

Multipartite circulant states with positive partial transposes

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We construct a large class of multipartite qudit states which are positive under the family of partial transpositions. The construction is based on certain direct sum decomposition of the total Hilbert space displaying characteristic circular structure and hence generalizes a class of bipartite circulant states proposed recently by the authors. This class contains many well known examples of multipartite quantum states from the literature and gives rise to a huge family of completely new states.

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I. INTRODUCTION

Quantum entanglement is one of the most remarkable features of quantum mechanics and it leads to powerful applications like quantum cryptography, dense coding and quantum computing [1, 2].

It is well known that it is very hard to check whether a given density matrix describing a quantum state of the composite system is separable or entangled. There are several operational criteria which enable one to detect quantum entanglement (see e.g. [2] for the recent review). The most famous Peres-Horodecki criterion [3, 4] is based on the partial transposition: if a state ρ is separable then its partial transposition $(\mathbb{1} \otimes \tau)\rho$ is positive (such states are called PPT state). The structure of this set is of primary importance in quantum information theory. Unfortunately, this structure is still unknown, that is, one may easily check whether a given state is PPT but we do not know how to construct a general quantum state with PPT property.

Recently [5] we proposed a large class of bipartite PPT states which are based on certain cyclic decomposition of the total Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$ — we called them circulant states. The crucial property of this class is that a partial transposition of the circulant state has again a circular structure corresponding to another cyclic decomposition of $\mathbb{C}^d \otimes \mathbb{C}^d$. Interestingly, many well known examples of PPT states fit the class of circulant states [5].

In the present paper we generalize the construction of circulant states to multipartite systems — N qudits living in $\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d$ (N copies). This space can be in a natural way decomposed into d^{N-1} subspaces of dimension d or d subspaces of dimension d^{N-1} . Multipartite circulant state is defined as a convex combination of positive operators supported on these orthogonal subspaces. It turns out that the family of partial

transpositions maps these states into circulant operators supported on another family of subspaces related by a circular multipartite structure. Again, we show that many well known examples of multipartite states belong to our class.

Recently, there is a considerable effort to explore multipartite systems [6]–[13] and multipartite circulant states introduced in this paper may shed new light on the more general investigation of multipartite entanglement.

The paper is organized as follows: for pedagogical reason we first illustrate our general method for $d = 2$ in Section II. We recall basic construction for 2 qubits from [5] and then analyze in details 3 qubit circulant states. We illustrate our construction with well known examples of 3 qubit states from the literature. Then we present a general construction of N qubit states. Section III discusses circulant states of N qudits. Final conclusions are collected in the last section.

II. N -QUBIT STATES

A. 2 qubits

Consider a density matrix living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ which is given by

$$\rho = \rho_0 + \rho_1, \quad (1)$$

where ρ_0 and ρ_1 are supported on two orthogonal subspaces

$$\begin{aligned} \Sigma_0 &= \text{span} \{e_0 \otimes e_0, e_1 \otimes e_1\}, \\ \Sigma_1 &= \text{span} \{e_0 \otimes e_1, e_1 \otimes e_0\}, \end{aligned} \quad (2)$$

and $\{e_0, e_1\}$ is a computational base in \mathbb{C}^2 . It is clear that $\{\Sigma_0, \Sigma_1\}$ defines the direct sum decomposition of

$\mathbb{C}^2 \otimes \mathbb{C}^2$, that is

$$\Sigma_0 \oplus \Sigma_1 = \mathbb{C}^2 \otimes \mathbb{C}^2 .$$

We call it a circulant decomposition because its structure is determined by the cyclic shift $S : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by

$$S e_i = e_{i+1} , \quad (\text{mod } 2) . \quad (3)$$

One finds that

$$\Sigma_1 = (\mathbb{I} \otimes S) \Sigma_0 , \quad (4)$$

and hence

$$\begin{aligned} \rho_0 &= \sum_{i,j=0}^1 a_{ij} e_{ij} \otimes e_{ij} , & (5) \\ \rho_1 &= \sum_{i,j=0}^1 b_{ij} e_{ij} \otimes S e_{ij} S^* , \\ &= (\mathbb{I} \otimes S) \left(\sum_{i,j=0}^1 b_{ij} e_{ij} \otimes e_{ij} \right) (\mathbb{I} \otimes S)^* , & (6) \end{aligned}$$

where $e_{ij} := |e_i\rangle\langle e_j|$, and one adds mod 2. Now, since ρ_0 and ρ_1 are supported on two orthogonal subspaces Σ_0 and Σ_1 one has an obvious

Proposition 1 ρ defined in (1) is a density matrix iff

- $a = [a_{ij}]$ and $b = [b_{ij}]$ are 2×2 semi-positive matrices, and
- $\text{Tr}(a + b) = 1$.

Now, the crucial observation is that partially transposed matrix $\rho^\tau = (\mathbb{I} \otimes \tau)\rho$ belongs to the same class as original ρ

$$\rho = \left(\begin{array}{cc|cc} a_{00} & \cdot & \cdot & a_{01} \\ \cdot & b_{00} & b_{01} & \cdot \\ \cdot & b_{10} & b_{11} & \cdot \\ a_{10} & \cdot & \cdot & a_{11} \end{array} \right) , \quad (7)$$

and

$$\rho^\tau = \left(\begin{array}{cc|cc} \tilde{a}_{00} & \cdot & \cdot & \tilde{a}_{01} \\ \cdot & \tilde{b}_{00} & \tilde{b}_{01} & \cdot \\ \cdot & \tilde{b}_{10} & \tilde{b}_{11} & \cdot \\ \tilde{a}_{10} & \cdot & \cdot & \tilde{a}_{11} \end{array} \right) , \quad (8)$$

where the matrices $\tilde{a} = [\tilde{a}_{ij}]$ and $\tilde{b} = [\tilde{b}_{ij}]$ read as follows

$$\tilde{a} = \begin{pmatrix} a_{00} & b_{01} \\ b_{10} & a_{11} \end{pmatrix} , \quad \tilde{b} = \begin{pmatrix} b_{00} & a_{01} \\ a_{10} & b_{11} \end{pmatrix} , \quad (9)$$

that is, both ρ and ρ^τ are circulant bipartite operators. Therefore, one arrives at

Proposition 2 A circulant state represented by (1) is PPT iff $\tilde{a} = [\tilde{a}_{ij}]$ and $\tilde{b} = [\tilde{b}_{ij}]$ are 2×2 semi-positive matrices.

Note, that matrices \tilde{a} and \tilde{b} may be rewritten in the following transparent way

$$\tilde{a} = a \circ \mathbb{I} + b \circ S , \quad (10)$$

and similarly

$$\tilde{b} = b \circ \mathbb{I} + a \circ S , \quad (11)$$

where $x \circ y$ denotes the Hadamard product of two matrices x and y .

Examples. 1. Bell states: $|\psi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$

$$|\psi^\pm\rangle\langle\psi^\pm| = \frac{1}{2} \left(\begin{array}{cc|cc} 1 & \cdot & \cdot & \pm 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \pm 1 & \cdot & \cdot & 1 \end{array} \right) , \quad (12)$$

and for $|\varphi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$

$$|\varphi^\pm\rangle\langle\varphi^\pm| = \frac{1}{2} \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \pm 1 & \cdot \\ \cdot & \pm 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right) . \quad (13)$$

2. Werner state [14]

$$\mathcal{W} = \frac{1}{4} \left(\begin{array}{cc|cc} 1-p & \cdot & \cdot & \cdot \\ \cdot & 1+p & -2p & \cdot \\ \cdot & -2p & 1+p & \cdot \\ \cdot & \cdot & \cdot & 1-p \end{array} \right) , \quad (14)$$

with $-1/3 \leq p \leq 1$. PPT condition implies well known result $p \leq 1/3$.

3. Isotropic state [15]

$$\mathcal{I} = \frac{1}{4} \left(\begin{array}{cc|cc} 1+p & \cdot & \cdot & 2p \\ \cdot & 1-p & \cdot & \cdot \\ \cdot & \cdot & 1-p & \cdot \\ 2p & \cdot & \cdot & 1+p \end{array} \right) , \quad (15)$$

with $-1/3 \leq p \leq 1$. Again PPT condition implies well known result $p \leq 1/3$.

4. $O(2) \otimes O(2)$ -invariant state

$$\mathcal{O} = \frac{1}{4} \left(\begin{array}{cc|cc} a+2b & \cdot & \cdot & 2b-a \\ \cdot & a+2c & a-2c & \cdot \\ \hline \cdot & a-2c & a+2c & \cdot \\ 2b-a & \cdot & \cdot & a+2b \end{array} \right), \quad (16)$$

with $a, b, c \geq 0$ and $a+b+c=1$. It is clear that \mathcal{O} is positive and \mathcal{O} is PPT iff

$$b \leq \frac{1}{2}, \quad c \leq \frac{1}{2}, \quad (17)$$

which reproduces well known result [16].

B. 3 qubits

There are in principle two ways to generalize the 2-qubit circulant decomposition for the case of three qubits. Either one decomposes $\mathcal{H}_{2^3} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ into four 2-dimensional subspaces or into two 4-dimensional ones.

1. “ $8 = 2 \oplus 2 \oplus 2 \oplus 2$ ”

Let us define 2-dimensional subspace

$$\Delta_{00} = \text{span} \{e_0 \otimes e_0 \otimes e_0, e_1 \otimes e_1 \otimes e_1\} \quad (18)$$

and for any two binaries μ and ν define

$$\Delta_{\mu\nu} = (\mathbb{I} \otimes S^\mu \otimes S^\nu) \Delta_{00}. \quad (19)$$

One easily finds

$$\begin{aligned} \Delta_{01} &= \text{span} \{e_0 \otimes e_0 \otimes e_1, e_1 \otimes e_1 \otimes e_0\}, \\ \Delta_{10} &= \text{span} \{e_0 \otimes e_1 \otimes e_0, e_1 \otimes e_0 \otimes e_1\}, \\ \Delta_{11} &= \text{span} \{e_0 \otimes e_1 \otimes e_1, e_1 \otimes e_0 \otimes e_0\}. \end{aligned} \quad (20)$$

It is clear that

$$\mathcal{H}_{2^3} = \Delta_{00} \oplus \Delta_{01} \oplus \Delta_{10} \oplus \Delta_{11}. \quad (21)$$

Now, we construct 3-qubit density matrix ρ of the following form

$$\rho = \rho_{00} + \rho_{01} + \rho_{10} + \rho_{11}, \quad (22)$$

where each $\rho_{\mu\nu}$ is supported on $\Delta_{\mu\nu}$. One has therefore

$$\rho_{\mu\nu} = \sum_{i,j=0}^1 x_{ij}^{(\mu\nu)} e_{ij} \otimes S^\mu e_{ij} S^{\mu*} \otimes S^\nu e_{ij} S^{\nu*}, \quad (23)$$

which generalizes 2-qubit construction (5). Positivity of ρ is guaranteed by positivity of each 2×2 matrix $[x^{(\mu\nu)}]$, and the normalization $\text{Tr} \rho = 1$ is equivalent to

$$\text{Tr} \left(x^{(00)} + x^{(01)} + x^{(10)} + x^{(11)} \right) = 1. \quad (24)$$

One obtains therefore the following block matrix

$$\rho = \left(\begin{array}{cc|cc} x_{00}^{(00)} & \cdot & \cdot & \cdot \\ \cdot & x_{00}^{(01)} & \cdot & \cdot \\ \hline \cdot & \cdot & x_{00}^{(10)} & \cdot \\ \cdot & \cdot & \cdot & x_{00}^{(11)} \\ \hline \cdot & \cdot & \cdot & x_{10}^{(11)} \\ \cdot & \cdot & x_{10}^{(10)} & \cdot \\ \hline \cdot & x_{10}^{(01)} & \cdot & \cdot \\ x_{10}^{(00)} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x_{11}^{(01)} & \cdot \\ \cdot & \cdot & \cdot & x_{11}^{(00)} \end{array} \right), \quad (25)$$

where vertical and horizontal lines remind us about the splitting into blocks corresponding to tensor product structure $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Double lines introduce split-

ting into four blocks corresponding to $\mathbb{C}^2 \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)$ and then single lines introduce splitting within each 4×4 block corresponding to the second tensor product

$(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Now, let us perform partial transposition. There are three independent transformations

$$\tau_{\alpha\beta} = \mathbb{1} \otimes \tau^\alpha \otimes \tau^\beta, \quad (26)$$

with binary indices α and β :

$$\begin{aligned} \tau_{01} &= \mathbb{1} \otimes \mathbb{1} \otimes \tau, \\ \tau_{10} &= \mathbb{1} \otimes \tau \otimes \mathbb{1}, \\ \tau_{11} &= \mathbb{1} \otimes \tau \otimes \tau. \end{aligned} \quad (27)$$

Now, it is easy to see that $\tau_{\alpha\beta}\rho$ has the same circular structure as original ρ defined in (60) with new 2×2 matrices $y^{(\mu\nu)[\alpha\beta]}$, that is,

$$\tau_{\alpha\beta}\rho = \sum_{\mu,\nu=0}^1 \sum_{i,j=0}^1 y_{ij}^{(\mu\nu)[\alpha\beta]} e_{ij} \otimes S^\mu e_{ij} S^{\mu*} \otimes S^\nu e_{ij} S^{\nu*}. \quad (28)$$

Proposition 3 *The set of 2×2 matrices $y^{(\mu\nu)[\alpha\beta]}$ is given by*

$$y^{(\mu\nu)[\alpha\beta]} = x^{(\mu\nu)} \circ \mathbb{I} + x^{(\mu+\alpha,\nu+\beta)} \circ S, \quad (29)$$

with addition modulo 2.

Using straightforward definition that ρ is $(\alpha\beta)$ -PPT if $\tau_{\alpha\beta}\rho \geq 0$, one has the following

Theorem 1 *A 3-qubit circulant state ρ is $(\alpha\beta)$ -PPT iff*

$$y^{(\mu\nu)[\alpha\beta]} \geq 0,$$

for all binary μ and ν .

It is clear that (25) generalizes 2-qubit circulant state (7). Note, however, that reducing 3-qubit state with respect to one subsystem one ends up with the following separable 2-qubit state

$$\text{Tr}_1 \rho = \sum_{\mu,\nu=0}^1 \sum_{i,j=0}^1 x_{ii}^{(\mu\nu)} e_{ii} \otimes e_{ii}. \quad (30)$$

It is again circulant but has very special structure: $[a_{ij}]$ is diagonal with

$$a_{ii} = \sum_{\mu,\nu=0}^1 x_{ii}^{(\mu\nu)},$$

and $[b_{ij}] = 0$. It is therefore clear that a general 2-qubit circulant state can not be obtained via reduction of (25).

Examples. 1. GHZ state [17]

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad (31)$$

does belong to circulant class which is easily seen from the corresponding density matrix

$$x^{(\mu\nu)} = \delta_{\mu 0} \delta_{\nu 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (32)$$

On the other hand the well known W state

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \quad (33)$$

is not circulant. The corresponding density matrix reads as follows

$$\rho_W = \frac{1}{3} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (34)$$

2. Bell states: the following 8 vectors

$$\psi_{\alpha\beta\gamma} = (-1)^\alpha (\mathbb{I} \otimes S^\beta \otimes S^\gamma) |\text{GHZ}\rangle, \quad (35)$$

with binary α, β, γ define circulant states. These are 3-qubit generalization of 2-qubit Bell states. Note that

$$\psi_{\alpha\beta\gamma} \in \Delta_{\beta\gamma}, \quad (36)$$

and the corresponding matrices $x^{(\mu\nu)}$ read as follows

$$x^{(\mu\nu)} = \delta_{\mu\beta} \delta_{\nu\gamma} \begin{pmatrix} 1 & (-1)^\alpha \\ (-1)^\alpha & 1 \end{pmatrix}. \quad (37)$$

3. Generalized 3-qubit isotropic state

$$\rho = \frac{1-s}{2^3} \mathbb{I}^{\otimes 3} + s |\text{GHZ}\rangle \langle \text{GHZ}| \quad (38)$$

with $s \in [-1/7, 1]$. One finds for $x^{(\mu\nu)}$ matrices

$$x^{(00)} = \frac{1}{8} \begin{pmatrix} 1+3s & 4s \\ 4s & 1+3s \end{pmatrix}, \quad (39)$$

and

$$x^{(01)} = x^{(10)} = x^{(11)} = \frac{1-s}{8} \mathbb{I}. \quad (40)$$

The only nontrivial PPT condition comes from the positivity of

$$y^{(01)[01]} = \frac{1}{8} \begin{pmatrix} 1-s & 4s \\ 4s & 1-s \end{pmatrix}, \quad (41) \quad \text{or}$$

which implies $s \leq 1/5$. Actually, it is well known [18] that ρ is fully separable iff $s \leq 1/5$.

4. 2-parameter 3-qubit state from [19]:

$$\rho(c, d) = \frac{1}{8} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & d & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad (42)$$

with $-1/8 \leq c, d \leq 1/8$. It was shown [19] that $\rho(c, d)$ has positive partial transposes iff $c = d$. Moreover, this condition implies full separability.

2. "8 = 4 ⊕ 4"

There are two ways to construct 4-dimensional circulant decompositions of 3-qubit Hilbert space out of

2-dimensional mutually orthogonal spaces $\Delta_{\mu\nu}$: either one introduces

$$\Sigma_0 = \Delta_{00} \oplus \Delta_{11}, \quad (43)$$

$$\Sigma_1 = \Delta_{01} \oplus \Delta_{10}, \quad (44)$$

$$\Xi_0 = \Delta_{00} \oplus \Delta_{10}, \quad (45)$$

$$\Xi_1 = \Delta_{01} \oplus \Delta_{11}. \quad (46)$$

The construction is clear:

$$\Sigma_0 = \bigoplus_{\mu+\nu=0} \Delta_{\mu\nu}, \quad (47)$$

$$\Sigma_1 = \bigoplus_{\mu+\nu=1} \Delta_{\mu\nu}, \quad (48)$$

whereas a second decomposition uses the following scheme

$$\Xi_\nu = \bigoplus_{\mu=0}^1 \Delta_{\mu\nu}. \quad (49)$$

Note, that using binary codes, Ξ_0 is constructed out of $\Delta_{\mu\nu}$ with $\mu\nu$ representing binary code for '0' (mod 2) and Ξ_1 is constructed out of $\Delta_{\mu\nu}$ with $\mu\nu$ representing binary code for '1' (mod 2). One easily finds

$$\begin{aligned} \Sigma_0 &= \text{span} \{e_0 \otimes e_0 \otimes e_0, e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_0 \otimes e_0, e_0 \otimes e_1 \otimes e_1\}, \\ \Sigma_1 &= \text{span} \{e_0 \otimes e_0 \otimes e_1, e_1 \otimes e_1 \otimes e_0, e_1 \otimes e_0 \otimes e_1, e_0 \otimes e_1 \otimes e_0\}, \end{aligned} \quad (50)$$

and

$$\begin{aligned} \Xi_0 &= \text{span} \{e_0 \otimes e_0 \otimes e_0, e_1 \otimes e_1 \otimes e_1, e_0 \otimes e_1 \otimes e_0, e_1 \otimes e_0 \otimes e_1\}, \\ \Xi_1 &= \text{span} \{e_0 \otimes e_0 \otimes e_1, e_1 \otimes e_1 \otimes e_0, e_0 \otimes e_1 \otimes e_1, e_1 \otimes e_0 \otimes e_0\}. \end{aligned} \quad (51)$$

It is clear that both decompositions are circulant, that is,

$$\Sigma_1 = (\mathbb{I} \otimes \mathbb{I} \otimes S) \Sigma_0 , \quad (52)$$

and

$$\Xi_1 = (\mathbb{I} \otimes \mathbb{I} \otimes S) \Xi_0 . \quad (53)$$

Now we are ready to construct the corresponding 3-qubit circulant states: a circulant state corresponding to

$$\Sigma_0 \oplus \Sigma_1 = \mathcal{H}_{2^3} ,$$

is defined by

$$\sigma = \sigma_0 + \sigma_1 ,$$

with σ_α supported on Σ_α . To define ρ_0 and ρ_1 one needs 4×4 matrices a and b . One has

$$\sigma_0 = \sum_{i,j=0}^1 \sum_{\mu,\nu=0}^1 a_{\mu i; \nu j} e_{\mu\nu} \otimes e_{ij} \otimes e_{ij} , \quad (54)$$

$$\sigma_1 = \sum_{i,j=0}^1 \sum_{\mu,\nu=0}^1 b_{\mu i; \nu j} e_{\mu\nu} \otimes e_{ij} \otimes S e_{ij} S^* , \quad (55)$$

where $a_{\mu i; \nu j}$ and $b_{\mu i; \nu j}$ are matrix elements of a and b considered as matrices in the tensor product $M_4 = M_2 \otimes M_2 = M_2(M_2)$. One obtains therefore the following block matrix

$$\sigma = \left(\begin{array}{cc|cc|cc} a_{00;00} & \cdot & \cdot & a_{00;01} & a_{00;10} & \cdot & \cdot & a_{00;11} \\ \cdot & b_{00;00} & b_{00;01} & \cdot & \cdot & b_{00;10} & b_{00;11} & \cdot \\ \cdot & b_{01;00} & b_{01;01} & \cdot & \cdot & b_{01;10} & b_{01;11} & \cdot \\ a_{01;00} & \cdot & \cdot & a_{01;01} & a_{01;10} & \cdot & \cdot & a_{01;11} \\ \hline a_{10;00} & \cdot & \cdot & a_{10;01} & a_{10;10} & \cdot & \cdot & a_{10;11} \\ \cdot & b_{10;00} & b_{10;01} & \cdot & \cdot & b_{10;10} & b_{10;11} & \cdot \\ \cdot & b_{11;00} & b_{11;01} & \cdot & \cdot & b_{11;10} & b_{11;11} & \cdot \\ a_{11;00} & \cdot & \cdot & a_{11;01} & a_{11;10} & \cdot & \cdot & a_{11;11} \end{array} \right) . \quad (56)$$

Let us note, that the density matrix defined by (25) is a special case of (56) where the matrices a and b are given by:

$$a = \left(\begin{array}{cc|cc} x_{00}^{(00)} & \cdot & \cdot & x_{01}^{(00)} \\ \cdot & x_{00}^{(11)} & x_{01}^{(11)} & \cdot \\ \cdot & x_{10}^{(11)} & x_{11}^{(11)} & \cdot \\ x_{10}^{(00)} & \cdot & \cdot & x_{11}^{(00)} \end{array} \right) , \quad (57)$$

and

$$b = \left(\begin{array}{cc|cc} x_{00}^{(01)} & \cdot & \cdot & x_{01}^{(01)} \\ \cdot & x_{00}^{(10)} & x_{01}^{(10)} & \cdot \\ \cdot & x_{10}^{(10)} & x_{11}^{(10)} & \cdot \\ x_{10}^{(01)} & \cdot & \cdot & x_{11}^{(01)} \end{array} \right) . \quad (58)$$

Similarly, a 3-qubit circulant state corresponding to

$$\Xi_0 \oplus \Xi_1 = \mathcal{H}_{2^3} ,$$

is defined by

$$\xi = \xi_0 + \xi_1 ,$$

with ξ_α supported on Ξ_α . It is defined via two 4×4 matrices c and d :

$$\xi_0 = \sum_{i,j=0}^1 \sum_{\mu,\nu=0}^1 c_{i\mu; j\nu} e_{ij} \otimes e_{\mu\nu} \otimes e_{ij} ,$$

$$\xi_1 = \sum_{i,j=0}^1 \sum_{\mu,\nu=0}^1 d_{i\mu; j\nu} e_{ij} \otimes e_{\mu\nu} \otimes S e_{ij} S^* . \quad (59)$$

One finds

$$\xi = \left(\begin{array}{cc|cc|cc} c_{00;00} & \cdot & c_{00;01} & \cdot & \cdot & c_{00;10} & \cdot & c_{00;11} \\ \cdot & d_{00;00} & \cdot & d_{00;01} & d_{00;10} & \cdot & d_{00;11} & \cdot \\ \hline c_{01;00} & \cdot & c_{01;01} & \cdot & \cdot & c_{01;10} & \cdot & c_{01;11} \\ \cdot & d_{01;00} & \cdot & d_{01;01} & d_{01;10} & \cdot & d_{01;11} & \cdot \\ \hline \cdot & d_{10;00} & \cdot & d_{10;01} & d_{10;10} & \cdot & d_{10;11} & \cdot \\ c_{10;00} & \cdot & c_{10;01} & \cdot & \cdot & c_{10;10} & \cdot & c_{10;11} \\ \hline \cdot & d_{11;00} & \cdot & d_{11;01} & d_{11;10} & \cdot & d_{11;11} & \cdot \\ c_{11;00} & \cdot & c_{11;01} & \cdot & \cdot & c_{11;10} & \cdot & c_{11;11} \end{array} \right). \quad (60)$$

It is clear that partially transposed $\tau_{\alpha\beta}\sigma$ and $\tau_{\alpha\beta}\xi$ have exactly the same circulant structure as original σ and ξ . One easily finds for the corresponding partial transpositions:

$$\begin{aligned} \tau_{\alpha\beta}\xi &= \sum_{i,j=0}^1 \sum_{\mu,\nu=0}^1 c_{i\mu;j\nu}^{[\alpha\beta]} e_{ij} \otimes e_{\mu\nu} \otimes e_{ij} \\ &+ \sum_{i,j=0}^1 \sum_{\mu,\nu=0}^1 d_{i\mu;j\nu}^{[\alpha\beta]} e_{ij} \otimes e_{\mu\nu} \otimes S e_{ij} S^*, \end{aligned} \quad (61)$$

where the new 4×4 matrices $c_{\mu i; \nu j}^{[\alpha\beta]}$ and $d_{\mu i; \nu j}^{[\alpha\beta]}$ are given by:

$$c^{[00]} = c, \quad d^{[00]} = d, \quad (62)$$

and

$$c^{[10]} = (\mathbb{1} \otimes \tau) c, \quad d^{[10]} = (\mathbb{1} \otimes \tau) d, \quad (63)$$

where

$$[(\mathbb{1} \otimes \tau) c]_{i\mu;j\nu} = c_{i\nu;j\mu}, \quad [(\mathbb{1} \otimes \tau) d]_{i\mu;j\nu} = d_{i\nu;j\mu}.$$

Moreover,

$$\begin{aligned} c^{[\alpha 1]} &= c^{[\alpha 0]} \circ (\mathbb{I} \otimes \tilde{\mathbb{I}}) + d^{[\alpha 0]} \circ (S \otimes \tilde{\mathbb{I}}), \\ d^{[\alpha 1]} &= d^{[\alpha 0]} \circ (\mathbb{I} \otimes \tilde{\mathbb{I}}) + c^{[\alpha 0]} \circ (S \otimes \tilde{\mathbb{I}}), \end{aligned} \quad (64)$$

where

$$\tilde{\mathbb{I}} = \mathbb{I} + S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (65)$$

The similar formulae one easily finds for $a^{[\alpha\beta]}$ and $b^{[\alpha\beta]}$ defined by

$$\begin{aligned} \tau_{\alpha\beta}\sigma &= \sum_{i,j=0}^1 \sum_{\mu,\nu=0}^1 a_{\mu i; \nu j}^{[\alpha\beta]} e_{\mu\nu} \otimes e_{ij} \otimes e_{ij} \\ &+ \sum_{i,j=0}^1 \sum_{\mu,\nu=0}^1 b_{\mu i; \nu j}^{[\alpha\beta]} e_{\mu\nu} \otimes e_{ij} \otimes S e_{ij} S^*. \end{aligned} \quad (66)$$

Theorem 2 *A circulant 3-qubit state σ is $(\alpha\beta)$ -PPT iff $a^{[\alpha\beta]}$ and $b^{[\alpha\beta]}$ are semi-positive matrices. Similarly, a circulant 3-qubit state ξ is $(\alpha\beta)$ -PPT iff $c^{[\alpha\beta]}$ and $d^{[\alpha\beta]}$ are semi-positive matrices.*

Let us observe that a 3-qubit circulant state may be reduced to the 2-qubit circulant state. Consider for example a density operator ξ defined in (59). Note that reduction with respect to the second factor gives

$$\begin{aligned} \text{Tr}_2 \xi_0 &= \sum_{i,j=0}^1 \sum_{\mu=0}^1 c_{i\mu;j\mu} e_{ij} \otimes e_{ij} \\ &= \sum_{i,j=0}^1 (\text{Tr}_2 c)_{ij} e_{ij} \otimes e_{ij}, \end{aligned} \quad (67)$$

$$\begin{aligned} \text{Tr}_2 \xi_1 &= \sum_{i,j=0}^1 \sum_{\mu=0}^1 d_{i\mu;j\mu} e_{ij} \otimes S e_{ij} S^* \\ &= \sum_{i,j=0}^1 (\text{Tr}_2 d)_{ij} e_{ij} \otimes S e_{ij} S^*, \end{aligned} \quad (68)$$

and hence

$$\text{Tr}_2 \xi = \text{Tr}_2 \xi_0 + \text{Tr}_2 \xi_1, \quad (69)$$

is a 2-qubit circulant state. It is no longer true for the remaining reductions with respect to the first and third factors. One obtains

$$\text{Tr}_1 \xi = \sum_{i=0}^1 \sum_{\mu,\nu=0}^1 [c + d]_{i\mu;i\nu} e_{\mu\nu} \otimes e_{ii}, \quad (70)$$

and

$$\text{Tr}_3 \xi = \sum_{i=0}^1 \sum_{\mu,\nu=0}^1 [c + d]_{i\mu;i\nu} e_{ii} \otimes e_{\mu\nu}, \quad (71)$$

which are not circulant states.

C. N qubits

Consider now a general case of N qubits living in $(\mathbb{C}^d)^{\otimes N}$. Again, there are two natural ways to decompose the corresponding Hilbert space \mathcal{H}_{2^N} : either into 2^{N-1} two-dimensional subspaces or into two 2^{N-1} -dimensional subspaces.

1. “ $2^N = 2 \oplus 2 \oplus \dots \oplus 2$ ”

Let us introduce a circulant decomposition of \mathcal{H}_{2^N} into 2^{N-1} two-dimensional subspaces. Now each integer from the set $\{0, 1, \dots, 2^{N-1} - 1\}$ may be represented by a string of $N - 1$ binaries $(\mu_1 \dots \mu_{N-1})$. Let us define 2-dimensional subspace

$$\Delta_{0\dots 0} = \text{span} \{e_0 \otimes \dots \otimes e_0, e_1 \otimes \dots \otimes e_1\}, \quad (72)$$

and for any string of binaries $(\mu_1 \dots \mu_{N-1})$ define

$$\Delta_{\mu_1 \dots \mu_{N-1}} = (\mathbb{1} \otimes S^{\mu_1} \otimes \dots \otimes S^{\mu_{N-1}}) \Delta_{0\dots 0}. \quad (73)$$

Introducing convenient vector notation

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_{N-1}),$$

one has

$$\Delta_{\boldsymbol{\mu}} = (\mathbb{1} \otimes S^{\boldsymbol{\mu}}) \Delta_{\mathbf{0}}, \quad (74)$$

with

$$S^{\boldsymbol{\mu}} = S^{\mu_1} \otimes \dots \otimes S^{\mu_{N-1}},$$

and $\Delta_{\mathbf{0}} = \Delta_{0\dots 0}$. One clearly has

$$\mathcal{H}_{2^N} = \bigoplus_{\boldsymbol{\mu}} \Delta_{\boldsymbol{\mu}}, \quad (75)$$

where the sum runs over all binary $(N - 1)$ -vectors $\boldsymbol{\mu}$.

Now, let us construct a circulant N -qubit state ρ based on (75):

$$\rho = \sum_{\boldsymbol{\mu}} \rho_{\boldsymbol{\mu}}, \quad (76)$$

where each $\rho_{\boldsymbol{\mu}}$ is supported on $\Delta_{\boldsymbol{\mu}}$. One has therefore

$$\rho_{\boldsymbol{\mu}} = (\mathbb{1} \otimes S^{\boldsymbol{\mu}}) \left[\sum_{i,j=0}^1 x_{ij}^{(\boldsymbol{\mu})} e_{ij} \otimes \dots \otimes e_{ij} \right] (\mathbb{1} \otimes S^{\boldsymbol{\mu}})^*, \quad (77)$$

where $[x^{(\boldsymbol{\mu})}]$ are 2×2 semi-positive matrices. Normalization of ρ implies

$$\sum_{\boldsymbol{\mu}} \text{Tr} x^{(\boldsymbol{\mu})} = 1. \quad (78)$$

Now, partial transpositions are labeled by a binary $(N - 1)$ -vectors $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_{N-1})$

$$\tau_{\boldsymbol{\sigma}} = \mathbb{1} \otimes \tau^{\sigma_1} \otimes \dots \otimes \tau^{\sigma_{N-1}}. \quad (79)$$

Note, that each partial transposition $\tau_{\boldsymbol{\sigma}} \rho$ belongs to the same class of circulant states

$$\tau_{\boldsymbol{\sigma}} \rho = \sum_{\boldsymbol{\mu}} \rho_{\boldsymbol{\mu}}^{[\boldsymbol{\sigma}]}, \quad (80)$$

with

$$\rho_{\boldsymbol{\mu}}^{[\boldsymbol{\sigma}]} = (\mathbb{1} \otimes S^{\boldsymbol{\mu}}) \left[\sum_{i,j=0}^1 y_{ij}^{(\boldsymbol{\mu})[\boldsymbol{\sigma}]} e_{ij} \otimes \dots \otimes e_{ij} \right] (\mathbb{1} \otimes S^{\boldsymbol{\mu}})^*, \quad (81)$$

where the new 2×2 matrices $y^{(\boldsymbol{\mu})[\boldsymbol{\sigma}]}$ are given by the following formula

$$y^{(\boldsymbol{\mu})[\boldsymbol{\sigma}]} = x^{[\boldsymbol{\mu}]} \circ \mathbb{1} + x^{[\boldsymbol{\mu} + \boldsymbol{\sigma}]} \circ S. \quad (82)$$

A state ρ is $\boldsymbol{\sigma}$ -PPT iff $\tau_{\boldsymbol{\sigma}} \rho \geq 0$ and hence one has

Theorem 3 *A circulant state ρ is $\boldsymbol{\sigma}$ -PPT iff $y^{(\boldsymbol{\mu})[\boldsymbol{\sigma}]}$ are semi-positive for all $\boldsymbol{\mu}$.*

Examples. 1. Generalized GHZ state [17]

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\dots 0\rangle + |1\dots 1\rangle), \quad (83)$$

does belong to circulant class which is easily seen from the corresponding density matrix

$$x^{(\boldsymbol{\mu})} = \delta_{\boldsymbol{\mu}, \boldsymbol{\nu}} \tilde{\mathbb{1}}. \quad (84)$$

2. Generalized Bell states: the following 2^N vectors

$$\psi_{\alpha \boldsymbol{\nu}} = (-1)^\alpha (\mathbb{1} \otimes S^{\boldsymbol{\nu}}) |\text{GHZ}\rangle, \quad (85)$$

with $\alpha = 0, 1$ and binary $(N - 1)$ -vector $\boldsymbol{\nu}$ define circulant states. These are N -qubit generalization of 2-qubit Bell states. Note that

$$\psi_{\alpha \boldsymbol{\nu}} \in \Delta_{\boldsymbol{\nu}}, \quad (86)$$

and the corresponding matrices $x^{(\boldsymbol{\mu})}$ read as follows

$$x^{(\boldsymbol{\mu})} = \delta_{\boldsymbol{\mu}, \boldsymbol{\nu}} \begin{pmatrix} 1 & (-1)^\alpha \\ (-1)^\alpha & 1 \end{pmatrix}. \quad (87)$$

3. Generalized N -qubit isotropic state

$$\rho = \frac{1-s}{2^N} \mathbb{I}^{\otimes N} + s |\text{GHZ}\rangle\langle\text{GHZ}| \quad (88)$$

with $s \in [-1/(2^N - 1), 1]$. One finds for $x^{(\boldsymbol{\mu})}$ matrices

$$x^{(\mathbf{0})} = \frac{1}{2^N} \begin{pmatrix} 1 + (2^N - 1)s & 2^{N-1}s \\ 2^{N-1}s & 1 + (2^N - 1)s \end{pmatrix}, \quad (89)$$

and

$$x^{(\boldsymbol{\mu})} = \frac{1-s}{2^N} \mathbb{I}, \quad (90)$$

for $\boldsymbol{\mu} \neq \mathbf{0}$. The only nontrivial PPT condition comes from the positivity of

$$\begin{pmatrix} 1-s & 2^{N-1}s \\ 2^{N-1}s & 1-s \end{pmatrix}, \quad (91)$$

which implies

$$s \leq \frac{1}{2^{N-1} + 1}.$$

The above condition guaranties full N -separability of ρ [18].

4. 2-parameter N -qubit state from [20]: for $-1/2^N \leq c, d \leq 1/2^N$ one defines a set of matrices $x^{(\boldsymbol{\mu})}$

$$x^{(\boldsymbol{\mu})} = \frac{1}{2^N} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (92)$$

for $\boldsymbol{\mu}$ corresponding to binary representation of $\{0, 1, \dots, 2^{N-2} - 1\}$,

$$x^{(\boldsymbol{\mu})} = \frac{1}{2^N} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, \quad (93)$$

for $\boldsymbol{\mu}$ corresponding to binary representation of $\{2^{N-2} - 1, \dots, 2^{N-1} - 1\}$, and

$$x^{(\boldsymbol{\mu})} = \frac{1}{2^N} \begin{pmatrix} 1 & d \\ d & 1 \end{pmatrix}, \quad (94)$$

for $\boldsymbol{\mu}$ corresponding to binary representation of $\{2^{N-1} - 1, \dots, 2^N - 1\}$. It generalizes a 3-qubit state defined in (42). It was shown [20] that the above N -qubit circulant state has positive partial transposes iff $c = d$. Moreover, this condition implies full separability.

2. “ $2^N = 2^{N-1} \oplus 2^{N-1}$ ”

There are several ways to construct 2^{N-1} -dimensional circulant decompositions of N -qubit Hilbert space out of 2-dimensional mutually orthogonal spaces $\Delta_{\boldsymbol{\mu}}$. The following choice

$$\Sigma_0 = \bigoplus_{|\boldsymbol{\mu}|=0} \Delta_{\boldsymbol{\mu}}, \quad (95)$$

$$\Sigma_1 = \bigoplus_{|\boldsymbol{\mu}|=1} \Delta_{\boldsymbol{\mu}}, \quad (96)$$

where

$$|\boldsymbol{\mu}| = \mu_1 + \dots + \mu_{N-2},$$

gives rise to the circulant structure

$$\Sigma_1 = (\mathbb{1}^{\otimes N-1} \otimes S) \Sigma_0. \quad (97)$$

Another construction goes as follows

$$\Xi_{(\alpha|k)} = \bigoplus_{\boldsymbol{\mu}}^{(\alpha|k)} \Delta_{\boldsymbol{\mu}}, \quad \alpha = 0, 1, \quad k = 1, 2, \dots, N-1, \quad (98)$$

where the sum $\bigoplus^{(\alpha|k)}$ runs over all $\boldsymbol{\mu}$ with $\mu_k = \alpha$. Note, that $\Xi_{(\alpha|k)}$ displays circulant structure defined by

$$\Xi_{(1|k)} = (\mathbb{1}^{\otimes k} \otimes S \otimes \mathbb{1}^{\otimes N-k-1}) \Xi_{(0|k)}. \quad (99)$$

We shall consider only one scheme with $k = N-1$ and to simplify notation let us define

$$\Xi_{\alpha} := \Xi_{(\alpha|N-1)}, \quad \alpha = 0, 1, \quad (100)$$

which satisfies

$$\Xi_1 = (\mathbb{1}^{\otimes N-1} \otimes S) \Xi_0. \quad (101)$$

This very choice has clear interpretation: to define Ξ_{α} we sum over all $\boldsymbol{\mu} = (\mu_1 \dots \mu_{N-1})$ which represent binary code for $\alpha \pmod{2}$.

Now, let us construct a circulant state

$$\sigma = \sigma_0 + \sigma_1,$$

with σ_{α} supported on Σ_{α} . It is clear that

$$\sigma_0 = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \sum_{i,j=0}^1 a_{\boldsymbol{\alpha}i; \boldsymbol{\beta}j} \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes e_{ij} \otimes e_{ij}, \quad (102)$$

$$\sigma_1 = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \sum_{i,j=0}^1 b_{\boldsymbol{\alpha}i; \boldsymbol{\beta}j} \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes e_{ij} \otimes S e_{ij} S^* \quad (103)$$

where α and β are binary $(N-2)$ -vectors with coordinates α_k and β_k , respectively, and $[a_{\alpha i; \beta j}]$ and $[b_{\alpha i; \beta j}]$ are $2^{N-1} \times 2^{N-1}$ semi-positive matrices.

Similarly, one constructs a circulant state

$$\xi = \xi_0 + \xi_1 ,$$

with ξ_α supported on Ξ_α . It is clear that

$$\begin{aligned} \xi_0 &= \sum_{\alpha, \beta} \sum_{i, j=0}^1 c_{i\alpha; j\beta} e_{ij} \otimes \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes e_{ij} , \quad (104) \\ \xi_1 &= \sum_{\alpha, \beta} \sum_{i, j=0}^1 d_{i\alpha; j\beta} e_{ij} \otimes \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes S e_{ij} S^* \end{aligned} \quad (105)$$

with $2^{N-1} \times 2^{N-1}$ semi-positive matrices $[c_{i\alpha; j\beta}]$ and $[d_{i\alpha; j\beta}]$.

Now, let us consider partially transposed N -qubit circulant operators. The corresponding partial transpositions are labeled by binary $(N-1)$ -vectors

$$\tau_\sigma := \mathbb{1} \otimes \tau^{\sigma_1} \otimes \dots \otimes \tau^{\sigma_{N-1}} . \quad (106)$$

Note, that both $\tau_\sigma \sigma$ and $\tau_\sigma \xi$ have exactly the same circulant structure as original σ and ξ . One easily finds for the corresponding partial transpositions:

$$\tau_\sigma \xi = \xi_0^{[\sigma]} + \xi_1^{[\sigma]} ,$$

where $\xi_\alpha^{[\sigma]}$ are again supported on Ξ_α :

$$\begin{aligned} \xi_0^{[\sigma]} &= \sum_{\alpha, \beta} \sum_{i, j=0}^1 c_{i\alpha; j\beta}^{[\sigma]} e_{ij} \otimes \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes e_{ij} , \quad (107) \\ \xi_1^{[\sigma]} &= \sum_{\alpha, \beta} \sum_{i, j=0}^1 d_{i\alpha; j\beta}^{[\sigma]} e_{ij} \otimes \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes S e_{ij} S^* \end{aligned} \quad (108)$$

with the new matrices $[c_{i\alpha; j\beta}^{[\sigma]}]$ and $[d_{i\alpha; j\beta}^{[\sigma]}]$ which are defined by the following formulae:

$$c^{[0]} = c , \quad d^{[0]} = d , \quad (109)$$

and

$$c^{[\gamma^0]} = \tau_\gamma c , \quad d^{[\gamma^0]} = \tau_\gamma d , \quad (110)$$

where γ is binary $(N-2)$ -vector and we treat c and d as a matrices living in the tensor product $M_2^{\otimes N-1}$. It is therefore clear that $\tau_\gamma c$ denotes the corresponding partial transposition of c in the tensor product $M_2^{\otimes N-1}$. Moreover,

$$\begin{aligned} c^{[\gamma^1]} &= c^{[\gamma^0]} \circ (\mathbb{I} \otimes \tilde{\mathbb{I}}^{\otimes N-2}) + d^{[\gamma^0]} \circ (S \otimes \tilde{\mathbb{I}}^{\otimes N-2}) , \\ d^{[\gamma^1]} &= d^{[\gamma^0]} \circ (\mathbb{I} \otimes \tilde{\mathbb{I}}^{\otimes N-2}) + c^{[\gamma^0]} \circ (S \otimes \tilde{\mathbb{I}}^{\otimes N-2}) . \end{aligned} \quad (111)$$

Theorem 4 *A circulant N -qubit state ξ is σ -PPT iff $c^{[\sigma]}$ and $d^{[\sigma]}$ are semi-positive matrices. Similarly, a circulant N -qubit state σ is σ -PPT iff $a^{[\sigma]}$ and $b^{[\sigma]}$ are semi-positive matrices.*

Let us observe that an N -qubit circulant state may be easily reduced to the $(N-L)$ -qubit circulant state (with $N-L \geq 2$): let l_1, \dots, l_L denote L distinct integers from the set $\{2, 3, \dots, N-1\}$. Then the partial trace $\text{Tr}_{l_1 \dots l_L} \xi$ defines $(N-L)$ -qubit circulant state with new $2^{N-L-1} \times 2^{N-L-1}$ matrices c' and d' defined by

$$c' = \text{Tr}_{l_1 \dots l_L} c , \quad d' = \text{Tr}_{l_1 \dots l_L} d . \quad (112)$$

III. N -QUDIT STATE

Consider now the most general case of N qudits living in $(\mathbb{C}^d)^{\otimes N}$. Again, there are two natural ways to decompose the corresponding Hilbert space \mathcal{H}_{d^N} : either into d^{N-1} — d -dimensional subspaces, or into $d - d^{N-1}$ -dimensional subspaces.

A. “ $d^N = d \oplus d \oplus \dots \oplus d$ ”

Let us introduce a circulant decomposition of \mathcal{H}_{d^N} into d^{N-1} two-dimensional subspaces. Now each integer from the set $\{0, 1, \dots, d^{N-1} - 1\}$ may be represented by a string of $N-1$ dinaries $(\mu_1 \dots \mu_{N-1})$, i.e. each $\mu_k \in \{0, 1, \dots, d-1\}$. Let us define d -dimensional subspace

$$\Delta_{0 \dots 0} = \text{span} \{e_0 \otimes \dots \otimes e_0 \dots, e_{d-1} \otimes \dots \otimes e_{d-1}\} , \quad (113)$$

and for any string of dinaries $(\mu_1 \dots \mu_{N-1})$ define

$$\Delta_{\mu_1 \dots \mu_{N-1}} = (\mathbb{I} \otimes S^{\mu_1} \otimes \dots \otimes S^{\mu_{N-1}}) \Delta_{0 \dots 0} . \quad (114)$$

Introducing convenient vector notation

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_{N-1}) ,$$

one has

$$\Delta_{\boldsymbol{\mu}} = (\mathbb{I} \otimes S^{\boldsymbol{\mu}}) \Delta_{\mathbf{0}} , \quad (115)$$

with

$$S^{\boldsymbol{\mu}} = S^{\mu_1} \otimes \dots \otimes S^{\mu_{N-1}} ,$$

and $\Delta_{\mathbf{0}} = \Delta_{0 \dots 0}$. One clearly has

$$\mathcal{H}_{d^N} = \bigoplus_{\boldsymbol{\mu}} \Delta_{\boldsymbol{\mu}} , \quad (116)$$

where the sum runs over all dinary $(N-1)$ -vectors $\boldsymbol{\mu}$.

Now, let us construct a circulant N -qubit state ρ based on (123):

$$\rho = \sum_{\boldsymbol{\mu}} \rho_{\boldsymbol{\mu}}, \quad (117)$$

where each $\rho_{\boldsymbol{\mu}}$ is supported on $\Delta_{\boldsymbol{\mu}}$. One has therefore

$$\rho_{\boldsymbol{\mu}} = (\mathbb{1} \otimes S^{\boldsymbol{\mu}}) \left[\sum_{i,j=0}^{d-1} x_{ij}^{(\boldsymbol{\mu})} e_{ij} \otimes \dots \otimes e_{ij} \right] (\mathbb{1} \otimes S^{\boldsymbol{\mu}})^*, \quad (118)$$

where $[x^{(\boldsymbol{\mu})}]$ are $d \times d$ semi-positive matrices. Normalization of ρ implies

$$\sum_{\boldsymbol{\mu}} \text{Tr } x^{(\boldsymbol{\mu})} = 1. \quad (119)$$

Now, let us look for the corresponding partial transpositions $\tau_{\boldsymbol{\sigma}}\rho$ with $\tau_{\boldsymbol{\sigma}}$ introduced in (79). There is a crucial difference between qubit and qudit case: for qubits partially transposed state have exactly the same structure as the original one. It is no longer true for qudits. It was shown in [5] that partial transposition gives rise to a new circulant structure governed by a certain permutation: let Π be a $d \times d$ permutation matrix defined by

$$\Pi e_0 = e_0, \quad \Pi e_k = e_{d-k}, \quad (120)$$

for $k = 1, \dots, d-1$. It turns out [5] that partially transposed matrix $\tau_{\boldsymbol{\sigma}}\rho$ is related to the following circulant structure:

$$\Delta_{\boldsymbol{\mu}}^{[\boldsymbol{\sigma}]} = (\mathbb{1} \otimes S^{\boldsymbol{\mu}}) \Delta_{\mathbf{0}}^{[\boldsymbol{\sigma}]}, \quad (121)$$

where

$$\Delta_{\mathbf{0}}^{[\boldsymbol{\sigma}]} = (\mathbb{1} \otimes \Pi^{\boldsymbol{\sigma}}) \Delta_{\mathbf{0}}, \quad (122)$$

and

$$\Pi^{\boldsymbol{\sigma}} = \Pi^{\sigma_1} \otimes \dots \otimes \Pi^{\sigma_{N-1}}.$$

One clearly has

$$\mathcal{H}_{d^N} = \bigoplus_{\boldsymbol{\mu}} \Delta_{\boldsymbol{\mu}}^{[\boldsymbol{\sigma}]}, \quad (123)$$

for each binary $(N-1)$ -vector $\boldsymbol{\sigma}$.

One finds therefore the following $\boldsymbol{\sigma}$ -circulant structure for $\tau_{\boldsymbol{\sigma}}\rho$

$$\tau_{\boldsymbol{\sigma}}\rho = \sum_{\boldsymbol{\mu}} \rho_{\boldsymbol{\mu}}^{[\boldsymbol{\sigma}]}, \quad (124)$$

with

$$\rho_{\boldsymbol{\mu}}^{[\boldsymbol{\sigma}]} = (\mathbb{1} \otimes S^{\boldsymbol{\mu}}) (\mathbb{1} \otimes \Pi^{\boldsymbol{\sigma}}) \left[\sum_{i,j=0}^{d-1} y_{ij}^{(\boldsymbol{\mu})[\boldsymbol{\sigma}]} e_{ij} \otimes \dots \otimes e_{ij} \right] (\mathbb{1} \otimes \Pi^{\boldsymbol{\sigma}})^* (\mathbb{1} \otimes S^{\boldsymbol{\mu}})^*, \quad (125)$$

where the new $d \times d$ matrices $y^{(\boldsymbol{\mu})[\boldsymbol{\sigma}]}$ are given by the following formula

$$y^{(\boldsymbol{\mu})[\boldsymbol{\sigma}]} = \sum_{k=0}^{d-1} x^{[\boldsymbol{\mu}+k\boldsymbol{\sigma}]} \circ (\Pi \cdot S^k). \quad (126)$$

For $d = 2$ one finds $\Pi = \mathbb{I}$ and the above sum reduces to two terms, only. One therefore recovers (82).

Theorem 5 *A circulant state ρ is $\boldsymbol{\sigma}$ -PPT iff $y^{(\boldsymbol{\mu})[\boldsymbol{\sigma}]}$ are semi-positive for all dinary $\boldsymbol{\mu}$.*

Examples. 1. Generalized GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e_k \otimes \dots \otimes e_k, \quad (127)$$

does belong to circulant class which is easily seen from the corresponding density matrix with

$$x^{(\boldsymbol{\mu})} = \delta_{\boldsymbol{\mu}, \mathbf{0}} \tilde{\mathbb{I}}, \quad (128)$$

where

$$\tilde{\mathbb{I}} = \sum_{\alpha=0}^{d-1} S^{\alpha}, \quad (129)$$

generalizes (65), that is, $\tilde{\mathbb{I}}_{ij} = 1$ for all $i, j = 0, 1, \dots, d-1$.

2. Generalized Bell states: the following d^N vectors

$$\psi_{\alpha\nu} = (\Omega^\alpha \otimes S^\nu)|\text{GHZ}\rangle, \quad (130)$$

where the phase operator Ω is defined via

$$\Omega e_k = \omega^k e_k, \quad k = 0, 1, \dots, d-1, \quad (131)$$

with $\omega = e^{2\pi i/d}$, define a circulant state for any $\alpha = 0, 1, \dots, d-1$ and arbitrary dinary $(N-1)$ -vector ν . These are N -qudit generalization of N -qubit Bell states (85). Note that

$$\psi_{\alpha\nu} \in \Delta_\nu, \quad (132)$$

and the corresponding matrices $x^{(\mu)}$ read as follows

$$x^{(\mu)} = \delta_{\mu,\nu} \tilde{\Omega}, \quad (133)$$

where the $d \times d$ matrix $\tilde{\Omega}$ is defined by

$$\tilde{\Omega}_{ij} = \omega^{j-i}, \quad (134)$$

and generalizes a 2×2 matrix from (87).

3. Generalized N -qudit isotropic state

$$\rho = \frac{1-s}{d^N} \mathbb{I}^{\otimes N} + s |\text{GHZ}\rangle\langle\text{GHZ}| \quad (135)$$

with $s \in [-1/(d^N - 1), 1]$. One finds for $x^{(\mu)}$ matrices

$$x^{(0)} = \frac{1}{d^N} \begin{pmatrix} 1 + (d^N - 1)s & d^{N-1}s \\ d^{N-1}s & 1 + (d^N - 1)s \end{pmatrix}, \quad (136)$$

and

$$x^{(\mu)} = \frac{1-s}{d^N} \mathbb{I}, \quad (137)$$

for $\mu \neq \mathbf{0}$. The only nontrivial PPT condition comes from the positivity of

$$\begin{pmatrix} 1-s & d^{N-1}s \\ d^{N-1}s & 1-s \end{pmatrix}, \quad (138)$$

which implies

$$s \leq \frac{1}{d^{N-1} + 1}.$$

The above condition guaranties full N -separability of ρ [18].

B. “ $d^N = d^{N-1} \oplus d^{N-1}$ ”

There are several ways to construct d^{N-1} -dimensional circulant decompositions of N -qudit Hilbert space out of d -dimensional mutually orthogonal spaces Δ_μ . The following choice

$$\Sigma_\alpha = \bigoplus_{|\mu|=\alpha} \Delta_\mu, \quad \alpha = 0, 1, \dots, d-1 \quad (139)$$

gives rise to the circulant structure

$$\Sigma_\alpha = (\mathbb{I}^{\otimes N-1} \otimes S^\alpha) \Sigma_0. \quad (140)$$

Another construction goes as follows

$$\Xi_{(\alpha|k)} = \bigoplus_{\mu}^{(\alpha|k)} \Delta_\mu, \quad (141)$$

for $\alpha = 0, 1, \dots, d-1$ and $k = 1, 2, \dots, N-1$. In the above formula the sum $\bigoplus^{(\alpha|k)}$ runs over all μ with $\mu_k = \alpha$. Note, that $\Xi_{(\alpha|k)}$ displays circulant structure defined by

$$\Xi_{(\alpha|k)} = (\mathbb{I}^{\otimes k} \otimes S^\alpha \otimes \mathbb{I}^{\otimes N-k-1}) \Xi_{(0|k)}. \quad (142)$$

We shall consider only one scheme with $k = N-1$ and to simplify notation let us define

$$\Xi_\alpha := \Xi_{(\alpha|N-1)}, \quad \alpha = 0, 1, \dots, d-1, \quad (143)$$

which satisfies

$$\Xi_\alpha = (\mathbb{I}^{\otimes N-1} \otimes S^\alpha) \Xi_0. \quad (144)$$

This very choice has clear interpretation: to define Ξ_α we sum over all $\mu = (\mu_1 \dots \mu_{N-1})$ which represent dinary code for $\alpha \pmod{d}$.

Now, let us construct a circulant state

$$\sigma = \sum_{\alpha=0}^{d-1} \sigma_\alpha,$$

with σ_α supported on Σ_α . It is clear that

$$\sigma_\alpha = \sum_{\alpha, \beta} \sum_{i, j=0}^{d-1} a_{\alpha i; \beta j}^{(\alpha)} \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes e_{ij} \otimes S^\alpha e_{ij} S^{\alpha*} \quad (145)$$

where α and β are dinary $(N-2)$ -vectors with coordinates α_k and β_k , respectively, and $[a_{\alpha i; \beta j}^{(\alpha)}]$ is a set of $d^{N-1} \times d^{N-1}$ semi-positive matrices. This set generalizes two matrices $a = a^{(0)}$ and $b = a^{(1)}$ in the qubit case, i.e. $d = 2$.

Similarly, one constructs a circulant state

$$\xi = \sum_{\alpha=0}^{d-1} \xi_{\alpha} ,$$

with ξ_{α} supported on Ξ_{α} . It is clear that

$$\xi_{\alpha} = \sum_{\alpha, \beta} \sum_{i, j=0}^{d-1} c_{i\alpha; j\beta}^{(\alpha)} e_{ij} \otimes \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes S^{\alpha} e_{ij} S^{\alpha*} \quad (146)$$

with $d^{N-1} \times d^{N-1}$ semi-positive matrices $[c_{i\alpha; j\beta}^{(\alpha)}]$.

Now, each partial transposition τ_{σ} gives rise to the new circulant structure: either

$$\Sigma_{\alpha}^{[\sigma]} = \bigoplus_{|\mu|=\alpha} \Delta_{\mu}^{[\sigma]} , \quad \alpha = 0, 1, \dots, d-1 \quad (147)$$

with the cyclic property

$$\Sigma_{\alpha}^{[\sigma]} = (\mathbb{1}^{\otimes N-1} \otimes S^{\alpha}) \Sigma_0^{[\sigma]} , \quad (148)$$

or

$$\Xi_{\alpha}^{[\sigma]} = \bigoplus_{\mu}^{(\alpha|N-1)} \Delta_{\mu}^{[\sigma]} , \quad (149)$$

with the same property, that is,

$$\Xi_{\alpha}^{[\sigma]} = (\mathbb{1}^{\otimes N-1} \otimes S^{\alpha}) \Xi_0^{[\sigma]} . \quad (150)$$

One easily finds for the corresponding partial transpositions:

$$\tau_{\sigma} \xi = \sum_{\alpha=0}^{d-1} \xi_0^{[\sigma]} ,$$

where $\xi_{\alpha}^{[\sigma]}$ are supported on $\Xi_{\alpha}^{[\sigma]}$:

$$\xi_{\alpha}^{[\sigma]} = \sum_{\alpha, \beta} \sum_{i, j=0}^{d-1} a_{i\alpha; j\beta}^{(\alpha)[\sigma]} e_{ij} \otimes \bigotimes_{k=1}^{N-2} e_{\alpha_k \beta_k} \otimes S^{\alpha} \Pi e_{ij} \Pi^* S^{\alpha*} , \quad (151)$$

with new set of matrices $[c_{i\alpha; j\beta}^{(\alpha)[\sigma]}]$ which are defined by the following formulae:

$$c^{(\alpha)[0]} = c^{(\alpha)} , \quad (152)$$

and

$$c^{(\alpha)[\gamma 0]} = \tau_{\gamma} c^{(\alpha)} , \quad (153)$$

where γ is binary $(N-2)$ -vector and we treat $c^{(\alpha)}$ matrices living in the tensor product $M_d^{\otimes N-1}$. It is therefore clear that $\tau_{\gamma} c^{(\alpha)}$ denotes the corresponding partial transposition of $c^{(\alpha)}$ in the tensor product $M_d^{\otimes N-1}$. Moreover,

$$c^{(\alpha)[\gamma 1]} = \sum_{\beta=0}^{d-1} c^{(\alpha+\beta)[\gamma 0]} \circ (\Pi S^{\beta} \otimes \tilde{\mathbb{I}}^{\otimes N-2}) . \quad (154)$$

Theorem 6 *A circulant N -qudit state ξ is σ -PPT iff $c^{(\alpha)[\sigma]}$ are semi-positive matrices for $\alpha = 0, 1, \dots, d-1$.*

IV. CONCLUSIONS

We have constructed a large class of PPT states which correspond to circular decompositions of $\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d$ into direct sums of d - and d^{N-1} -dimensional subspaces. This class generalizes bipartite circulant states introduced in [5]. It contains several known examples from the literature and produces a highly nontrivial family of new states.

There are several open problems related to this new class: the basic question is how to detect entanglement within this class of multipartite states. One may expect that there is special class of entanglement witnesses which are sensitive to entanglement encoded into circular decompositions, that is, circulant Hermitian operators $W \in M_d^{\otimes N}$ such that

$$\text{Tr}(W \rho_1 \otimes \dots \otimes \rho_N) \geq 0 , \quad (155)$$

for all product states $\rho_1 \otimes \dots \otimes \rho_N$, and

$$\text{Tr}(W \xi) < 0 , \quad (156)$$

for some circulant state ξ . It is interesting to explore the possibility of other decompositions leading to new classes of multipartite states. Let us note, that so called W state of 3 qubits (34) does not belong to our class. Another important family of states which does not fit circulant class was introduced in [12]: these are N -qudit states satisfying

$$U \otimes \dots \otimes U \rho = \rho U \otimes \dots \otimes U , \quad (157)$$

for all unitaries $U \in U(d)$. For $N = 2$ it reduces to the Werner state [14] which belongs to bipartite circulant class. However, it is easy to check that for $N \geq 3$ states satisfying (157) are not circulant. One may expect the existence of other characteristic decompositions which are responsible for the structure of symmetric states governed by (157). Anyway, multipartite

circulant states introduced in this paper may shed new light on the more general investigation of multipartite entanglement.

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