Parameterizing density matrices for composite quantum systems

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Abstract

A parametrization of density operators for bipartite quantum systems is proposed. It is based on the particular parametrization of the unitary group found recently by Jarlskog. It is expected that this parametrization will find interesting applications in the study of quantum properties of many partite systems.

1 Introduction

Density operators represent states of quantum systems. They are crucial to describe the dynamics of open quantum systems [1, 2]. Recently, there is a considerable interest in the structure of the set of density operators due to the emerging field of quantum information theory [3]. It turns out that quantum entanglement may be used as basic resource in quantum information processing and communication. The prominent examples are quantum cryptography, quantum teleportation, quantum error correction codes and quantum computation. It is, therefore, clear that the proper description of the set of density operators is highly important.

It turns out that the structure of the set of density operators is nontrivial and it is well understood only for 2-level systems [4, 5]. This problem was studied by many authors from different perspectives and various parameterizations of the set of density matrices were proposed [6]–[12], see also the recent paper [13] for the generalized Bloch vector approach. Recently, two of us proposed another parametrization [14] which is based on the particular parametrization of the unitary group found by Jarlskog [15]. This parametrization was already applied for modeling quantum gates in the context of quantum computing [16] and to investigate possible time evolutions for density matrices [14].

In the present paper we propose a similar parametrization for composite systems. This problem is of significant importance for analyzing quantum entanglement. Although the analysis of states is
parametrization invariant (as much of the problems in physics) a clever parametrization may considerably simplify the problem and sheds new light into the structure of quantum states. Moreover, as usual, the particular parametrization depends very much upon the problem one would like to analyze.

Recently, a simple parametrization enabling one to analyze the PPT property (positive partial transpose) was proposed in [17]. This special parametrization was used to define a new class of states (so called SPPT states) which are PPT but of a very special form. It was conjectured that SPPT states are separable. Now, we study the problem of parametrization of composite systems from a more general perspective. It is expected that the proposed parametrization will be helpful in analyzing the intricate structure of quantum entanglement.

The paper is organized as follows: in Section 2 we recall the basic ingredients of the parametrization of \( n \)-level systems used in [14]. Then in Section 3 it is shown how to generalize it for arbitrary \( n \otimes m \) composed systems. The procedure is illustrated by explicit examples where it is easy to check for separability. Finally, we end with some conclusions.

2 Parametrization for \( n \)-level system

The parametrization used in [14] is defined as follows: any density matrix \( \rho_n \) may be written as

\[
\rho_n = U_n D(\lambda_1, \ldots, \lambda_n) U_n^\dagger,
\]

where the matrix \( D(\lambda_1, \ldots, \lambda_n) \) is diagonal with \( \lambda_k \) on the main diagonal and \( U_n \) is a unitary matrix from \( SU(n) \). Now, following [15] any element \( U_n \) from \( SU(n) \) may be factorized as follows

\[
U_n = A_n^n A_n^{n-1} \cdots A_2 A_1,
\]

where

\[
A_j = e^{X_j},
\]

with \( X_j \in su(n) \) for \( j = 1, 2, \ldots, n \). The antihermitian traceless matrices \( X_j \) entering (3) are defined as follows: \( X_1 \) is diagonal and the \( X_j \), for \( j = 2, \ldots, n \), are given by

\[
X_j = \begin{pmatrix}
O_{j-1} & |z_j| & 0 \\
-(z_j) & 0 & 0 \\
0 & 0 & O_{n-j}
\end{pmatrix}
\]

where \( O_k \) denotes \( k \times k \) null matrix, and \( |z_j| \) denotes a complex vector from \( \mathbb{C}^{j-1} \)

\[
|z_j| = \begin{pmatrix}
z_{1j} \\
\vdots \\
z_{j-1,j}
\end{pmatrix},
\]

together with \( (z_j) = (z_{1,j}, \ldots, z_{j-1,j}) \). Taking into account that \( A_1 \) is diagonal, formula (1) implies

\[
\rho_n = A_n^n \cdots A_2 D(\lambda_1, \ldots, \lambda_n) A_2^\dagger \cdots A_1^\dagger.
\]

Hence, to parameterize \( \rho_n \) one needs \( (n - 1) \) complex vectors \( z_2, \ldots, z_n \), with \( z_j \in \mathbb{C}^{j-1} \), and \( (n - 1) \) real parameters \( \lambda_1, \ldots, \lambda_{n-1} \). All together \( (n^2 - 1) \) real parameters (to parameterize \( n \times n \) Hermitian
matrix one needs indeed \( n^2 \) real parameters – \( n \) on the diagonal and \( n(n - 1) \) off-diagonal – and the normalization eliminates one parameter). Now, a simple calculation \( [16] \) gives

\[
A^j_n = \left( \begin{array}{cc} V^j_n & 0 \\ 0 & \mathbb{I}_{n-j} \end{array} \right),
\]

(7)

with \( \mathbb{I}_k \) being the \( k \times k \) unit matrix and \( V^j_n \) being the \( j \times j \) unitary matrix given by

\[
V^j_n = \left( \begin{array}{cc} \mathbb{I}_{j-1} - (1 - c_j)|\tilde{z}_j\rangle\langle \tilde{z}_j| & s_j|\tilde{z}_j\rangle \\ -s_j\langle \tilde{z}_j| & c_j \end{array} \right),
\]

(8)

where \( |\tilde{z}_j\rangle \) denotes the unit vector

\[
|\tilde{z}_j\rangle = \frac{|z_j\rangle}{||z_j||},
\]

(9)

that is,

\[
\langle \tilde{z}_j|\tilde{z}_j\rangle = ||\tilde{z}_j||^2 = 1,
\]

(10)

and

\[
c_j := \cos \theta_j, \quad s_j := \sin \theta_j,
\]

(11)

with

\[
\theta_j := ||z_j||.
\]

(12)

As usual, in Eq. (8) \(|\tilde{z}_j\rangle\langle \tilde{z}_j|\) denotes the \((j - 1) \times (j - 1)\) matrix defined by

\[
(|\tilde{z}_j\rangle\langle \tilde{z}_j|)_{kl} := \tilde{z}_k\tilde{z}_l,
\]

(13)

for \( k, l = 1, \ldots, j - 1 \).

Therefore, \( \rho_n \) is parameterized by \((n - 1)\) eigenvalues \( \lambda_1 \geq \ldots \geq \lambda_{n-1} \), \((n - 1)\) unit vectors \( \tilde{z}_2, \ldots, \tilde{z}_n \), i.e., \( \tilde{z}_j \) defines a point on the \((2j - 1)\)–dimensional unit sphere \( S^{2j-1} \), and \((n - 1)\) angles \( \theta_2, \ldots, \theta_n \) from the hyperoctant of \((n - 1)\)–dimensional space. All together we have the correct number of independent parameters of a \( n \times n \) density matrix

\[
\sum_{j=1}^{n-1} (2j - 1) + 2(n - 1) = n^2 - 1.
\]

As an example consider the simplest system, i.e., a qubit corresponding to \( n = 2 \). One obtains \( [14] \)

\[
A^2_2 = V^2_2 = \left( \begin{array}{cc} c|\tilde{z}\rangle\langle \tilde{z}| & s|\tilde{z}\rangle \\ -s\langle \tilde{z}| & c \end{array} \right),
\]

(14)

with \(|\tilde{z}\rangle = e^{i\varphi} \) and \( c = \cos \theta, s = \sin \theta \). One has therefore

\[
\rho^2 = \left( \begin{array}{cc} c^2\lambda_1 + s^2\lambda_2 & sc e^{i\varphi}(\lambda_1 - \lambda_2) \\ sc e^{-i\varphi}(\lambda_1 - \lambda_2) & c^2\lambda_2 + s^2\lambda_1 \end{array} \right).
\]

(15)

The above parametrization reproduces the standard Bloch ball: \( \varphi \) and \( \vartheta = 2\theta \) are nothing but the spherical angles on the unit Bloch sphere, and \( r = \lambda_1 - \lambda_2 \in [0, 1] \) is the radial coordinate. For \( \lambda_1 = 1 \) one obtains the celebrated Bloch sphere of pure states

\[
|\psi\rangle\langle \psi| = \left( \begin{array}{cc} c^2 e^{i\varphi} & sc \\ -sc e^{-i\varphi} & s^2 \end{array} \right),
\]

(16)
that is

$$|\psi\rangle = \begin{pmatrix} \cos \theta \\ e^{i\phi} \sin \theta \end{pmatrix},$$  \hspace{1cm} (17)$$

up to an overall phase factor.

3 Composite $n \otimes m$ systems

Consider now a density operator for the composite system living in $\mathbb{C}^n \otimes \mathbb{C}^m$. It is clear that we may parameterize it as a density operator living in $\mathbb{C}^{nm}$. However, this way we lose information about the particular tensor product structure of the total Hilbert space $\mathbb{C}^{nm}$. To control the division into subsystems $\mathbb{C}^{nm} = \mathbb{C}^n \otimes \mathbb{C}^m$ let us consider $\rho$ as an $n \times n$ matrix with $m \times m$ blocks, i.e.

$$\rho_{n,m} = \sum_{i,j=1}^{n} |i\rangle \langle j| \otimes \rho_{ij},$$  \hspace{1cm} (18)$$

with $\rho_{ij}$ being $m \times m$ complex matrices. Our aim is to provide a suitable parametrization for positive block matrices. Let $D(\lambda_1, \ldots, \lambda_{nm})$ denote a diagonal $nm \times nm$ matrix with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. It is clear that

$$\rho_{n,m} = U_{n,m} \cdot D(\lambda_1, \ldots, \lambda_{nm}) \cdot U_{n,m}^\dagger,$$  \hspace{1cm} (19)$$

where $U_{n,m} \in SU(nm)$. Any special unitary matrix $U_{n,m}$ may be written as

$$U_{n,m} = e^{X},$$  \hspace{1cm} (20)$$

where $X$ is an $nm \times nm$ anti-hermitian matrix and hence it may be represented as follows

$$X = X_1 + X_2 + \ldots + X_n,$$  \hspace{1cm} (21)$$

where $X_1$ is anti-hermitian block-diagonal and $X_j$ for $j \geq 2$ are $n \times n$ block anti-hermitian matrices with $m \times m$ blocks defined as follows:

$$X_j = \begin{pmatrix} \mathbb{I}_{j-1} \otimes \mathbb{O}_m & |Z_j\rangle \\ \langle Z_j| & \mathbb{O}_m \\ 0 & 0 & \mathbb{I}_{n-j} \otimes \mathbb{O}_m \end{pmatrix},$$  \hspace{1cm} (22)$$

where, instead of $(n-1)$ column vectors $z_j$ we take $(n-1)$ column block vectors

$$|Z_j\rangle = \begin{pmatrix} Z_{1,j} \\ \vdots \\ Z_{j-1,j} \end{pmatrix},$$  \hspace{1cm} (23)$$

with $Z_{i,j}$ being $m \times m$ matrices. Similarly

$$\langle Z_j| = (Z_{1,j}^\dagger, Z_{2,j}^\dagger, \ldots, Z_{j-1,j}^\dagger).$$

Using the parametrization of block anti-hermitian matrices (21) and (22) we are ready to define the following parametrization of the unitary group:

$$A_{n,m}^j = e^{X_j},$$  \hspace{1cm} (24)$$
Hence, we consider unitary matrices from $SU(nm)$ of the following form

$$U_{n,m} = A_{n,m}^n A_{n,m}^{n-1} \ldots A_{n,m}^2 A_{n,m}^1,$$

(25)

where $A_{n,m}^j$ are unitary block matrices and $A_{n,m}^1$ is unitary block diagonal, i.e.,

$$A_{n,m}^1 = \begin{pmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_n \end{pmatrix},$$

(26)

where $U_k$ are $m \times m$ unitary matrices. It is clear that

$$A_{n,m}^1 D(\lambda_1, \ldots, \lambda_{nm}) A_{n,m}^1 \dagger$$

is a positive block diagonal matrix. Let us denote it by $D(\Lambda_1 \ldots \Lambda_n)$, where $\Lambda_k$ stand for $m \times m$ diagonal positive blocks

$$\Lambda_k = U_k D(\lambda_{km}, \ldots, \lambda_{km+m-1}) U_k \dagger.$$  

(27)

If $m = 1$ one has $\Lambda_k = \lambda_k \geq 0$. Moreover, we add the normalization condition

$$\text{Tr}(\Lambda_1 + \ldots + \Lambda_n) = 1.$$  

(28)

One has finally

$$\rho_{n,m} = A_{n,m}^n \ldots A_{n,m}^2 D(\Lambda_1 | \Lambda_n) A_{n,m}^2 \dagger \ldots A_{n,m}^1 \dagger.$$  

(29)

To apply the above formula one needs the explicit form of the unitary components $A_{n,m}^j$. A straightforward calculation gives (we follow [10])

$$A_{n,m}^j = \begin{pmatrix} V_{n,m}^j & 0 & \cdots & 0 \\ 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_m \end{pmatrix},$$

(30)

where the unit block $I_m$ appears $n - j$ times. In the above formula $V_{n,m}^j$ is a $j \times j$ block unitary matrix with $m \times m$ blocks defined as follows:

$$V_{n,m}^j = \begin{pmatrix} I_j \otimes I_m - |\bar{Z}_j\rangle \langle \bar{Z}_j| & I_j \otimes (I_m - C_j) \langle \bar{Z}_j|S_j \\ -S_j \langle \bar{Z}_j| & C_j \end{pmatrix},$$

(31)

and $|\bar{Z}_j\rangle$ denotes the normalized block vectors, that is,

$$\bar{Z}_{k,j} := \frac{Z_{k,j}}{||Z_j||},$$

(32)

where

$$||Z_j||^2 = Z_{1,j}^\dagger Z_{1,j} + \ldots + Z_{j-1,j}^\dagger Z_{j-1,j}.$$  

(33)
Moreover, \(|\widetilde{Z}_j (\mathbb{I}_{j-1} \otimes (\mathbb{I}_m - C_j)) \rangle \langle \widetilde{Z}_j|\) stands for the following \((j - 1) \times (j - 1)\) block matrix

\[
\sum_{k,l=1}^{j-1} |k\rangle \langle l| \otimes \widetilde{Z}_{k,j}^\dagger C_j \widetilde{Z}_{l,j},
\]

and

\[
C_j = \cos \Xi_j, \quad S_j = \sin \Xi_j,
\]

with

\[
\Xi_j := ||Z_j||.
\]

It is clear that for \(m = 1\) we recover the parametrization used in [14].

4 Examples

Class 1. Let us consider a \(2 \otimes 2\) system to illustrate our parametrization for the well known 2-qubit states. Taking

\[
\Lambda_1 = \mathbb{O}_2, \quad \Lambda_2 = \frac{1}{2}(\mathbb{I}_2 - \sigma_z), \quad S = \sin \alpha \mathbb{I}_2, \quad C = \cos \alpha \mathbb{I}_2, \quad U = \sigma_x,
\]

one obtains a family of rank-1 projectors

\[
P(\alpha) = \begin{pmatrix}
\sin^2 \alpha & 0 & 0 & \sin \alpha \cos \alpha \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\sin \alpha \cos \alpha & 0 & 0 & \cos^2 \alpha
\end{pmatrix},
\]

which corresponds to a pure state

\[
\psi_\alpha = \sin \alpha \langle 00| + \cos \alpha \langle 11|.
\]

Note, that this state is separable if and only if \(S = 0\) or \(C = 0\). For \(S = C = \mathbb{I}_2/\sqrt{2}\), one obtains a maximally entangled state. It shows that a nontrivial rotation by \(\alpha\) does produce quantum entanglement. As a second example in this class let us take \(S = C = \mathbb{I}_2/\sqrt{2}, U = \sigma_x\) and

\[
\Lambda_1 = \frac{1}{4} \begin{pmatrix}
1 - p & 0 \\
0 & 1 - p
\end{pmatrix}, \quad \Lambda_2 = \frac{1}{4} \begin{pmatrix}
1 - p & 0 \\
0 & 1 + 3p
\end{pmatrix},
\]

with \(-1/3 \leq p \leq 1\) to guarantee positivity of the matrices \(\Lambda\). One obtains the following 1-parameter family of 2-qubit states

\[
I(p) = \frac{1}{4} \begin{pmatrix}
1 + p & 0 & 0 & 2p \\
0 & 1 - p & 0 & 0 \\
0 & 0 & 1 - p & 0 \\
2p & 0 & 0 & 1 + p
\end{pmatrix}.
\]

This is the well known family of isotropic states which is known to be separable if and only if \(p \leq 1/3\). Actually, a point \(p = 1/3\) is not distinguished by our parametrization.
Taking $S = \sin \alpha \mathbb{I}_2$ and $C = \cos \alpha \mathbb{I}_2$ one obtains a more general 2-parameter family
\[
I(p, \alpha) = \frac{1-p}{4} \mathbb{I}_2 \otimes \mathbb{I}_2 + pP(\alpha)
\]
which is separable if and only if
\[
p \leq \frac{1}{1+2\sin(2\alpha)}.
\]

The above example may be generalized as follows. Instead of (39) let us consider
\[
\Lambda_1 = \begin{pmatrix} p_2 & 0 \\ 0 & p_4 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} p_3 & 0 \\ 0 & p_1 \end{pmatrix},
\]
with $p_k \geq 0$ and $p_1 + p_2 + p_3 + p_4 = 1$. Taking
\[
S = \begin{pmatrix} \sin \alpha & 0 \\ 0 & \sin \beta \end{pmatrix}, \quad C = \begin{pmatrix} \cos \alpha & 0 \\ 0 & \cos \beta \end{pmatrix}, \quad \alpha, \beta \in [0, \pi/2],
\]
and $U = \sigma_x$ one obtains the following family
\[
\rho(p; \alpha, \beta) = \begin{pmatrix} p_1 c_{\alpha}^2 + p_2 s_{\alpha}^2 & 0 & 0 & (p_1 - p_2)s_{\beta} c_{\beta} \\ 0 & p_3 c_{\beta}^2 + p_4 s_{\beta}^2 & (p_3 - p_4)s_{\alpha} c_{\alpha} & 0 \\ 0 & (p_3 - p_4)s_{\alpha} c_{\alpha} & p_3 s_{\alpha}^2 + p_4 s_{\beta}^2 & 0 \\ (p_1 - p_2)s_{\beta} c_{\beta} & 0 & 0 & p_1 s_{\alpha}^2 + p_2 s_{\beta}^2 \end{pmatrix},
\]
where
\[
s_{\alpha} = \sin \alpha, \quad c_{\alpha} = \cos \alpha,
\]
and similarly for $s_{\beta}, c_{\beta}$. We stress that one has $\rho(p; \alpha, \beta) \geq 0$ and $\text{Tr} \rho(p; \alpha, \beta) = 1$ for any $\alpha, \beta$ and the arbitrary probability vector $p = (p_1, p_2, p_3, p_4)$ by construction. Interestingly, the above family belongs to the class of $2 \otimes 2$ circulant states considered in [18]. The Peres PPT criterion [19] gives the following separability conditions
\[
p_3 s_{\beta}^2 + p_4 s_{\beta}^2 \geq |p_1 - p_2| s_{\beta} c_{\beta},
\]
\[
p_1 c_{\alpha}^2 + p_2 s_{\alpha}^2 \geq |p_3 - p_4| s_{\alpha} c_{\alpha}.
\]
For $\alpha = \beta = \pi/4$ the above family reduces to the family of Bell diagonal states
\[
\rho(p) = \frac{1}{2} \begin{pmatrix} p_1 + p_2 & 0 & 0 & p_1 - p_2 \\ 0 & p_3 + p_4 & p_3 - p_4 & 0 \\ 0 & p_3 - p_4 & p_3 + p_4 & 0 \\ p_1 - p_2 & 0 & 0 & p_1 + p_2 \end{pmatrix}.
\]
Moreover, separability conditions (44)–(45) reduce to $p_k \leq 1/2$ for $k = 1, 2, 3, 4$. Note, that even if $\rho(p)$ is entangled $\rho(p; \alpha, \beta)$ might be separable. Consider e.g. $p_1 = p_2 = p_3 = 1/8$ and $p_4 = 5/8$, that is, $\rho(p)$ is entangled. Now, (44) is trivially satisfied and (45) implies $\sin 2\alpha \leq 1/2$. Hence, $\rho(p; \alpha, \beta)$ is separable for $\alpha \leq \pi/12$ and arbitrary $\beta$.

**Class 2.** An arbitrary state of a $2 \otimes m$ system corresponds to
\[
A_{2,m}^2 = V_{2,m}^2 = \begin{pmatrix} ZC & \tilde{Z}^\dagger \\ -S\tilde{Z}^\dagger & C \end{pmatrix},
\]
with $\tilde{Z} = U \in U(m)$ and again $C = \cos \Xi_2$, $S = \sin \Xi_2$. One finds

$$\rho_{2,m} = \frac{U(CU^\dagger \Lambda_1 UC + S \Lambda_2 S)U^\dagger}{(C \Lambda_2 S - SU^\dagger \Lambda_1 UC)U^\dagger} \left( \frac{U(S \Lambda_2 C - CU^\dagger \Lambda_1 US)}{C \Lambda_2 C + SU^\dagger \Lambda_1 US} \right) .$$

(48)

Note that for $S = 0$ or $C = 0$ one obtains a class of block-diagonal matrices

$$\left( \begin{array}{c|c} \Lambda_1 & 0_m \\ \hline 0_m & \Lambda_2 \end{array} \right) \quad \text{or} \quad \left( \begin{array}{c|c} \Lambda_2 & 0_m \\ \hline 0_m & \Lambda_1 \end{array} \right) ,$$

(49)

respectively. Being block-diagonal these matrices represent separable $2 \otimes m$ states. It shows that quantum entanglement arises only for nontrivial $\Xi_2$ corresponding to $C \neq 0$ and $S \neq 0$.

Note, that for

1. $\Lambda_1 = \Lambda_2 = \Lambda$ ,
2. $[\Lambda, U] = 0$ ,

one obtains the following class of $2 \otimes m$ states:

$$\left( \begin{array}{c|c} U & 0_m \\ \hline 0_m & I_m \end{array} \right) \left( \begin{array}{c|c} A & B \\ \hline B^\dagger & A \end{array} \right) \left( \begin{array}{c|c} U^\dagger & 0_m \\ \hline 0_m & I_m \end{array} \right) ,$$

(50)

with

$$A = C \Lambda C + SAS , \quad B = SAC - CAS .$$

Now, if $UAU^\dagger = A$, then one gets

$$\left( \begin{array}{c|c} A & UB \\ \hline (UB)^\dagger & A \end{array} \right) .$$

(51)

These are block Toeplitz positive matrices and it is well known that they are separable [20]. In this way we define huge family of bipartite separable states. Another class of separable states is defined by block Hankel positive matrices [20]: taking $U, \Lambda_1, \Lambda_2$ and $\Xi_2$ satisfying

1. $[U^\dagger \Lambda_1 U, \Xi_2] = 0$ ,
2. $[\Lambda_2, \Xi_2] = 0$ ,

one obtains the following class of $2 \otimes m$ states:

$$\left( \begin{array}{c|c} U & 0_m \\ \hline 0_m & I_m \end{array} \right) \left( \begin{array}{c|c} A_1 & B' \\ \hline B'^\dagger & A_2 \end{array} \right) \left( \begin{array}{c|c} U^\dagger & 0_m \\ \hline 0_m & I_m \end{array} \right) ,$$

(52)

with

$$B' = SC(\Lambda_2 - U^\dagger \Lambda_1 U) .$$

Now, if

$$UB' = B'U^\dagger ,$$

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then one gets
\[
\begin{pmatrix}
UA_1U^\dagger & X \\
X & A_2
\end{pmatrix},
\]
with \(X := UB'\). These are block Hankel positive matrices and hence separable [20].

**Class 3.** An interesting class of bipartite \(n \otimes m\) states corresponds to \(\Lambda_1 = \ldots = \Lambda_{n-1} = O_m\) and \(\Lambda_n = \frac{1}{m}I_m\). In this case one has
\[
\rho_{n,m} = \frac{1}{m} A_{n,m}^n D(O_m| \ldots |O_m|I_m) A_{n,m}^{n\dagger}.
\]
This class is parameterized by a positive matrix \(\Xi_n\) and \((n-1)\) complex matrices \(\tilde{Z}_{1,n}, \ldots, \tilde{Z}_{n-1,n}\) satisfying \(\sum_{k=1}^{n-1} \tilde{Z}_{k,n}^\dagger \tilde{Z}_{k,n} = I_m\). Taking into account a polar decomposition \(\tilde{Z}_{k,n} = P_k U_k\), with positive \(P_k\) and unitary \(U_k\), one replaces the above constraint by
\[
P_1^2 + \ldots + P_{n-1}^2 = I_m.
\]
The above equation defines a **nonabelian sphere** [21]. Therefore, to parameterize the class defined by (54) one may use \((n-1)\) unitaries \(\{U_1, \ldots, U_{n-1}\}\) and \((n-1)\) positive operators \(\{P_1, \ldots, P_{n-2}, \Xi_n\}\) (since \(P_{n-1}\) may be calculated from (55)). Note, that for fixed \(\{P_1, \ldots, P_{n-2}, \Xi_n\}\) one obtains a \((n-1)m^2\)-dimensional subspace which may be called a nonabelian \((n-1)\)-torus. It is, therefore, clear that a class (54) generalizes the subspace of pure states for \(n\)-level single system and hence it may be regarded as a nonabelian generalization of the complex projective space \(\mathbb{C}P^{n-1}\) [4, 5]. Let us observe that if \(Z_{k,j}\) are normal, that is
\[
Z_{k,j} Z_{k,j}^\dagger = Z_{k,j}^\dagger Z_{k,j},
\]
then (54) defines a rank-\(m\) projector which generalizes rank-1 projector (a pure state) for a single \(n\)-level system. In particular for \(n = 2\) one generalizes a 2-dimensional Bloch sphere (one may call it a nonabelian Bloch sphere):
\[
\frac{1}{m} \begin{pmatrix}
S^2 & USC \\
CSU^\dagger & C^2
\end{pmatrix},
\]
which is parameterized by two nonabelian angles: unitary \(U\) and positive \(\Xi_2\) \((C = \cos \Xi_2, S = \sin \Xi_2)\). All together \(2m^2\) parameters. Note that for \(n = m\) the class (54) defines a set of extremal states of the extended quantum theory proposed recently by \(\dot{Z}\)yczkowski [22].

**5 Conclusions**

We proposed a parametrization of density matrices of composed \(n \otimes m\) quantum system. For \(m = 1\) this parametrization reduces to the one used recently in [14]. Note, that it may be generalized for multipartite systems living in \(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_N}\). Indeed, instead of dealing with \(n \times n\) block matrices with \(m \times m\) blocks, in the multipartite case one has to consider \(n_1 \times n_1\) block matrices with blocks being \(n_2 \times n_2\) block matrices with blocks being \(n_3 \times n_3\) block matrices and so on. Although the strategy seems to be simple the technical part of the story is quite involved. It is anticipated that the presented parametrization will find interesting applications in the study of quantum properties of many partite systems.
Acknowledgement This work is based upon research supported by the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation. One of us (DC) was partially supported by the Polish Ministry of Science and Higher Education Grant No 3004/B/H03/2007/33 and by the Polish Research Network Laboratory of Physical Foundations of Information Processing. DC thanks Francesco Petruccione for the warm hospitality in Durban.

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