Spectral conditions for positive maps

Dariusz Chruściński and Andrzej Kossakowski
Institute of Physics, Nicolaus Copernicus University,
Grudziądzka 5/7, 87–100 Toruń, Poland

Abstract

We provide a partial classification of positive linear maps in matrix algebras which is based on a family of spectral conditions. This construction generalizes celebrated Choi example of a map which is positive but not completely positive. It is shown how the spectral conditions enable one to construct linear maps on tensor products of matrix algebras which are positive but only on a convex subset of separable elements. Such maps provide basic tools to study quantum entanglement in multipartite systems.

1 Introduction

One of the most important problems of quantum information theory [1] is the characterization of mixed states of composed quantum systems. In particular it is of primary importance to test whether a given quantum state exhibits quantum correlation, i.e. whether it is separable or entangled. For low dimensional systems there exists simple necessary and sufficient condition for separability. The celebrated Peres-Horodecki criterium [2, 3] states that a state of a bipartite system living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable iff its partial transpose is positive. Unfortunately, for higher-dimensional systems there is no single *universal* separability condition.

It turns out that the above problem may be reformulated in terms of positive linear maps in operator algebras: a state ρ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is separable iff $(\mathrm{id} \otimes \varphi)\rho$ is positive for any positive map φ which sends positive operators on $\mathcal{H}_2$ into positive operators on $\mathcal{H}_1$. Therefore, a classification of positive linear maps between operator algebras $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ is of primary importance. Unfortunately, in spite of the considerable effort, the structure of positive maps is rather poorly understood [4]–[26]. Positive maps play important role both in physics and mathematics providing generalization of $*$-homomorphism, Jordan homomorphism and conditional expectation. Normalized positive maps define an affine mapping between sets of states of $\mathbb{C}^*$-algebras.

In the present paper we perform partial classification of positive linear maps which is based on spectral conditions. Actually, presented method enables one to construct maps with a desired degree of positivity — so called k-positive maps with $k = 1, 2, \ldots, d = \min\{\dim \mathcal{H}_1, \dim \mathcal{H}_2\}$. Completely positive (CP) maps correspond to d-positive maps, i.e. maps with the highest degree of positivity. These maps are fully classified due to Stinespring theorem [27, 28]. Now, any positive map which is not CP can be written as $\varphi = \varphi_+ - \varphi_-$, with $\varphi_\pm$ being CP maps. However, there is no general method to recognize the positivity of φ from $\varphi_+ - \varphi_-$. We show that suitable spectral conditions satisfied by a pair (φ_+, φ_-) guarantee k-positivity of $\varphi_+ - \varphi_-$. This construction generalizes celebrated Choi example of a map which is $(d-1)$-positive but not CP [6].

From the physical point of view our method leads to partial classification of entanglement witnesses. Recall, that en entanglement witness is a Hermitian operator $W \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ which is not positive but satisfies $(h_1 \otimes h_2, W\, h_1 \otimes h_2) \geq 0$ for any $h_i \in \mathcal{H}_i$.

Interestingly, our construction may be easily generalized for multipartite case, i.e. for constructing entanglement witnesses in $\mathcal{B}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n)$. Translated into language of linear maps from $\mathcal{B}(\mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n)$ into $\mathcal{B}(\mathcal{H}_1)$ presented method enables one to construct maps which are not positive but which are positive when restricted to separable elements in $\mathcal{B}(\mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n)$. To the best of our knowledge we provide the first nontrivial example of such a map (nontrivial means that it is not a tensor product of positive maps).

2 Preliminaries

Consider a space $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ of linear operators $a : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, or equivalently a space of $d_1 \times d_2$ matrices, where $d_i = \dim \mathcal{H}_i < \infty$. Let us recall that $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is equipped with a family of Ky Fan k-norms [29]: for any $a \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ one defines

$$|| \, a \, ||_k := \sum_{i=1}^{k} s_i(a) \ , \tag{2.1}$$

where $s_1(a) \geq \ldots \geq s_d(a)$ ($d = \min\{d_1, d_2\}$) are singular values of a. Clearly, for $k = 1$ one recovers an operator norm $||\, a \,||_1 = ||\, a \,||$ and if $d_1 = d_2 = d$, then for $k = d$ one reproduces a trace norm $||\, a \,||_d = ||\, a \,||_{\rm tr}$. The family of k-norms satisfies:

1. $||\, a \,||_k \leq ||\, a \,||_{k+1}$,
2. $||\, a \,||_k = ||\, a \,||_{k+1}$ if and only if $\operatorname{rank} a = k$,
3. if $\operatorname{rank} a \geq k+1$, then $||\, a \,||_k < ||\, a \,||_{k+1}$.

Note, that a family of Ky Fan norms may be equivalently introduced as follows: let us define the following subset of $\mathcal{B}(\mathcal{H})$

$$\mathcal{P}_k(\mathcal{H}) = \{\, p \in \mathcal{B}(\mathcal{H}) : \ p = p^* = p^2 \ , \ \operatorname{tr} p = k \,\} \ . \tag{2.2}$$

Now, for any $p \in \mathcal{P}_k(\mathcal{H}_2)$ define the following inner product in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$

$$\langle a, b \rangle_p := \operatorname{tr}\left[(pa)^*(pb)\right] = \operatorname{tr}\left(a^* p b\right) = \operatorname{tr}\left(p b a^*\right) \ . \tag{2.3}$$

It is easy to show that

$$||\, a \,||_k^2 = \max_{p \in \mathcal{P}_k(\mathcal{H}_2)} \langle a, a \rangle_p = \max_{p \in \mathcal{P}_k(\mathcal{H}_2)} \operatorname{tr}\left(p a a^*\right) \ . \tag{2.4}$$

Thought out the paper we shall consider only finite dimensional Hilbert spaces. We denote by M_d a space of $d \times d$ complex matrices and $\mathbb{I}_d$ is a identity matrix from M_d.

Proposition 1 *For arbitrary projectors P and Q in $\mathcal{H}$*

$$||\, QPQ \,|| = ||\, PQP \,|| \ . \tag{2.5}$$

Proof. One obviously has

$$|| \, QPQ \, || = || \, QP(QP)^* \, || = || \, (QP)^2 \, || \ , \tag{2.6}$$

and

$$|| \, PQP \, || = || \, PQ(PQ)^* \, || = || \, (PQ)^2 \, || \ . \tag{2.7}$$

Now, due to $|| \, A^2 \, || = || \, A^{*2} \, || = || \, A \, ||^2$ one obtains

$$|| \, (QP)^2 \, || = || \, (QP)^{*2} \, || = || \, (PQ)^2 \, || \ , \tag{2.8}$$

which ends the proof. □

Consider now a Hilbert space being a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let us observe that any rank-1 projector P in $\mathcal{H}_1 \otimes \mathcal{H}_2$ may be represented in the following way

$$P = \sum_{i,j=1}^{d_1} e_{ij} \otimes F e_{ij} F^* \ , \tag{2.9}$$

where $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ and $\mathrm{tr}\, FF^* = 1$. Moreover, $\{e_1, \ldots, e_{d_1}\}$ denotes an arbitrary orthonormal basis in $\mathcal{H}_1$, and $e_{ij} := |e_i\rangle\langle e_j| \in \mathcal{B}(\mathcal{H}_1)$. Note, that $P = |\psi\rangle\langle\psi|$, where

$$\psi = \sum_{i=1}^{d_1} e_i \otimes F e_i \ . \tag{2.10}$$

It is easy to see that

$$\mathrm{SR}(\psi) = \mathrm{rank}\, F \ , \tag{2.11}$$

where $\mathrm{SR}(\psi)$ denotes the Schmidt rank of ψ ($1 \leq \mathrm{SR}(\psi) \leq d$), i.e. the number of non-vanishing Schmidt coefficients in the Schmidt decomposition of ψ. It is clear that F does depend upon the chosen basis $\{e_1, \ldots, e_{d_1}\}$. Note, however, that FF^* is basis-independent and, therefore, it has physical meaning being a reduction of P with respect to the first subsystem,

$$FF^* = \mathrm{tr}_1 P \ . \tag{2.12}$$

Proposition 2 *Let P be a projector in $\mathcal{H}_1 \otimes \mathcal{H}_2$ represented as in (2.9) and $Q = \mathbb{I}_{d_1} \otimes p$, where $p \in \mathcal{P}_k(\mathcal{H}_2)$. Then the following formula holds*

$$|| \, (\mathbb{I}_{d_1} \otimes p) P (\mathbb{I}_{d_1} \otimes p) \, || = \mathrm{tr}(pFF^*) \ , \tag{2.13}$$

and hence

$$|| \, (\mathbb{I}_{d_1} \otimes p) P (\mathbb{I}_{d_1} \otimes p) \, || \leq || \, F \, ||_k^2 \ . \tag{2.14}$$

Proof. Due to Proposition 1 one has

$$|| \, (\mathbb{I}_{d_1} \otimes p) P (\mathbb{I}_{d_1} \otimes p) \, || = || \, P (\mathbb{I}_{d_1} \otimes p) P \, || \ , \tag{2.15}$$

and hence

$$|| (\mathbb{I}_{d_1} \otimes p)P(\mathbb{I}_{d_1} \otimes p) || = \mathrm{tr}[P(\mathbb{I}_{d_1} \otimes p)] = \sum_{i=1}^{d_1} \mathrm{tr}(Fe_{ii}F^*p) = \mathrm{tr}(FF^*p) \ , \tag{2.16}$$

where we have used $\sum_{i=1}^{d_1} e_{ii} = \mathbb{I}_{d_1}$. □

Note, that if $F = V/\sqrt{d_1}$, where V is an isometry $VV^* = \mathbb{I}_{d_2}$, then P is a maximally entangled state

$$P = \frac{1}{d_1} \sum_{i,j=1}^{d_1} e_{ij} \otimes Ve_{ij}V^* \ , \tag{2.17}$$

and one obtains in this case

$$|| (\mathbb{I}_{d_1} \otimes p)P(\mathbb{I}_{d_1} \otimes p) || = \frac{k}{d_1} = || F ||_k^2 \ . \tag{2.18}$$

3 Entangled states vs. positive maps

Let us recall that a state of a quantum system living in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is separable iff the corresponding density operator σ is a convex combination of product states $\sigma_1 \otimes \sigma_2$. For any normalized positive operator σ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ one may define its Schmidt number

$$\mathrm{SN}(\sigma) = \min_{\alpha_k,\psi_k} \left\{ \max_k \ \mathrm{SR}(\psi_k) \right\} \ , \tag{3.1}$$

where the minimum is taken over all possible pure states decompositions

$$\sigma = \sum_k \alpha_k \, |\psi_k\rangle\langle\psi_k| \ , \tag{3.2}$$

with $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$ and ψ_k are normalized vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$. This number characterizes the minimum Schmidt rank of the pure states that are needed to construct such density matrix. It is evident that $1 \leq \mathrm{SN}(\sigma) \leq d = \min\{d_1, d_2\}$. Moreover, σ is separable iff $\mathrm{SN}(\sigma) = 1$. It was proved [30] that the Schmidt number is non-increasing under local operations and classical communication. Now, the notion of the Schmidt number enables one to introduce a natural family of convex cones in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)^+$ (a set of semi-positive elements in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$):

$$\mathbf{V}_r = \{ \sigma \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)^+ \mid \mathrm{SN}(\sigma) \leq r \} \ . \tag{3.3}$$

One has the following chain of inclusions

$$\mathbf{V}_1 \subset \ldots \subset \mathbf{V}_d = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)^+ \ . \tag{3.4}$$

Clearly, $\mathbf{V}_1$ is a cone of separable (unnormalized) states and $\mathbf{V}_d \smallsetminus \mathbf{V}_1$ stands for a set of entangled states.

Let $\varphi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$ be a linear map such that $\varphi(a)^* = \varphi(a^*)$. A map φ is positive iff $\varphi(a) \geq 0$ for any $a \geq 0$.

Definition 1 *A linear map φ is k-positive if*

$$\mathrm{id}_k \otimes \varphi \ : \ M_k \otimes \mathcal{B}(\mathcal{H}_1) \ \longrightarrow \ M_k \otimes \mathcal{B}(\mathcal{H}_2) \ ,$$

is positive. A map which is k-positive for $k = 1, \ldots, d = \min\{d_1, d_2\}$ is called completely positive (CP map).

Due to the Choi-Jamiołkowski isomorphism [6, 8] any linear adjoint-preserving map $\varphi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2)$ corresponds to a Hermitian operator $\widehat{\varphi} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$

$$\widehat{\varphi} := \sum_{i,j=1}^{d_1} e_{ij} \otimes \varphi(e_{ij}) \ . \tag{3.5}$$

Proposition 3 *A linear map φ is k-positive if and only if*

$$(\mathbb{I}_{d_1} \otimes p)\widehat{\varphi}(\mathbb{I}_{d_1} \otimes p) \geq 0 \ , \tag{3.6}$$

for all $p \in \mathcal{P}_k(\mathcal{H}_2)$. Equivalently, φ is k-positive iff $\mathrm{tr}(\sigma\widehat{\varphi}) \geq 0$ for any $\sigma \in \mathbf{V}_k$.

Corollary 1 *A linear map φ is positive iff $\mathrm{tr}(\sigma\widehat{\varphi}) \geq 0$ for any $\sigma \in \mathbf{V}_1$, i.e. or all separable states σ. Moreover, φ is CP iff $\mathrm{tr}(\sigma\widehat{\varphi}) \geq 0$ for any $\sigma \in \mathbf{V}_d$, i.e. $\widehat{\varphi} \geq 0$.*

4 Main result

It is well known that any CP map may be represented in the so called Kraus form [31]

$$\varphi_{\mathrm{CP}}(a) = \sum_{\alpha} K_\alpha a K_\alpha^* \ , \tag{4.1}$$

where (Kraus operators) $K_\alpha \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Any positive map is a difference of two CP maps $\varphi = \varphi_+ - \varphi_-$. However, there is no general method to recognize the positivity of φ from $\varphi_+ - \varphi_-$. Consider now a special class when $\widehat{\varphi}_+$ and $\widehat{\varphi}_-$ are orthogonally supported and $\widehat{\varphi}_- = \lambda_1 P_1$, with P_1 being a rank-1 projector. Let

$$\varphi(a) = \sum_{\alpha=2}^{D} \lambda_\alpha F_\alpha a F_\alpha^* - \lambda_1 F_1 a F_1^* \ , \tag{4.2}$$

such that

1. all rank-1 projectors $P_\alpha = d_1^{-1} \sum_{i,j=1}^{d_1} e_{ij} \otimes F_\alpha e_{ij} F_\alpha^*$, are mutually orthogonal,
2. $\lambda_\alpha > 0$, for $\alpha = 1, \ldots, D$, with $D := d_1 d_2$.

Theorem 1 *Let $||\, F_1 \,||_k < 1$. If*

$$\widehat{\varphi}_+ \ \geq \ \frac{\lambda_1 \, ||F_1||_k^2}{1 - ||F_1||_k^2} \, (\mathbb{I}_{d_1} \otimes \mathbb{I}_{d_2} - P_1) \ , \tag{4.3}$$

then φ is k-positive.

Proof. Let $p \in \mathcal{P}_k(\mathcal{H}_2)$. Take a unit vector $\xi \in (\mathbb{I}_{d_1} \otimes p)\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and set

$$\mu = \frac{\lambda_1 \, ||F_1||_k^2}{1 - ||F_1||_k^2} \ . \tag{4.4}$$

One obtains

$$(\xi, (\mathbb{I}_{d_1} \otimes p)\widehat{\varphi}(\mathbb{I}_{d_1} \otimes p)\xi) \geq \mu - (\mu + \lambda_1)(\xi, (\mathbb{I}_{d_1} \otimes p)P_1(\mathbb{I}_{d_1} \otimes p)\xi) \ . \tag{4.5}$$

Now, using Proposition 2 one has

$$(\xi, (\mathbb{I}_{d_1} \otimes p)P_1(\mathbb{I}_{d_1} \otimes p)\xi) \leq || \, (\mathbb{I}_{d_1} \otimes p)P_1(\mathbb{I}_{d_1} \otimes p) \, || \leq ||F_1||_k^2 \ , \tag{4.6}$$

and hence

$$(\xi, (\mathbb{I}_{d_1} \otimes p)\widehat{\varphi}(\mathbb{I}_{d_1} \otimes p)\xi) \geq 0 \ , \tag{4.7}$$

which proves k-positivity of φ. □

Remark 1 Note, that condition (4.3) may be equivalently rewritten as follows

$$\lambda_\alpha \geq \mu \ ; \quad \alpha = 2, \ldots, D \ , \tag{4.8}$$

with μ defined in (4.4).

Remark 2 If $d_1 = d_2 = d$ and P_1 is a maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$, i.e. $F = U/\sqrt{d}$ with unitary U, then the above theorem reproduces 25 years old result by Takasaki and Tomiyama [11].

Remark 3 For $d_1 = d_2 = d$, $k = 1$ and arbitrary P_1 the formula (4.8) was derived by Benatti et. al. [21].

The above theorem may be easily generalized for maps where rank $\widehat{\varphi}_- = m > 1$. Consider

$$\varphi(a) = \sum_{\alpha=m+1}^{D} \lambda_\alpha F_\alpha a F_\alpha^* - \sum_{\alpha=1}^{m} \lambda_\alpha F_\alpha a F_\alpha^* \ , \tag{4.9}$$

with $\lambda_\alpha > 0$.

Theorem 2 *Let* $\sum_{\alpha=1}^m || \, F_\alpha \, ||_k^2 < 1$. *If*

$$\widehat{\varphi}_+ \ \geq \ \frac{\sum_{\alpha=1}^m \lambda_\alpha ||F_\alpha||_k^2}{1 - \sum_{\alpha=1}^m ||F_\alpha||_k^2} \left(\mathbb{I}_{d_1} \otimes \mathbb{I}_{d_2} - \sum_{\alpha=1}^m P_\alpha \right) \ , \tag{4.10}$$

then φ *is* k*-positive.*

The proof is analogous.

Remark 4 Note, that condition (4.3) may be equivalently rewritten as follows

$$\lambda_\alpha \geq \nu \ ; \quad \alpha = m+1,\ldots,D \ , \tag{4.11}$$

with ν defined by

$$\nu = \frac{\sum_{\alpha=1}^m \lambda_\alpha ||F_\alpha||_k^2}{1-\sum_{\alpha=1}^m ||F_\alpha||_k^2} \ . \tag{4.12}$$

Let us note that the condition $\lambda_\alpha > 0$ may be easily relaxed. One has the following

Corollary 2 *Consider a map (4.9) such that* $\lambda_1 = \ldots = \lambda_\ell = 0$ *(*$\ell < m$*) and* $\lambda_{\ell+1},\ldots,\lambda_D > 0$*. If*

$$\widehat{\varphi}_+ \ \geq \ \frac{\sum_{\alpha=\ell}^m \lambda_\alpha ||F_\alpha||_k^2}{1-\sum_{\alpha=1}^m ||F_\alpha||_k^2} \left(\mathbb{I}_{d_1}\otimes\mathbb{I}_{d_2} - \sum_{\alpha=1}^m P_\alpha \right) \ , \tag{4.13}$$

then φ *is k-positive.*

Consider again the map (4.2).

Theorem 3 *Let* $||\, F_1\,||_k < 1$*. If*

$$\widehat{\varphi}_+ \ < \ \frac{\lambda_1\, ||F_1||_k^2}{1-||F_1||_k^2}\, (\mathbb{I}_{d_1}\otimes\mathbb{I}_{d_2} - P_1) \ , \tag{4.14}$$

then φ *is not k-positive.*

Proof. To prove that φ is not k positive we construct a vector $\xi_0 \in \mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$ such that

$$(\xi_0,(\mathbb{I}_{d_1}\otimes p_0)\widehat{\varphi}(\mathbb{I}_{d_1}\otimes p_0)\xi_0) < 0 \ , \tag{4.15}$$

for some $p_0 \in \mathcal{P}_k(\mathbb{C}^{d_2})$. Now, take any $p \in \mathcal{P}_k(\mathbb{C}^{d_2})$ such that

$$N^2 = \mathrm{tr}(pF_1F_1^*) \ , \tag{4.16}$$

is finite. Define

$$\xi = N^{-1}\sum_{i=1}^{d_1} e_i \otimes pF_1e_i \ . \tag{4.17}$$

Assuming (4.14) one finds

$$\begin{aligned} (\xi,(\mathbb{I}_{d_1}\otimes p)\widehat{\varphi}(\mathbb{I}_{d_1}\otimes p)\xi) \ &< \ \mu - (\mu+\lambda_1)(\xi,(\mathbb{I}_{d_1}\otimes p)P_1(\mathbb{I}_{d_1}\otimes p)\xi) \\ &= \ \frac{\mu}{||F_1||_k^2}\left[||F_1||_k^2 - (\xi,(\mathbb{I}_{d_1}\otimes p)P_1(\mathbb{I}_{d_1}\otimes p)\xi)\right] \ , \end{aligned} \tag{4.18}$$

with μ defined by (4.4). Now, it is easy to show that

$$(\xi,(\mathbb{I}_{d_1}\otimes p)P_1(\mathbb{I}_{d_1}\otimes p)\xi) = \mathrm{tr}(pF_1F_1^*) \ , \tag{4.19}$$

and therefore

$$(\xi, (\mathbb{I}_{d_1} \otimes p)\widehat{\varphi}(\mathbb{I}_{d_1} \otimes p)\xi) < \frac{\mu}{||F_1||_k^2} \left[||F_1||_k^2 - \mathrm{tr}(pF_1F_1^*) \right] . \tag{4.20}$$

Finally, let us observe that since $\mathcal{P}_k(\mathbb{C}^{d_2})$ is compact there exists a point $p_0 \in \mathcal{P}_k(\mathbb{C}^{d_2})$ such that

$$\mathrm{Tr}(p_0F_1F_1^*) = ||F_1||_k^2 . \tag{4.21}$$

Hence

$$(\xi_0, (\mathbb{I}_{d_1} \otimes p_0)\widehat{\varphi}(\mathbb{I}_{d_1} \otimes p_0)\xi_0) < 0 , \tag{4.22}$$

with $\xi_0 = ||F_1||_k^{-1} \sum_{i=1}^{d_1} e_i \otimes p_0 F_1 e_i$. □

Corollary 3 *Let $||F_1||_{k+1} < 1$. A map (4.2) is k-positive but not $(k+1)$-positive if*

$$\frac{\lambda_1 \, ||F_1||_{k+1}^2}{1 - ||F_1||_{k+1}^2} \, (\mathbb{I}_{d_1} \otimes \mathbb{I}_{d_2} - P_1) \ > \ \widehat{\varphi}_+ \ \geq \ \frac{\lambda_1 \, ||F_1||_k^2}{1 - ||F_1||_k^2} \, (\mathbb{I}_{d_1} \otimes \mathbb{I}_{d_2} - P_1) \ . \tag{4.23}$$

5 Example: generalized Choi maps

Let us consider a family of maps

$$\varphi_\lambda \ : \ M_d \ \longrightarrow \ M_d \ ,$$

defined as follows

$$\varphi_\lambda(a) := \mathbb{I}_d \, \mathrm{tr} a - \lambda F_1 a F_1^* \ . \tag{5.1}$$

It generalizes celebrated Choi map which is $(d-1)$-positive but not CP

$$\varphi_{\mathrm{Choi}}(a) := \mathbb{I}_d \, \mathrm{tr} a - \frac{d}{d-1} a \ , \tag{5.2}$$

which follows from (5.1) with $F_1 = \mathbb{I}_d/\sqrt{d}$ and $\lambda = d/(d-1)$. If $\lambda = d$, then (5.1) reproduces the so called reduction map

$$\varphi_{\mathrm{red}}(a) := \mathbb{I}_d \, \mathrm{tr} a - a \ , \tag{5.3}$$

which is known to be completely co-positive. One easily finds

$$\widehat{\varphi}_\lambda = \mathbb{I}_d \otimes \mathbb{I}_d - \lambda P_1 \ , \tag{5.4}$$

where

$$P_1 = \sum_{i,j=1}^{d} e_{ij} \otimes F_1 e_{ij} F_1^* \ . \tag{5.5}$$

Let $f_k := || \, F_1 \, ||_k$ and assume that $f_{k+1} < 1$. A map φ_λ is k-positive but not $(k+1)$-positive iff

$$\frac{1}{d \, f_k} \ \geq \ \lambda \ > \ \frac{1}{d \, f_{k+1}} \ . \tag{5.6}$$

Consider a family of states

$$\rho_\mu = \frac{1-\mu}{d^2-1} \, (\mathbb{I}_d \otimes \mathbb{I}_d - P_1) + \mu P_1 \ . \tag{5.7}$$

Computing $\mathrm{tr}(\widehat{\varphi}_\lambda \rho_\mu)$ one finds that $\mathrm{SN}(\rho_\mu) = k$ iff

$$f_k \ \geq \ \mu \ > \ f_{k-1} \ . \tag{5.8}$$

In particular ρ_μ is separable iff $\mu \geq f_1 = ||\, F_1 \,||^2$. Note, that if P_1 is a maximally entangled state then ρ_μ defines a family of isotropic state. In this case $f_k = k/d$ and one recovers well know result [30]: $\mathrm{SN}(\rho_\mu) = k$ iff $k/d \geq \mu > (k-1)/d$.

Consider now the following generalization of (5.1):

$$\varphi_\lambda(a) := \mathbb{I}_d \, \mathrm{tr} a - \lambda \sum_{\alpha=1}^{m} F_\alpha a F_\alpha^* \ , \tag{5.9}$$

and the corresponding operator

$$\widehat{\varphi}_\lambda = \mathbb{I}_d \otimes \mathbb{I}_d - \lambda P \ , \tag{5.10}$$

where P is a rank-m projector given by

$$P = \sum_{i,j=1}^{d} \sum_{\alpha=1}^{m} e_{ij} \otimes F_\alpha e_{ij} F_\alpha^* \ . \tag{5.11}$$

A map φ_λ is k-positive if

$$\lambda \ \leq \ \frac{1}{d\, \widetilde{f}_k} \ , \tag{5.12}$$

where now $\widetilde{f}_k = \sum_{\alpha=1}^{m-1} ||\, F_\alpha \,||_k^2$ and we assume that $\widetilde{f}_k < 1$. Consider a family of states

$$\rho_\mu = \frac{1 - m\mu}{d^2 - m} \, (\mathbb{I}_d \otimes \mathbb{I}_d - P) + \frac{\mu}{m} \, P \ . \tag{5.13}$$

Computing $\mathrm{tr}(\widehat{\varphi}_\lambda \rho_\mu)$ one finds that $\mathrm{SN}(\rho_\mu) = k$ iff

$$\widetilde{f}_k \ \geq \ \mu \ > \ \widetilde{f}_{k-1} \ . \tag{5.14}$$

In particular ρ_μ is separable iff $\mu \geq \widetilde{f}_1 = \sum_{\alpha=1}^{m-1} ||\, F_\alpha \,||^2$. Note, that if P is a sum of m maximally entangled state then ρ_μ defines a generalization of a family of isotropic state. In this case $\widetilde{f}_k = mk/d$ and one obtains: $\mathrm{SN}(\rho_\mu) = k$ iff $mk/d \geq \mu > m(k-1)/d$.

6 Multipartite setting

Consider now an n-partite state ρ living in $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$. Recall

Definition 2 *A state ρ is separable iff it can be represented as the convex combination of product states $\rho_1 \otimes \ldots \otimes \rho_n$.*

Theorem 4 *An n-partite state ρ in $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$ is separable iff*

$$(\mathrm{id} \otimes \varphi)\, \rho \geq 0 \ , \tag{6.1}$$

for all linear maps $\varphi : \mathcal{B}(\mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n) \longrightarrow \mathcal{B}(\mathcal{H}_1)$ satisfying

$$\varphi(p_2 \otimes \ldots \otimes p_n) \geq 0 \ , \tag{6.2}$$

where p_k is a rank-1 projector in $\mathcal{H}_k$.

Definition 3 (Generalized Choi-Jamiołkowski isomorphism) *For any linear map*

$$\varphi : \mathcal{B}(\mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n) \longrightarrow \mathcal{B}(\mathcal{H}_1) \ ,$$

define an operator $\widehat{\varphi}$ *in* $\mathcal{B}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n)$

$$\widehat{\varphi} := d_1(\mathrm{id} \otimes \varphi^\sharp)\, P^+ \ , \tag{6.3}$$

where P^+ *is the canonical maximally entangled state in* $\mathcal{H}_1 \otimes \mathcal{H}_1$*, and* $\varphi^\sharp$ *denotes a dual map.*

Proposition 4 *A linear map*

$$\varphi : \mathcal{B}(\mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n) \longrightarrow \mathcal{B}(\mathcal{H}_1) \ ,$$

satisfies (6.2) iff

$$\mathrm{tr}[(p_1 \otimes \ldots \otimes p_n)\, \widehat{\varphi}] \geq 0 \ , \tag{6.4}$$

for any rank-1 projectors p_k*.*

Proof. One has

$$\mathrm{tr}[(p_1 \otimes \ldots \otimes p_n)\, \widehat{\varphi}] = d_1 \mathrm{tr}[(p_1 \otimes \ldots \otimes p_n)\, (\mathrm{id} \otimes \varphi^\sharp) P^+] = d_1 \mathrm{tr}[P^+ \cdot p_1 \otimes \varphi(p_2 \otimes \ldots \otimes p_n)] \ . \tag{6.5}$$

Now, using $P^+ = d_1^{-1} \sum_{i,j=1}^{d_1} e_{ij} \otimes e_{ij}$ and obtains

$$\mathrm{tr}[P^+ \cdot p_1 \otimes \varphi(p_2 \otimes \ldots \otimes p_n)] = d_1^{-1} \sum_{i,j=1}^{d_1} \mathrm{tr}(e_{ij} p_1)\, \mathrm{tr}[e_{ij} \varphi(p_2 \otimes \ldots \otimes p_n)] \ . \tag{6.6}$$

Finally, due to $\sum_{i,j} \mathrm{tr}(e_{ij} a) e_{ij} = a^T$, one finds

$$\mathrm{tr}[(p_1 \otimes \ldots \otimes p_n)\, \widehat{\varphi}] = \mathrm{tr}[p_1^T \varphi(p_2 \otimes \ldots \otimes p_n)] \ , \tag{6.7}$$

from which the Proposition immediately follows. □

Corollary 4 *A linear map*

$$\varphi : \mathcal{B}(\mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n) \longrightarrow \mathcal{B}(\mathcal{H}_1) \ ,$$

satisfies (6.2) iff

$$(\mathbb{I} \otimes p_2 \otimes \ldots \otimes p_n)\, \widehat{\varphi}\, (\mathbb{I} \otimes p_2 \otimes \ldots \otimes p_n) \geq 0 \ , \tag{6.8}$$

for any rank-1 projectors p_k*.*

To construct linear maps which are positive on separable states let us define the following norm: let

$$\mathcal{P}_{\mathrm{sep}} = \{ p_2 \otimes \ldots \otimes p_n \ : \ p_k = p_k^* = p_k^2 \ , \ \mathrm{tr}\, p_k = 1 \} \ , \tag{6.9}$$

and define an inner product in the space of linear operators $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n)$

$$\langle A, B\rangle_P := \mathrm{tr}[(PA)^*(PB)] \ , \tag{6.10}$$

with $P \in \mathcal{P}_{\rm sep}$. Finally, let

$$||\, A\,||^2_{\rm sep} := \max_{P\in\mathcal{P}_{\rm sep}} \langle A, A\rangle_P \ . \tag{6.11}$$

It is clear that

$$||\, A\,||_{\rm sep} \leq ||\, A\,|| \ . \tag{6.12}$$

Consider now a linear map defined by

$$\varphi(a) = \sum_{\alpha=2}^{D} \lambda_\alpha F_\alpha a F_\alpha^* - \lambda_1 F_1 a F_1^* \ , \tag{6.13}$$

where $D = d_1 \ldots d_n$, $\mathrm{tr}(F_\alpha^* F_\beta) = \delta_{\alpha\beta}$ and $\lambda_\alpha > 0$. One finds for the corresponding $\widehat{\varphi}$

$$\widehat{\varphi} = \sum_{\alpha=2}^{D} \lambda_\alpha P_\alpha - \lambda_1 P_1 \ , \tag{6.14}$$

where the rank-1 projectors read as follows

$$P_\alpha = \sum_{i,j=1}^{d_1} e_{ij} \otimes F_\alpha e_{ij} F_\alpha^* \ . \tag{6.15}$$

In analogy to Theorems 2 and 3 one easily proves

Theorem 5 *Let $||\, F_1\,||_{\rm sep} < 1$. Then φ is positive on separable states if and only if*

$$\lambda_\alpha \ \geq \ \frac{\lambda_1 ||F_1||^2_{\rm sep}}{1 - ||F_1||^2_{\rm sep}} \ , \tag{6.16}$$

for $\alpha = 2, \ldots, D$.

Corollary 5 *Let $||\, F_1\,||_{\rm sep} < ||\, F_1\,|| < 1$. Then φ is positive on separable states but not positive if and only if*

$$\frac{\lambda_1 ||F_1||^2}{1 - ||F_1||^2} \ > \ \lambda_\alpha \ \geq \ \frac{\lambda_1 ||F_1||^2_{\rm sep}}{1 - ||F_1||^2_{\rm sep}} \ , \tag{6.17}$$

for $\alpha = 2, \ldots, D$.

Example. Consider a map

$$\varphi_\lambda \ : \ M_d \otimes M_d \ \longrightarrow \ M_{d^2} \equiv M_d \otimes M_d \ , \tag{6.18}$$

defined by

$$\varphi_\lambda(a) = \lambda(\mathbb{I}_d \otimes \mathbb{I}_d \,\mathrm{tr} a - F_0 a F_0) - F_0 a F_0 \ , \tag{6.19}$$

with

$$F_0 = F_0^* = \frac{1}{\sqrt{2d(d-1)}} \left[\mathbb{I}_d \otimes \mathbb{I}_d - \sum_{i,j=1}^{d} e_{ij} \otimes e_{ij}^* \right] . \tag{6.20}$$

Note that $\mathrm{tr}F_0^2 = 1$ and $\sqrt{d(d-1)/2} \cdot F_0$ is a projector (see [32, 33] for more details). Hence

$$|| \, F_0 \, ||^2 = \frac{2}{d(d-1)} . \tag{6.21}$$

Now, for any rank-1 projectors $p, q \in M_d$ one has

$$\mathrm{tr}\Big[(p \otimes q) F_0^2\Big] = \frac{1}{d(d-1)} \, (1 - \mathrm{tr}pq) \, , \tag{6.22}$$

and therefore

$$|| \, F_0 ||_{\mathrm{sep}}^2 := \max_{p,q \in \mathcal{P}_{\mathrm{sep}}} \mathrm{tr}\Big[(p \otimes q) F_0^2\Big] = \frac{1}{d(d-1)} < || \, F_0 \, ||^2 . \tag{6.23}$$

Corollary 6 *Let $d > 2$, i.e. $|| \, F_0 \, ||_{\mathrm{sep}} < || \, F_0 \, || < 1$. For*

$$\frac{2}{d(d-1) - 2} \; > \; \lambda \; \geq \; \frac{1}{d(d-1) - 1} \tag{6.24}$$

φ_λ is positive on separable elements in $M_d \otimes M_d$ but it is not a positive map.

Remark 5 To the best of our knowledge φ_λ provides the first nontrivial example of a map which is not positive but it is positive on separable states. Nontrivial means that it is not a tensor product of two positive maps.

7 Conclusions

We provide partial classification of positive linear maps based on spectral conditions. Presented method generalizes celebrated Choi example of a map which is positive but not CP. From the physical point of view our scheme provides simple method for constructing entanglement witnesses. Moreover, this scheme may be easily generalized for multipartite setting.

Presented method guarantees k-positivity but says nothing about indecomposability and/or optimality. We stress that both indecomposable and optimal positive maps are crucial in detecting and classifying quantum entanglement. Therefore, the analysis of positive maps based on spectral properties deserves further study.

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