Geometry of quantum states: new construction of positive maps

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Abstract

We provide a new class of positive maps in matrix algebras. The construction is based on the family of balls living in the space of density matrices of n-level quantum system. This class generalizes the celebrated Choi map and provide a wide family of entanglement witnesses which define a basic tool for analyzing quantum entanglement.

1 Introduction

One of the most important problems of quantum information theory [1] is the characterization of mixed states of composed quantum systems. In particular it is of primary importance to test whether a given quantum state exhibits quantum correlation, i.e. whether it is separable or entangled. For low dimensional systems there exists simple necessary and sufficient condition for separability. The celebrated Peres-Horodecki criterium [2, 3] states that a state of a bipartite system living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable iff its partial transpose is positive. Unfortunately, for higher-dimensional systems there is no single universal separability condition.

The most general approach to separability problem is based on the following observation [4]: a state ρ of a bipartite system living in $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable iff $\mathrm{Tr}(W\rho) \geq 0$ for any Hermitian operator W satisfying $\mathrm{Tr}(WP_A \otimes P_B) \geq 0$, where P_A and P_B are projectors acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Recall, that a Hermitian operator $W \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is an entanglement witness [4, 5] iff: i) it is not positively defined, i.e. $W \ngeq 0$, and ii) $\mathrm{Tr}(W\sigma) \geq 0$ for all separable states σ. A bipartite state ρ living in $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled iff there exists an entanglement witness W detecting ρ, i.e. such that $\mathrm{Tr}(W\rho) < 0$. Clearly, the construction of entanglement witnesses is a hard task. It is easy to construct W which is not positive, i.e. has at leat one negative eigenvalue, but it is very difficult to check that $\mathrm{Tr}(W\sigma) \geq 0$ for all separable states σ.

The separability problem may be equivalently formulated in terms positive maps [4]: a state ρ is separable iff $(\mathbb{1} \otimes \Lambda)\rho$ is positive for any positive map Λ which sends positive operators on $\mathcal{H}_B$ into positive operators on $\mathcal{H}_A$. Positive maps play important role both in physics and mathematics providing generalization of $*$-homomorphism, Jordan homomorphism and conditional expectation. Normalized positive maps define an affine mapping between sets of states of $\mathbb{C}^*$-algebras. Unfortunately, in spite of the considerable effort (see e.g. [6]–[16]), the structure of positive maps (and hence also the set of entanglement witnesses) is rather poorly understood.

In the present paper we construct a new class of positive maps using the family of balls contained in the space of density matrices of n-level quantum system. Our construction generalizes a class of maps introduced in [12].

The paper is organized as follows: in the next Section we introduce a family of balls in the space of quantum states. We show that each faithful state (i.e. strictly positive density operator) serves as a center of the ball. In particular ball centered at the maximally mixed state $\rho_0 = \mathbb{I}/n$ possesses a maximal radius $[n(n-1)]^{-1/2}$. Section 3 provides positive maps with values in the corresponding ball. Composing with affine maps they give rise to the wide class of positive maps discussed in Section 4. Finally, in Section 5 we illustrate our construction for $n = 3$ and provide generalization of the celebrated Choi map [7]. A brief discussion is included in the last section.

2 A family of balls

Let us consider the space of quantum states $\mathcal{S}_n$ corresponding to n-level quantum system, i.e. the space of density operators living in the Hilbert space $\mathcal{H} = \mathbb{C}^n$. It defines a convex subset of the linear space of Hermitian operators

$$H_n = \{\, a \in M_n(\mathbb{C}) \,|\, a^* = a \,\} \ , \tag{2.1}$$

where $M_n(\mathbb{C})$ denotes the space of $n \times n$ complex matrices. Recall, that H_n is a real Hilbert space equipped with the scalar product $(a, b) = \mathrm{tr}(ab)$ and the norm $\|a\|^2 = (a, a)$. Now, let $\widetilde{\rho} \in \mathcal{S}_n$ be a strictly positive density matrix, i.e. its spectral decomposition has the following form

$$\widetilde{\rho} = \widetilde{\lambda}_1 P_1 + \widetilde{\lambda}_2 P_2 + \ldots + \widetilde{\lambda}_n P_n \ , \tag{2.2}$$

where

$$\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \ldots \geq \widetilde{\lambda}_n > 0 \ . \tag{2.3}$$

A set of rank 1 projectors $\mathbf{P} = \{P_1, P_2, \ldots, P_n\}$ defines a simplex $\Sigma(\mathbf{P}) \subset \mathcal{S}_n$, and the condition (2.3) implies that $\widetilde{\rho}$ belongs to the interior of $\Sigma(\mathbf{P})$. Note, that $\widetilde{\rho}$ may be rewritten as follows

$$\widetilde{\rho} = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_{n-1} P_{n-1} + \lambda_n \frac{\mathbb{I}}{n} \ , \tag{2.4}$$

where

$$\lambda_i = \widetilde{\lambda}_i - \widetilde{\lambda}_n \geq 0 \ ; \qquad i = 1, \ldots, n-1 \ , \tag{2.5}$$

and

$$\lambda_n = n\widetilde{\lambda}_n > 0 \ . \tag{2.6}$$

Let F_i be a $(n-2)$-dimensional face of $\Sigma(\mathbf{P})$, i.e. a set

$$\mathrm{F}_i(\mathbf{P}) = \left\{ \sum_{k=1}^{n} p_k P_k \subset \Sigma(\mathbf{P}) \,\middle|\, p_i = 0 \right\} \ , \tag{2.7}$$

and for any $a \in H_n$ and $r > 0$ denote by $B_n(a, r)$ the following ball

$$B_n(a, r) = \{ x \in H_n \,|\, ||a - x|| \leq r \,\} \,\subset\, H_n \ . \tag{2.8}$$

Theorem 1 *For any $r \leq r_{\max} := \lambda_n/\sqrt{n(n-1)}$ one has $B_n(\widetilde{\rho}, r) \subset \Sigma(\mathbf{P})$. Moreover, a maximal ball $B_n(\widetilde{\rho}, r_{\max})$ is tangent to the face* $\mathrm{F}_n(\mathbf{P})$.

Remark. In the special case when $\widetilde{\rho} \equiv \rho_0 = \mathbb{I}/n$, one has $\lambda_n = 1$ and $r_{\max} = 1/\sqrt{n(n-1)}$ defines a ball $B_n(\rho_0, r_{\max})$ inscribed in each simplex $\mathbf{P} = \{P_1, \ldots, P_n\}$ [12], that is, this ball is tangent to each face $\mathrm{F}_i(\mathbf{P})$.

Proof. Take an arbitrary point $\rho_{\boldsymbol{\alpha}} \in \mathrm{F}_n(\mathbf{P})$, i.e.

$$\rho_{\boldsymbol{\alpha}} = \alpha_1 P_1 + \ldots + \alpha_{n-1} P_{n-1} \ , \tag{2.9}$$

with $\alpha_i \geq 0$, and $\alpha_1 + \ldots + \alpha_{n-1} = 1$. Let us compute a distance between $\widetilde{\rho}$ and $\rho_{\boldsymbol{\alpha}}$

$$D(\boldsymbol{\alpha}) := ||\widetilde{\rho} - \rho_{\boldsymbol{\alpha}}||^2 \ . \tag{2.10}$$

One finds

$$D(\boldsymbol{\alpha}) = (\alpha_1 - \lambda_1)^2 + \ldots + (\alpha_{n-1} - \lambda_{n-1})^2 - \frac{\lambda_n^2}{n} \ . \tag{2.11}$$

To find a minimum of $D(\boldsymbol{\alpha})$ we treat $\alpha_1, \ldots, \alpha_{n-2}$ as independent variables ($\alpha_{n-1} = 1 - \alpha_1 - \ldots - \alpha_{n-2}$). The condition for a local extremum

$$\frac{\partial D(\boldsymbol{\alpha})}{\partial \alpha_i} = 0 \ ; \quad i = 1, \ldots, n-2 \ , \tag{2.12}$$

gives rise to the following system of linear equations

$$\sum_{j=1}^{n-2} \mathbf{A}_{ij}\, \alpha_j^* = \beta_i \ ; \quad i = 1, \ldots, n-2 \ , \tag{2.13}$$

where the $(n-2) \times (n-2)$ matrix $\mathbf{A}$ reads as follows

$$\mathbf{A}_{ij} = \left\{ \begin{array}{ccc} n-2 & ; & i = j \\ 1 & ; & i \neq j \end{array} \right. \ , \tag{2.14}$$

and

$$\beta_i = 1 + \lambda_i - \lambda_{n-1} \ ; \quad i = 1, \ldots, n-2 \ . \tag{2.15}$$

Finding the inverse matrix

$$\mathbf{A}^{-1}_{ij} = \left\{ \begin{array}{ccc} \frac{n-2}{n-1} & ; & i = j \\ \frac{-1}{n-1} & ; & i \neq j \end{array} \right. \ , \tag{2.16}$$

one obtains for the solution

$$\alpha_i^* = \lambda_i + \frac{\lambda_n}{n-1} \ ; \quad i = 1, \ldots, n-1 \ . \tag{2.17}$$

Inserting $\boldsymbol{\alpha}^* = (\alpha_1^*, \ldots, \alpha_{n-1}^*)$ into (2.11) one finds

$$r_{\max}^2 := D(\boldsymbol{\alpha}^*) = \frac{\lambda_n^2}{n(n-1)} \ , \tag{2.18}$$

which ends the proof. $\Box$

3 From balls to positive maps

Let us consider the following linear map

$$\varphi_\mu \ : \ M_n(\mathbb{C}) \ \longrightarrow \ M_n(\mathbb{C}) \ , \tag{3.1}$$

defined by

$$\varphi_\mu(a) := \mu a + (1-\mu)\widetilde{\rho}\,\mathrm{tr}\, a \ , \tag{3.2}$$

with a real parameter μ. Note, that

$$\varphi_\mu(\widetilde{\rho}) = \widetilde{\rho} \ , \tag{3.3}$$

and

$$\mathrm{tr}\,\varphi_\mu(a) = \mathrm{tr}\, a \ . \tag{3.4}$$

It is clear that if $\mu \in [0,1]$ then φ_μ is a CP map being a convex combination of two CP maps. Our aim is to prove the following

Theorem 2 *If μ satisfies*

$$|\mu| \leq \mu_{\max} \ , \tag{3.5}$$

where

$$\mu_{\max} := \frac{r_{\max}}{\sqrt{1 + \lambda_1^2 + \ldots + \lambda_{n-1}^2 - \lambda_n^2/n}} \ , \tag{3.6}$$

and $r_{\max}$ is defined in (2.18), then φ_μ is a positive map.

Proof. For any rank 1 projector P one has

$$\frac{\lambda_n}{n} \leq \mathrm{tr}(\widetilde{\rho}P) \leq \frac{\lambda_n}{n} + (\lambda_1 + \ldots + \lambda_{n-1}) \ . \tag{3.7}$$

Now, for any $\rho \in \mathcal{S}_n$

$$||\widetilde{\rho} - \rho|| \leq \max_P ||\widetilde{\rho} - P|| \ , \tag{3.8}$$

where the maximum is taken over all rank 1 projectors $P \in \mathcal{S}_n$. Now

$$||\widetilde{\rho} - P||^2 = ||\widetilde{\rho}||^2 + ||P||^2 - 2\mathrm{tr}(\widetilde{\rho}P) \leq ||\widetilde{\rho}||^2 + 1 - 2\frac{\lambda_n}{n} \ . \tag{3.9}$$

Moreover, one easily finds

$$\begin{aligned} ||\widetilde{\rho}||^2 &= \lambda_1^2 + \ldots \lambda_{n-1}^2 + \frac{\lambda_n^2}{n} + 2\frac{\lambda_n}{n}(\lambda_1 + \ldots + \lambda_{n-1}) \\ &= \lambda_1^2 + \ldots \lambda_{n-1}^2 + 2\frac{\lambda_n}{n} - \frac{\lambda_n^2}{n} \ , \end{aligned} \tag{3.10}$$

and hence one obtains the following bound for the distance between $\widetilde{\rho}$ and P

$$||\widetilde{\rho} - P||^2 \leq \lambda_1^2 + \ldots \lambda_{n-1}^2 + 1 - \frac{\lambda_n^2}{n} \ . \tag{3.11}$$

Now, let us compute the corresponding distance between $\widetilde{\rho}$ and $\varphi_\mu(\rho)$ for an arbitrary state $\rho \in \mathcal{S}_n$. Since

$$\widetilde{\rho} - \varphi_\mu(\rho) = \mu(\widetilde{\rho} - \rho) \ , \tag{3.12}$$

one has

$$\max_\rho ||\widetilde{\rho} - \varphi_\mu(\rho)||^2 \ = \ \mu^2 \max_\rho ||\widetilde{\rho} - \rho||^2 \leq \mu^2 \left(1 - \frac{\lambda_n^2}{n} + \lambda_1^2 + \ldots + \lambda_{n-1}^2 \right) \ . \tag{3.13}$$

Now, assume that

$$\mu^2 \left(1 - \frac{\lambda_n^2}{n} + \lambda_1^2 + \ldots + \lambda_{n-1}^2 \right) \leq r_{\rm max}^2 = \frac{\lambda_n^2}{n(n-1)} \ . \tag{3.14}$$

It implies that for any $\rho \in \mathcal{S}_n$ an image $\varphi_\mu(\rho) \in B_n(\widetilde{\rho}, r_{\rm max})$ and hence $\tilde{\varphi}$ is a positive map. Formula (3.14) is equivalent to (3.5) which ends the proof. □

Remark. In the special case when $\widetilde{\rho} \equiv \rho_0 = \mathbb{I}/n$, one has $\lambda_n = 1$ and

$$\mu_{\rm max} = \frac{1}{n-1} \ , \tag{3.15}$$

which reproduces the result of [12].

Figure 1 shows the action of φ_μ with $|\mu| = \mu_{\rm max}$ for $n = 3$, i.e. $\varphi_\mu(P_k) = P_k'$. The figure on the left corresponds to $\mu > 0$ and the map φ_μ is completely positive being a sum of two completely positive maps. The figure on the right corresponds to $\mu < 0$ and the the map φ_μ is positive but not CP.

4 Composing with affine maps

Having define a map φ_μ with a property that $\varphi_\mu(\rho) \in B_n(\widetilde{\rho}, r_{\rm max})$ for all density operators $\rho \in \mathcal{S}_n$ let us observe that we may compose it with an arbitrary affine map which maps a ball $B_n(\widetilde{\rho}, r_{\rm max})$ into itself, i.e. if

$$\psi \ : \ B_n(\widetilde{\rho}, r_{\rm max}) \ \longrightarrow \ B_n(\widetilde{\rho}, r_{\rm max}) \ , \tag{4.1}$$

then $\psi \circ \varphi_\mu$ maps all density matrices from $\mathcal{S}_n$ into $B_n(\widetilde{\rho}, r_{\rm max})$. Denote by Aff_n a set of affine maps $(\mathrm{T}, \mathbf{t}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which map the closed unit balls into itself, i.e.

$$(\mathrm{T}, \mathbf{t})\mathbf{x} := \mathrm{T}\mathbf{x} + \mathbf{t} \ , \tag{4.2}$$

where $\mathrm{T} \in M_n(\mathbb{R})$ and $\mathbf{t} \in \mathbb{R}^n$ represents translation. Now, Aff_n being a compact convex set it is entirely determined by its extremal elements.

Proposition 1 *The extremal elements* $\mathrm{Extr}\,\mathrm{Aff}_n$ *are defined by*

$$\mathrm{T} = R_1 \Lambda R_2 \ , \quad \mathbf{t} = R_1 \mathbf{c} \ , \tag{4.3}$$

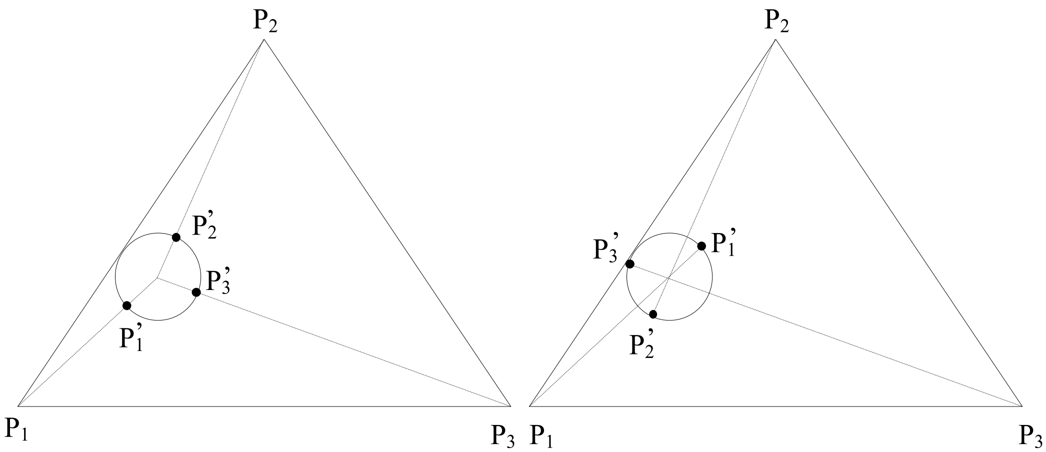

Figure 1: The action of φ_μ for $n = 3$. It maps P_k into P'_k. On the left $\mu > 0$ and φ_μ is CP, on the right $\mu < 0$ and φ_μ is positive but not CP.

where $R_1, R_2 \in O(n)$, Λ *is diagonal with eigenvalues*

$$\lambda_1 = \ldots = \lambda_{n-1} = \frac{\lambda_n}{\kappa} = \sqrt{1 - \delta^2(1-\kappa^2)} \ ,$$

with $0 \leq \kappa \leq 1$ *and* $0 < \delta \leq 1$. *Finally,* $\mathbf{c} = (c_1, \ldots, c_n)$ *reads as follows*

$$c_1 = \ldots = c_{n-1} = 0 \ , \quad c_n = \delta(1-\kappa^2) \ .$$

For the proof see [17]. Note, that $(\mathrm{T}, r\mathbf{t})$ maps a ball with radius r into itself provided $(\mathrm{T}, \mathbf{t}) \in \mathrm{Aff}_n$. Denote by Aff^0_n a subset of Aff_n corresponding to $\kappa = 0$. It is clear that

$$\mathrm{Extr}\,\mathrm{Aff}^0_n = \{ (\mathrm{T}, \mathbf{t}) \in \mathrm{Aff}_n \ : \ \mathrm{T} \in O(n) \ , \mathbf{t} = 0 \} \ . \tag{4.4}$$

It is convenient to introduce an orthonormal basis in H_n: $\mathbf{f} = (f_1, \ldots, f_{n^2-1})$ and $f_{n^2} = \mathbb{I}/\sqrt{n}$, such that $(f_\alpha, f_\beta) = \delta_{\alpha\beta}$. It implies that $\mathrm{tr}\, f_\alpha = 0$ for $\alpha = 1, \ldots, n^2 - 1$. Now, any element $a \in H_n$ may be decomposed as follows

$$a = \frac{\mathbb{I}}{n}\,\mathrm{tr}\, a + \langle \mathbf{f}, \mathbf{a} \rangle \ , \tag{4.5}$$

with $\mathbf{a} = (a_1, \ldots, a_{n^2-1}) \in \mathbb{R}^{n^2-1}$, $a_\alpha = \mathrm{tr}(f_\alpha a)$, and $\langle \mathbf{f}, \mathbf{a} \rangle = \sum_{\alpha=1}^{n^2-1} f_\alpha a_\alpha$. In particular one has

$$\widetilde{\rho} = \frac{\mathbb{I}}{n} + \langle \mathbf{f}, \widetilde{\mathbf{x}} \rangle \ , \tag{4.6}$$

and

$$a' := \varphi_\mu(a) = \frac{\mathbb{I}}{n} \,\mathrm{tr}\, a + \langle \mathbf{f}, \mathbf{a}' \rangle \ , \tag{4.7}$$

due to $\mathrm{tr}\, \varphi_\mu(a) = \mathrm{tr}\, a$. Now, if $a' \in B(\widetilde{\rho}, r_{\max})$ we may shift $\mathbf{a}'$ by '$-\widetilde{\mathbf{x}}\,\mathrm{tr} a$', apply an affine map $(\mathrm{T}, r_{\max}\mathbf{t})$ and then shift back by '$\widetilde{\mathbf{x}}\,\mathrm{tr} a$'. As a result one obtains again an element $a'' \in B(\widetilde{\rho}, r_{\max})$. Therefore, the main result of this section may be summarized by the following

Theorem 3 *For $|\mu| \leq \mu_{\max}$ every affine map $(\mathrm{T}, \mathbf{t}) \in \mathrm{Aff}_{n^2-1}$ induces a positive trace preserving map*

$$\varphi_\mu[\mathrm{T}, \mathbf{t}] \ : \ M_n(\mathbb{C}) \ \longrightarrow \ M_n(\mathbb{C}) \ ,$$

defined by

$$\varphi_\mu[\mathrm{T}, \mathbf{t}](a) = \widetilde{\rho}\,\mathrm{tr}\, a + \langle \mathbf{f}, (\mathrm{T}, r_{\max}\mathbf{t})(\mathbf{a}' - \widetilde{\mathbf{x}}\mathrm{tr} a) \rangle \ , \tag{4.8}$$

where $\widetilde{\mathbf{x}}$ and $\mathbf{a}'$ are given by (4.6) and (4.7), respectively.

Remark 1 Actually, we have constructed the action of $\varphi_\mu[\mathrm{T}, \mathbf{t}]$ only for hermitian elements. However, due to the linearity one obviously has

$$\varphi_\mu[\mathrm{T}, \mathbf{t}](a) = \varphi_\mu[\mathrm{T}, \mathbf{t}](a_1) + i\, \varphi_\mu[\mathrm{T}, \mathbf{t}](a_2) \ , \tag{4.9}$$

where $a = a_1 + i a_2$ is an arbitrary element from $M_n(\mathbb{C})$ with $a_1, a_2 \in H_n$.

Remark 2 If $\widetilde{\rho} = \mathbb{I}/n$, then one recovers a family of positive maps constructed in [12].

5 Example: generalized Choi map

Our basic formula (4.8) does depend upon an orthonormal basis f_α. Now, let $\{e_1, \ldots, e_n\}$ denote the eigen-basis of $\widetilde{\rho}$, that is, $\widetilde{\rho} e_i = \widetilde{\lambda}_i e_i$. Let us construct $\mathbf{f} = (f_1, \ldots, f_{n^2-1})$ as the following generators of $SU(n)$

$$(f_1, \ldots, f_{n^2-1}) = (d_\ell, u_{ij}, v_{ij}) \ ,$$

with $\ell = 1, \ldots, n-1$, and $1 \leq i < j \leq n$: d_ℓ generate Cartan subalgebra

$$d_\ell = \frac{1}{\sqrt{\ell(\ell+1)}} \Big(\sum_{k=1}^{\ell} e_{kk} - \ell e_{\ell+1,\ell+1} \Big) \ , \tag{5.1}$$

and

$$u_{ij} = \frac{1}{\sqrt{2}} (e_{ij} + e_{ji}) \ , \qquad v_{ij} = \frac{1}{\sqrt{2}\, i} (e_{ij} - e_{ji}) \ , \tag{5.2}$$

where $e_{ij} := |e_i\rangle\langle e_j|$.

To illustrate our general scheme let us consider $n = 3$ and take an affine transformation from a set $\mathrm{Extr}\,\mathrm{Aff}_8^0$, i.e. $(\mathrm{T}, \mathbf{t})$ with $\mathrm{T} \in O(8)$ and $\mathbf{t} = 0$. Let us introduce the following set of coordinates in $\mathbb{R}^8$:

$$x_\ell = \mathrm{tr}(a d_\ell) \ , \quad \ell = 1, 2 \ , \tag{5.3}$$

and

$$x_{ij} = \mathrm{tr}(au_{ij}), \qquad y_{ij} = \mathrm{tr}(av_{ij}) \ , \quad 1 \leq i < j \leq 3 \ . \tag{5.4}$$

Now, let T be a rotation from $O(8)$ given by

$$\begin{aligned} x_1' &= x_1 \cos\alpha - x_2 \sin\alpha \ , \\ x_2' &= x_1 \sin\alpha + x_2 \cos\alpha \ , \\ x_{ij}' &= -x_{ij} \ , \\ y_{ij}' &= -y_{ij} \ . \end{aligned} \tag{5.5}$$

In this parametrization the map

$$\varphi_{\mu_{\max}}[\alpha] \ : \ M_3(\mathbb{C}) \ \longrightarrow \ M_3(\mathbb{C}) \ ,$$

has the following form

$$\varphi_{\mu_{\max}}[\alpha](e_{ii}) = \sum_{j=1}^{3} \Lambda_{ij} e_{jj} \ , \tag{5.6}$$

$$\varphi_{\mu_{\max}}[\alpha](e_{ij}) = -\mu_{\max} e_{ij} \ , \quad i \neq j \ , \tag{5.7}$$

where

$$\Lambda_{ij} = \mu_{\max} \Lambda^0_{ij} + \Lambda^1_{ij} \ , \tag{5.8}$$

with Λ^0 being a circulant matrix defined by

$$\Lambda^0 = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_3 & \eta_1 & \eta_2 \\ \eta_2 & \eta_3 & \eta_1 \end{pmatrix} \ , \tag{5.9}$$

where the matrix elements η_i depend upon the parameter α in the following way

$$\begin{aligned} \eta_1(\alpha) &= \frac{2}{3} \cos\alpha \ , \\ \eta_2(\alpha) &= -\frac{1}{3}(\cos\alpha + \sqrt{3}\,\sin\alpha) \ , \\ \eta_3(\alpha) &= \frac{1}{3}\,(-\cos\alpha + \sqrt{3}\,\sin\alpha) \ , \end{aligned} \tag{5.10}$$

and

$$\Lambda^1 = \begin{pmatrix} \xi_1 & \xi_1 & \xi_1 \\ \xi_2 & \xi_2 & \xi_2 \\ \xi_3 & \xi_3 & \xi_3 \end{pmatrix} \ , \tag{5.11}$$

with

$$\begin{aligned} \xi_1 &= \lambda_1 + \frac{\lambda_3}{3} - \mu_{\max}\Big[\lambda_1\eta_1(\alpha) + \lambda_2\eta_2(\alpha)\Big] \ , \\ \xi_2 &= \lambda_2 + \frac{\lambda_3}{3} - \mu_{\max}\Big[\lambda_1\eta_3(\alpha) + \lambda_2\eta_1(\alpha)\Big] \ , \\ \xi_3 &= \frac{\lambda_3}{3} - \mu_{\max}\Big[\lambda_1\eta_2(\alpha) + \lambda_2\eta_3(\alpha)\Big] \ . \end{aligned} \tag{5.12}$$

Note that

$$\eta_1(\alpha) + \eta_2(\alpha) + \eta_2(\alpha) = 0 \ , \tag{5.13}$$

and

$$\xi_1 + \xi_2 + \xi_3 = \lambda_1 + \lambda_2 + \lambda_3 = 1 \ . \tag{5.14}$$

The matrix Λ^0 is *universal*, i.e. it does not depend upon the invariant state $\widetilde{\rho}$.

Remark 3 If $\widetilde{\rho} = \mathbb{I}/3$, then

$$\xi_1 = \xi_2 = \xi_3 = \frac{1}{3} \ , \tag{5.15}$$

and the matrix Λ is circulant and stochastic (hence doubly stochastic). For $\widetilde{\rho} \neq \mathbb{I}/3$, it is no longer circulant but Λ^T is stochastic.

Remark 4 The map $\varphi_{\mu_{\max}}[\alpha = \pi/3]$ reduces for $\widetilde{\rho} = \mathbb{I}/3$ to the celebrated Choi map [7] defined by

$$\varphi_{\rm Choi}(e_{ii}) \ = \ \sum_{j=1}^{3} \Lambda^{\rm Choi}_{ij} e_{jj} \ , \tag{5.16}$$

$$\varphi_{\rm Choi}(e_{ij}) \ = \ -\frac{1}{2}\, e_{ij} \ , \quad i \neq j \ , \tag{5.17}$$

where the doubly stochastic matrix $\Lambda^{\rm Choi}$ is defined by

$$\Lambda^{\rm Choi} = \frac{1}{2} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right) \ . \tag{5.18}$$

Finally, let us note that the corresponding entanglement witness

$$W[\alpha] = 3(\mathrm{id} \otimes \varphi_{\mu_{\max}}[\alpha])P^+_3 \ ,$$

where P^+_3 denotes the maximally entangled state in $\mathbb{C}^3 \otimes \mathbb{C}^3$, reads as follows

$$W[\alpha] \ = \ \mu_{\max} \left(\begin{array}{ccc|ccc|ccc} a_1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a_2 & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & b_2 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a_3 \end{array} \right) \ , \tag{5.19}$$

where the α-dependent coefficients are given by

$$a_i = \frac{\eta_1(\alpha) + \xi_i}{\mu_{\max}} \ , \qquad b_i = \frac{\eta_2(\alpha) + \xi_i}{\mu_{\max}} \ , \qquad c_i = \frac{\eta_3(\alpha) + \xi_i}{\mu_{\max}} \ . \tag{5.20}$$

It is clear that $a_i, b_i, c_i \geq 0$ and

$$a_i + b_{i+1} + c_{i+2} = \frac{1}{\mu_{\max}} \ , \tag{5.21}$$

for $i = 1, 2, 3 \pmod 3$. The above class of entanglement witnesses belongs to a class of bipartite operators studied in [18]. Note, that $W[\alpha]$ defines true entanglement witness iff it is not positive, i.e. possesses at least one negative eigenvalue, that is, the following 3×3 matrix

$$\begin{pmatrix} a_1 & -1 & -1 \\ -1 & a_2 & -1 \\ -1 & -1 & a_3 \end{pmatrix}$$

is not positive. It is easy to see that if $\widetilde{\rho} = \mathbb{I}/3$, then $W[\alpha]$ is never positive. However, it is no longer true for the general invariant state $\widetilde{\rho}$.

6 Conclusions

We introduced a new class of positive maps in the matrix algebra $M_n(\mathbb{C})$ using a family of balls in the space of density operators of n-level quantum system. Each map has an invariant state $\widetilde{\rho}$ which defines the center of the ball. If $\widetilde{\rho} = \mathbb{I}/n$, i.e. the map is unital, our construction generalizes the family of positive maps introduced in [12]. In particular for $n = 3$ it generalizes the celebrated Choi map [7]. As is well know positive maps which are not completely positive provide a basic tool to study quantum entanglement. Therefore our method provides new class of entanglement witnesses.

Presented construction guarantees positivity but says nothing about indecomposability and/or optimality [19]. Both indecomposable and optimal positive maps are crucial in detecting and classifying quantum entanglement. Therefore, the analysis of positive maps based on the family of balls deserves further study.

We stress that the structure off balls discussed in this paper may be easily introduced for the composed $n \otimes n$ system. In this case it generalizes well known ball of separable states centered at $\mathbb{I}/n^2$ [20]. It would be interesting to investigate properties of quantum states belonging to other (not necessary central) balls.

Acknowledgement

This work was partially supported by the Polish Ministry of Science and Higher Education Grant No 3004/B/H03/2007/33 and by the Polish Research Network *Laboratory of Physical Foundations of Information Processing*.

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