Parameterizing density matrices for composite quantum systems

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Abstract

A parametrization of density operators for bipartite quantum systems is proposed. It is based on the particular parametrization of the unitary group found recently by Jarlskog. It is expected that this parametrization will find interesting applications in the study of quantum properties of many partite systems.

1 Introduction

Density operators represent states of quantum systems. They are crucial to describe the dynamics of open quantum systems [1, 2]. Recently, there is a considerable interest in the structure of the set of density operators due to the emerging field of quantum information theory [3]. It turns out that quantum entanglement may be used as basic resource in quantum information processing and communication. The prominent examples are quantum cryptography, quantum teleportation, quantum error correction codes and quantum computation. It is, therefore, clear that the proper description of the set of density operators is highly important.

It turns out that the structure of the set of density operators is nontrivial and it is well understood only for 2-level systems [4, 5]. This problem was studied by many authors from different perspectives and various parameterizations of the set of density matrices were proposed [6]–[12], see also the recent paper [13] for the generalized Bloch vector approach. Recently, two of us proposed another parametrization [14] which is based on the particular parametrization of the unitary group found by Jarlskog [15]. This parametrization was already applied for modeling quantum gates in the context of quantum computing [16] and to investigate possible time evolutions for density matrices [14].

In the present paper we propose a similar parametrization for composite systems. This problem is of significant importance for analyzing quantum entanglement. Although the analysis of states is parametrization invariant (as much of the problems in physics) a clever parametrization may considerably simplify the problem and sheds new light into the structure of quantum states. Moreover, as usual, the particular parametrization depends very much upon the problem one would like to analyze.

Recently, a simple parametrization enabling one to analyze the PPT property (positive partial transpose) was proposed in [17]. This special parametrization was used to define a new class of states (so called SPPT states) which are PPT but of a very special form. It was conjectured that SPPT states are separable. Now, we study the problem of parametrization of composite systems from a more general perspective. It is expected that the proposed parametrization will be helpful in analyzing the intricate structure of quantum entanglement.

The paper is organized as follows: in Section 2 we recall the basic ingredients of the parametrization of n-level systems used in [14]. Then in Section 3 it is shown how to generalize it for arbitrary $n \otimes m$ composed systems. The procedure is illustrated by explicit examples where it is easy to check for separability. Finally, we end with some conclusions.

2 Parametrization for n-level system

The parametrization used in [14] is defined as follows: any density matrix ρ_n may be written as

$$\rho_n = U_n D(\lambda_1, \ldots, \lambda_n) U_n^\dagger \ , \tag{1}$$

where the matrix $D(\lambda_1, \ldots, \lambda_n)$ is diagonal with λ_k on the main diagonal and U_n is a unitary matrix from $SU(n)$. Now, following [15] any element U_n from $SU(n)$ may be factorized as follows

$$U_n = A_n^n A_n^{n-1} \ldots A_n^2 A_n^1 \ , \tag{2}$$

where

$$A_n^j = e^{X_j} \ , \tag{3}$$

with $X_j \in su(n)$ for $j = 1, 2, \ldots, n$. The antihermitian traceless matrices X_j entering (3) are defined as follows: X_1 is diagonal and the X_j, for $j = 2, \ldots, n$, are given by

$$X_j = \left(\begin{array}{c|c|c} \mathbb{O}_{j-1} & |z_j\rangle & 0 \\ \hline -\langle z_j| & 0 & 0 \\ \hline 0 & 0 & \mathbb{O}_{n-j} \ , \end{array} \right) \tag{4}$$

where $\mathbb{O}_k$ denotes $k \times k$ null matrix, and $|z_j\rangle$ denotes a complex vector from $\mathbb{C}^{j-1}$

$$|z_j\rangle = \left(\begin{array}{c} z_{1j} \\ \vdots \\ z_{j-1,j} \end{array} \right) \ , \tag{5}$$

together with $\langle z_j| = (\overline{z}_{1,j}, \ldots, \overline{z}_{j-1,j})$. Taking into account that A_n^1 is diagonal, formula (1) implies

$$\rho_n = A_n^n \ldots A_n^2 \, D(\lambda_1, \ldots, \lambda_n) \, A_n^{2\,\dagger} \ldots A_n^{n\dagger} \ . \tag{6}$$

Hence, to parameterize ρ_n one needs $(n-1)$ complex vectors $z_2, \ldots, z_n$, with $z_j \in \mathbb{C}^{j-1}$, and $(n-1)$ real parameters $\lambda_1, \ldots, \lambda_{n-1}$. All together $(n^2 - 1)$ real parameters (to parameterize $n \times n$ Hermitian matrix one needs indeed n^2 real parameters – n on the diagonal and $n(n-1)$ off-diagonal – and the normalization eliminates one parameter). Now, a simple calculation [16] gives

$$A_n^j = \left(\begin{array}{cc} V_n^j & 0 \\ 0 & \mathbb{I}_{n-j} \end{array} \right) , \tag{7}$$

with $\mathbb{I}_k$ being the $k \times k$ unit matrix and V_n^j being the $j \times j$ unitary matrix given by

$$V_n^j = \left(\begin{array}{cc} \mathbb{I}_{j-1} - (1-c_j)|\widetilde{z}_j\rangle\langle\widetilde{z}_j| & s_j|\widetilde{z}_j\rangle \\ -s_j\langle\widetilde{z}_j| & c_j \end{array} \right) , \tag{8}$$

where $|\widetilde{z}_j\rangle$ denotes the unit vector

$$|\widetilde{z}_j\rangle = \frac{|z_j\rangle}{||z_j||} , \tag{9}$$

that is,

$$\langle\widetilde{z}_j|\widetilde{z}_j\rangle = ||\widetilde{z}_j||^2 = 1 , \tag{10}$$

and

$$c_j := \cos\theta_j , \quad s_j := \sin\theta_j , \tag{11}$$

with

$$\theta_j := ||z_j|| . \tag{12}$$

As usual, in Eq. (8) $|\widetilde{z}_j\rangle\langle\widetilde{z}_j|$ denotes the $(j-1)\times(j-1)$ matrix defined by

$$(|\widetilde{z}_j\rangle\langle\widetilde{z}_j|)_{kl} := \overline{\widetilde{z}}_k \widetilde{z}_l , \tag{13}$$

for $k, l = 1, \ldots, j-1$.

Therefore, ρ_n is parameterized by $(n-1)$ eigenvalues $\lambda_1 \geq \ldots \geq \lambda_{n-1}$, $(n-1)$ unit vectors $\widetilde{z}_2, \ldots, \widetilde{z}_n$, i.e., $\widetilde{z}_j$ defines a point on the $(2j-1)$–dimensional unit sphere S^{2j-1}, and $(n-1)$ angles $\theta_2, \ldots, \theta_n$ from the hyperoctant of $(n-1)$–dimensional space. All together we have the correct number of independent parameters of a $n \times n$ density matrix

$$\sum_{j=1}^{n-1}(2j-1) + 2(n-1) = n^2 - 1 .$$

As an example consider the simplest system, i.e., a qubit corresponding to $n = 2$. One obtains [14]

$$A_2^2 = V_2^2 = \left(\begin{array}{cc} c|\widetilde{z}\rangle\langle\widetilde{z}| & s|\widetilde{z}\rangle \\ -s\langle\widetilde{z}| & c \end{array} \right) , \tag{14}$$

with $|\widetilde{z}\rangle = e^{i\varphi}$ and $c = \cos\theta$, $s = \sin\theta$. One has therefore

$$\rho_2 = \left(\begin{array}{cc} c^2\lambda_1 + s^2\lambda_2 & sc\, e^{i\varphi}(\lambda_1 - \lambda_2) \\ sc\, e^{-i\varphi}(\lambda_1 - \lambda_2) & c^2\lambda_2 + s^2\lambda_1 \end{array} \right) . \tag{15}$$

The above parametrization reproduces the standard Bloch ball: φ and $\vartheta = 2\theta$ are nothing but the spherical angles on the unit Bloch sphere, and $r = \lambda_1 - \lambda_2 \in [0, 1]$ is the radial coordinate. For $\lambda_1 = 1$ one obtains the celebrated Bloch sphere of pure states

$$|\psi\rangle\langle\psi| = \left(\begin{array}{cc} c^2 & e^{i\varphi}\, sc \\ e^{-i\varphi}\, sc & s^2 \end{array} \right) , \tag{16}$$

that is

$$|\psi\rangle = \begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix} , \tag{17}$$

up to an overall phase factor.

3 Composite $n \otimes m$ systems

Consider now a density operator for the composite system living in $\mathbb{C}^n \otimes \mathbb{C}^m$. It is clear that we may parameterize it as a density operator living in $\mathbb{C}^{nm}$. However, this way we loose information about the particular tensor product structure of the total Hilbert space $\mathbb{C}^{nm}$. To control the division into subsystems $\mathbb{C}^{nm} = \mathbb{C}^n \otimes \mathbb{C}^m$ let us consider ρ as an $n \times n$ matrix with $m \times m$ blocks, i.e.

$$\rho_{n,m} = \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes \rho_{ij} , \tag{18}$$

with ρ_{ij} being $m \times m$ complex matrices. Our aim is to provide a suitable parametrization for positive block matrices. Let $D(\lambda_1, \ldots, \lambda_{nm})$ denote a diagonal $nm \times nm$ matrix with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. It is clear that

$$\rho_{n,m} = U_{n,m} \cdot D(\lambda_1, \ldots, \lambda_{nm}) \cdot U_{n,m}^\dagger , \tag{19}$$

where $U_{n,m} \in SU(nm)$. Any special unitary matrix $U_{n,m}$ may be written as

$$U_{n,m} = e^X , \tag{20}$$

where X is an $nm \times nm$ anti-hermitian matrix and hence it may be represented as follows

$$X = X_1 + X_2 + \ldots + X_n , \tag{21}$$

where X_1 is anti-hermitian block-diagonal and X_j for $j \geq 2$ are $n \times n$ block anti-hermitian matrices with $m \times m$ blocks defined as follows:

$$X_j = \left(\begin{array}{c|c|c} \mathbb{I}_{j-1} \otimes \mathbb{O}_m & |Z_j\rangle & 0 \\ \hline -\langle Z_j| & \mathbb{O}_m & 0 \\ \hline 0 & 0 & \mathbb{I}_{n-j} \otimes \mathbb{O}_m \end{array} \right) , \tag{22}$$

where, instead of $(n-1)$ column vectors z_j we take $(n-1)$ column block vectors

$$|Z_j\rangle = \begin{pmatrix} Z_{1,j} \\ \vdots \\ Z_{j-1,j} \end{pmatrix} , \tag{23}$$

with $Z_{i,j}$ being $m \times m$ matrices. Similarly

$$\langle Z_j| = (Z_{1,j}^\dagger, Z_{2,j}^\dagger, \ldots, Z_{j-1,j}^\dagger) .$$

Using the parametrization of block anti-hermitian matrices (21) and (22) we are ready to define the following parametrization of the unitary group:

$$A_{n,m}^j = e^{X_j} . \tag{24}$$

Hence, we consider unitary matrices from $SU(nm)$ of the following form

$$U_{n,m} = A^{n}_{n,m} A^{n-1}_{n,m} \ldots A^{2}_{n,m} A^{1}_{n,m} \ , \tag{25}$$

where $A^{j}_{n,m}$ are unitary block matrices (24) and $A^{1}_{n,m}$ is unitary block diagonal, i.e.,

$$A^{1}_{n,m} = \left(\begin{array}{c|c|c|c} U_1 & \mathbb{O}_m & \ldots & \mathbb{O}_m \\ \hline \mathbb{O}_m & U_2 & \ldots & \mathbb{O}_m \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbb{O}_m & \mathbb{O}_m & \ldots & U_n \end{array} \right) \ , \tag{26}$$

where U_k are $m \times m$ unitary matrices. It is clear that

$$A^{1}_{n,m} D(\lambda_1, \ldots, \lambda_{nm}) A^{1\,\dagger}_{n,m} \ ,$$

is a positive block diagonal matrix. Let us denote it by $D(\Lambda_1|\ldots|\Lambda_n)$, where Λ_k stand for $m \times m$ diagonal positive blocks

$$\Lambda_k = U_k D(\lambda_{km}, \ldots, \lambda_{km+m-1}) U_k^{\dagger} \ . \tag{27}$$

If $m = 1$ one has $\Lambda_k = \lambda_k \geq 0$. Moreover, we add the normalization condition

$$\mathrm{Tr}\,(\Lambda_1 + \ldots + \Lambda_n) = 1 \ . \tag{28}$$

One has finally

$$\rho_{n,m} = A^{n}_{n,m} \ldots A^{2}_{n,m} D(\Lambda_1|\ldots|\Lambda_n) A^{2\dagger}_{n,m} \ldots A^{n\dagger}_{n,m} \ . \tag{29}$$

To apply the above formula one needs the explicit form of the unitary components $A^{j}_{n,m}$. A straightforward calculation gives (we follow [16])

$$A^{j}_{n,m} = \left(\begin{array}{c|c|c|c} V^{j}_{n,m} & \mathbb{O}_m & \ldots & \mathbb{O}_m \\ \hline \mathbb{O}_m & \mathbb{I}_m & \ldots & \mathbb{O}_m \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbb{O}_m & \mathbb{O}_m & \ldots & \mathbb{I}_m \end{array} \right) \ , \tag{30}$$

where the unit block $\mathbb{I}_m$ appears $n-j$ times. In the above formula $V^{j}_{n,m}$ is a $j \times j$ block unitary matrix with $m \times m$ blocks defined as follows:

$$V^{j}_{n,m} = \left(\begin{array}{c|c} \mathbb{I}_{j-1} \otimes \mathbb{I}_m - |\widetilde{Z}_j\rangle \Big[\mathbb{I}_{j-1} \otimes (\mathbb{I}_m - C_j) \Big] \langle \widetilde{Z}_j| & |\widetilde{Z}_j\rangle S_j \\ \hline -S_j \langle \widetilde{Z}_j| & C_j \end{array} \right) \ , \tag{31}$$

and $|\widetilde{Z}_j\rangle$ denotes the normalized block vectors, that is,

$$\widetilde{Z}_{k,j} := \frac{Z_{k,j}}{||Z_j||} \ , \tag{32}$$

where

$$||Z_j||^2 = Z^{\dagger}_{1,j} Z_{1,j} + \ldots + Z^{\dagger}_{j-1,j} Z_{j-1,j} \ . \tag{33}$$

Moreover, $|\widetilde{Z}_j\rangle\Big[\mathbb{I}_{j-1}\otimes(\mathbb{I}_m - C_j)\Big]\langle\widetilde{Z}_j|$ stands for the following $(j-1)\times(j-1)$ block matrix

$$\sum_{k,l=1}^{j-1} |k\rangle\langle l| \otimes \widetilde{Z}^\dagger_{k,j} C_j \, \widetilde{Z}_{l,j} \ , \tag{34}$$

and

$$C_j = \cos\Xi_j \ , \qquad S_j = \sin\Xi_j \ , \tag{35}$$

with

$$\Xi_j := ||Z_j|| \ . \tag{36}$$

It is clear that for $m = 1$ we recover the parametrization used in [14].

4 Examples

Class 1. Let us consider a $2\otimes 2$ system to illustrate our parametrization for the well known 2-qubit states. Taking

$$\Lambda_1 = \mathbb{O}_2 \ , \quad \Lambda_2 = \frac{1}{2}(\mathbb{I}_2 - \sigma_z) \ , \quad S = \sin\alpha\,\mathbb{I}_2 \ , \quad C = \cos\alpha\,\mathbb{I}_2 \ , \quad U = \sigma_x \ , \tag{37}$$

one obtains a family of rank-1 projectors

$$P(\alpha) = \left(\begin{array}{cc|cc} \sin^2\alpha & 0 & 0 & \sin\alpha\cos\alpha \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \sin\alpha\cos\alpha & 0 & 0 & \cos^2\alpha \end{array}\right) \ , \tag{38}$$

which corresponds to a pure state

$$\psi_\alpha = \sin\alpha\,|00\rangle + \cos\alpha\,|11\rangle \ .$$

Note, that this state is separable if and only if $S = 0$ or $C = 0$. For $S = C = \mathbb{I}_2/\sqrt{2}$, one obtains a maximally entangled state. It shows that a nontrivial rotation by α does produce quantum entanglement. As a second example in this class let us take $S = C = \mathbb{I}_2/\sqrt{2}$, $U = \sigma_x$ and

$$\Lambda_1 = \frac{1}{4}\left(\begin{array}{cc} 1-p & 0 \\ 0 & 1-p \end{array}\right) \ , \qquad \Lambda_2 = \frac{1}{4}\left(\begin{array}{cc} 1-p & 0 \\ 0 & 1+3p \end{array}\right) \ , \tag{39}$$

with $-1/3 \le p \le 1$ to guarantee positivity of the matrices Λ. One obtains the following 1-parameter family of 2-qubit states

$$I(p) = \frac{1}{4}\left(\begin{array}{cc|cc} 1+p & 0 & 0 & 2p \\ 0 & 1-p & 0 & 0 \\ \hline 0 & 0 & 1-p & 0 \\ 2p & 0 & 0 & 1+p \end{array}\right) \ . \tag{40}$$

This is the well known family of isotropic states which is known to be separable if and only if $p \le 1/3$. Actually, a point $p = 1/3$ is not distinguished by our parametrization.

Taking $S = \sin\alpha\,\mathbb{I}_2$ and $C = \cos\alpha\,\mathbb{I}_2$ one obtains a more general 2-parameter family

$$I(p,\alpha) = \frac{1-p}{4}\,\mathbb{I}_2 \otimes \mathbb{I}_2 + pP(\alpha)\ , \tag{41}$$

which is separable if and only if

$$p \ \leq\ \frac{1}{1+2\sin(2\alpha)}\ .$$

The above example may be generalized as follows. Instead of (39) let us consider

$$\Lambda_1 = \left(\begin{array}{cc} p_2 & 0 \\ 0 & p_4 \end{array} \right)\ , \qquad \Lambda_2 = \left(\begin{array}{cc} p_3 & 0 \\ 0 & p_1 \end{array} \right)\ , \tag{42}$$

with $p_k \geq 0$ and $p_1 + p_2 + p_3 + p_4 = 1$. Taking

$$S = \left(\begin{array}{cc} \sin\alpha & 0 \\ 0 & \sin\beta \end{array} \right)\ , \qquad C = \left(\begin{array}{cc} \cos\alpha & 0 \\ 0 & \cos\beta \end{array} \right)\ , \qquad \alpha,\beta \in [0,\pi/2]\ ,$$

and $U = \sigma_x$ one obtains the following family

$$\rho(\mathbf{p};\alpha,\beta) = \left(\begin{array}{cc|cc} p_1c_\alpha^2 + p_2s_\alpha^2 & 0 & 0 & (p_1-p_2)s_\beta c_\beta \\ 0 & p_3c_\beta^2 + p_4s_\beta^2 & (p_3-p_4)s_\alpha c_\alpha & 0 \\ \hline 0 & (p_3-p_4)s_\alpha c_\alpha & p_3s_\beta^2 + p_4c_\beta^2 & 0 \\ (p_1-p_2)s_\beta c_\beta & 0 & 0 & p_1s_\alpha^2 + p_2c_\alpha^2 \end{array} \right)\ , \tag{43}$$

where

$$s_\alpha = \sin\alpha\ , \qquad c_\alpha = \cos\alpha\ ,$$

and similarly for s_β, c_β. We stress that one has $\rho(\mathbf{p};\alpha,\beta) \geq 0$ and $\mathrm{Tr}\,\rho(\mathbf{p};\alpha,\beta) = 1$ for any α,β and the arbitrary probability vector $\mathbf{p} = (p_1,p_2,p_3,p_4)$ by construction. Interestingly, the above family belongs to the class of $2\otimes 2$ circulant states considered in [18]. The Peres PPT criterion [19] gives the following separability conditions

$$p_3c_\beta^2 + p_4s_\beta^2 \ \ \geq\ \ |p_1-p_2|s_\beta c_\beta\ , \tag{44}$$

$$p_1c_\alpha^2 + p_2s_\alpha^2 \ \ \geq\ \ |p_3-p_4|s_\alpha c_\alpha\ . \tag{45}$$

For $\alpha = \beta = \pi/4$ the above family reduces to the family of Bell diagonal states

$$\rho(\mathbf{p}) = \frac{1}{2}\left(\begin{array}{cc|cc} p_1+p_2 & 0 & 0 & p_1-p_2 \\ 0 & p_3+p_4 & p_3-p_4 & 0 \\ \hline 0 & p_3-p_4 & p_3+p_4 & 0 \\ p_1-p_2 & 0 & 0 & p_1+p_2 \end{array} \right)\ . \tag{46}$$

Moreover, separability conditions (44)–(45) reduce to $p_k \leq 1/2$ for $k = 1,2,3,4$. Note, that even if $\rho(\mathbf{p})$ is entangled $\rho(\mathbf{p};\alpha,\beta)$ might be separable. Consider e.g. $p_1 = p_2 = p_3 = 1/8$ and $p_4 = 5/8$, that is, $\rho(\mathbf{p})$ is entangled. Now, (44) is trivially satisfied and (45) implies $\sin 2\alpha \leq 1/2$. Hence, $\rho(\mathbf{p};\alpha,\beta)$ is separable for $\alpha \leq \pi/12$ and arbitrary β.

Class 2. An arbitrary state of a $2\otimes m$ system corresponds to

$$A^2_{2,m} = V^2_{2,m} = \left(\begin{array}{cc} \widetilde{Z}C\,\widetilde{Z}^\dagger & \widetilde{Z}S \\ -S\widetilde{Z}^\dagger & C \end{array} \right)\ , \tag{47}$$

with $\widetilde{Z} = U \in U(m)$ and again $C = \cos\Xi_2$, $S = \sin\Xi_2$. One finds

$$\begin{aligned}\rho_{2,m} &= \left(\begin{array}{c|c} U(CU^\dagger\Lambda_1UC + S\Lambda_2S)U^\dagger & U(S\Lambda_2C - CU^\dagger\Lambda_1US) \\ \hline (C\Lambda_2S - SU^\dagger\Lambda_1UC)U^\dagger & C\Lambda_2C + SU^\dagger\Lambda_1US \end{array}\right) \\ &= \left(\begin{array}{c|c} U & \mathbb{O}_m \\ \hline \mathbb{O}_m & \mathbb{I}_m \end{array}\right)\left(\begin{array}{c|c} CU^\dagger\Lambda_1UC + S\Lambda_2S & S\Lambda_2C - CU^\dagger\Lambda_1US \\ \hline C\Lambda_2S - SU^\dagger\Lambda_1UC & C\Lambda_2C + SU^\dagger\Lambda_1US \end{array}\right)\left(\begin{array}{c|c} U^\dagger & \mathbb{O}_m \\ \hline \mathbb{O}_m & \mathbb{I}_m \end{array}\right) . \end{aligned} \tag{48}$$

Note that for $S = 0$ or $C = 0$ one obtains a class of block-diagonal matrices

$$\left(\begin{array}{c|c} \Lambda_1 & \mathbb{O}_m \\ \hline \mathbb{O}_m & \Lambda_2 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} \Lambda_2 & \mathbb{O}_m \\ \hline \mathbb{O}_m & \Lambda_1 \end{array}\right) , \tag{49}$$

respectively. Being block-diagonal these matrices represent separable $2 \otimes m$ states. It shows that quantum entanglement arises only for nontrivial Ξ_2 corresponding to $C \neq 0$ and $S \neq 0$.

Note, that for

1. $\Lambda_1 = \Lambda_2 = \Lambda$,
2. $[\Lambda, U] = 0$,

one obtains the following class of $2 \otimes m$ states:

$$\left(\begin{array}{c|c} U & \mathbb{O}_m \\ \hline \mathbb{O}_m & \mathbb{I}_m \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline B^\dagger & A \end{array}\right)\left(\begin{array}{c|c} U^\dagger & \mathbb{O}_m \\ \hline \mathbb{O}_m & \mathbb{I}_m \end{array}\right) , \tag{50}$$

with

$$A = C\Lambda C + S\Lambda S \ , \quad B = S\Lambda C - C\Lambda S \ .$$

Now, if $UAU^\dagger = A$, then one gets

$$\left(\begin{array}{c|c} A & UB \\ \hline (UB)^\dagger & A \end{array}\right) . \tag{51}$$

These are block Toeplitz positive matrices and it is well known that they are separable [20]. In this way we define huge family of bipartite separable states. Another class of separable states is defined by block Hankel positive matrices [20]: taking U, Λ_1, Λ_2 and Ξ_2 satisfying

1. $[U^\dagger\Lambda_1U, \Xi_2] = 0$,
2. $[\Lambda_2, \Xi_2] = 0$,

one obtains the following class of $2 \otimes m$ states:

$$\left(\begin{array}{c|c} U & \mathbb{O}_m \\ \hline \mathbb{O}_m & \mathbb{I}_m \end{array}\right)\left(\begin{array}{c|c} A_1 & B' \\ \hline B' & A_2 \end{array}\right)\left(\begin{array}{c|c} U^\dagger & \mathbb{O}_m \\ \hline \mathbb{O}_m & \mathbb{I}_m \end{array}\right) , \tag{52}$$

with

$$B' = SC(\Lambda_2 - U^\dagger\Lambda_1U) \ .$$

Now, if

$$UB' = B'U^\dagger \ ,$$

then one gets

$$\left(\begin{array}{c|c} UA_1U^\dagger & X \\ \hline X & A_2 \end{array} \right) , \tag{53}$$

with $X := UB'$. These are block Hankel positive matrices and hence separable [20].

Class 3. An interesting class of bipartite $n \otimes m$ states corresponds to $\Lambda_1 = \ldots = \Lambda_{n-1} = \mathbb{O}_m$ and $\Lambda_n = \frac{1}{m}\mathbb{I}_m$. In this case one has

$$\rho_{n,m} = \frac{1}{m} A^n_{n,m}\, D(\mathbb{O}_m|\ldots|\mathbb{O}_m|\mathbb{I}_m)\, A^{n\dagger}_{n,m} \ . \tag{54}$$

This class is parameterized by a positive matrix Ξ_n and $(n-1)$ complex matrices $\widetilde{Z}_{1,n}, \ldots, \widetilde{Z}_{n-1,n}$ satisfying $\sum_{k=1}^{n-1} \widetilde{Z}^\dagger_{k,n}\widetilde{Z}_{k,n} = \mathbb{I}_m$. Taking into account a polar decomposition $\widetilde{Z}_{k,n} = P_kU_k$, with positive P_k and unitary U_k, one replaces the above constraint by

$$P_1^2 + \ldots + P_{n-1}^2 = \mathbb{I}_m \ . \tag{55}$$

The above equation defines a *nonabelian sphere* [21]. Therefore, to parameterize the class defined by (54) one may use $(n-1)$ unitaries $\{U_1, \ldots, U_{n-1}\}$ and $(n-1)$ positive operators $\{P_1, \ldots, P_{n-2}, \Xi_n\}$ (since P_{n-1} may be calculated from (55)). Note, that for fixed $\{P_1, \ldots, P_{n-2}, \Xi_n\}$ one obtains a $(n-1)m^2$–dimensional subspace which may be called a nonabelian $(n-1)$–torus. It is, therefore, clear that a class (54) generalizes the subspace of pure states for n-level single system and hence it may be regarded as a nonabelian generalization of the complex projective space $\mathbb{C}P^{n-1}$ [4, 5]. Let us observe that if $Z_{k,j}$ are normal, that is

$$Z_{k,j}Z^\dagger_{k,j} = Z^\dagger_{k,j}Z_{k,j} \ , \tag{56}$$

then (54) defines a rank-m projector which generalizes rank-1 projector (a pure state) for a single n-level system. In particular for $n = 2$ one generalizes a 2-dimensional Bloch sphere (one may call it a nonabelian Bloch sphere):

$$\frac{1}{m} \left(\begin{array}{c|c} S^2 & USC \\ \hline CSU^\dagger & C^2 \end{array} \right) , \tag{57}$$

which is parameterized by two nonabelian angles: unitary U and positive Ξ_2 ($C = \cos\Xi_2$, $S = \sin\Xi_2$). All together $2m^2$ parameters. Note that for $n = m$ the class (54) defines a set of extremal states of the extended quantum theory proposed recently by Życzkowski [22].

5 Conclusions

We proposed a parametrization of density matrices of composed $n \otimes m$ quantum system. For $m = 1$ this parametrization reduces to the one used recently in [14]. Note, that it may be generalized for multipartite systems living in $\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_N}$. Indeed, instead of dealing with $n \times n$ block matrices with $m \times m$ blocks, in the multipartite case one has to consider $n_1 \times n_1$ block matrices with blocks being $n_2 \times n_2$ block matrices with blocks being $n_3 \times n_3$ block matrices and so on. Although the strategy seems to be simple the technical part of the story is quite involved. It is anticipated that the presented parametrization will find interesting applications in the study of quantum properties of many partite systems.

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