widetilde{d}_{22} & \cdot & \cdot \\
\widetilde{a}_{20} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{22} & \cdot & \cdot & \widetilde{a}_{23} & \cdot \\
\hline
\cdot & \cdot & \widetilde{c}_{30} & \cdot & \cdot & \cdot & \cdot & \widetilde{c}_{31} & \cdot & \widetilde{c}_{32} & \cdot & \cdot & \widetilde{c}_{33} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \widetilde{d}_{30} & \widetilde{d}_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{d}_{32} & \cdot & \cdot & \widetilde{d}_{32} & \cdot & \cdot \\
\widetilde{a}_{30} & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{a}_{32} & \cdot & \cdot & \widetilde{a}_{33} & \cdot \\
\cdot & \widetilde{b}_{30} & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{31} & \cdot & \widetilde{b}_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \widetilde{b}_{33}
\end{array}\right) . \tag{A.9}
$$

where the matrices $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ are given by

$$
\widetilde{a} = \begin{pmatrix} a_{00} & b_{01} & d_{02} & c_{03} \\ b_{10} & c_{11} & a_{12} & d_{13} \\ d_{20} & a_{21} & c_{22} & b_{23} \\ c_{30} & d_{31} & b_{32} & a_{33} \end{pmatrix} , \quad
\widetilde{b} = \begin{pmatrix} b_{00} & c_{01} & a_{02} & d_{03} \\ c_{10} & d_{11} & b_{12} & a_{13} \\ a_{20} & b_{21} & d_{22} & c_{23} \\ d_{30} & a_{31} & c_{32} & b_{33} \end{pmatrix} , \quad
\widetilde{c} = \begin{pmatrix} c_{00} & d_{01} & b_{02} & a_{03} \\ d_{10} & a_{11} & c_{12} & b_{13} \\ b_{20} & c_{21} & a_{22} & d_{23} \\ a_{30} & b_{31} & d_{32} & c_{33} \end{pmatrix} , \quad
\widetilde{d} = \begin{pmatrix} d_{00} & a_{01} & c_{02} & b_{03} \\ a_{10} & b_{11} & d_{12} & c_{13} \\ c_{20} & d_{21} & b_{22} & a_{23} \\ b_{30} & c_{31} & a_{32} & d_{33} \end{pmatrix} .
$$

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[21] A matrix $X \in M_n$ is circulant if

$$X = \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \dots & \alpha_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \dots & \alpha_0 \end{pmatrix} .$$

[22] A Hadamard (or Schur) product of two $n \times n$ matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ is defined by

$$(A \circ B)_{ij} = A_{ij} B_{ij} \ .$$

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