Estimating Concurrence via Entanglement Witnesses

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We show that each entanglement witness detecting given bipartite entangled state provides an estimation of its concurrence. We illustrate our result with several well known examples of entanglement witnesses and compare the corresponding estimation of concurrence with other estimations provided by the trace norm of partial transposition and realignment.

I. INTRODUCTION

The interest on quantum entanglement has dramatically increased during the last two decades due to the emerging field of quantum information theory [1, 2]. It turns out that quantum entanglement may be used as basic resources in quantum information processing and communication. The prominent examples are quantum cryptography, quantum teleportation, quantum error correction codes and quantum computation. Hence it is of basic importance from both experimental and theoretical point of view to provide methods of detecting and quantifying entanglement [3–5]. There are no universal criteria to detect quantum entanglement and there are few measures of entanglement (based on the notion of entropy [6], entanglement of formation [7], concurrence [8], robustness [9, 10], geometrical measures [11, 12] and others). They can be calculated for pure quantum states or for the very limited class of mixed states possessing some symmetry properties [13–16]. Therefore, the great effort is directed to obtain methods of estimation of particular entanglement measures and to find relations between them [17–22].

On the other hand, what is measured in an experiment it is an expectation value of some observables, hence the estimations based on such quantities are most welcome [23–25]. For example, it was recently shown [25] that the concurrence for pure bipartite state $|\psi\rangle$ living in a Hilbert space $\mathcal{H}$ can be obtained as follows

$$C(|\psi\rangle) \ = \ 2\sqrt{\langle\psi| \otimes \langle\psi|\hat{\Pi}|\psi\rangle \otimes |\psi\rangle}\,,$$

where $\hat{\Pi} = \hat{\Pi}_+ \otimes \hat{\Pi}_+$ and $\hat{\Pi}_+$ is the projector onto the symmetric subspace of $\mathcal{H}$, i.e., concurrence is expressible by the mean value of the observable $\hat{\Pi}$ acting on the two-copy space $\mathcal{H} \otimes \mathcal{H}$.

It turns out that useful candidates for this purpose are entanglement witnesses [24] (see [26] for the recent review on entanglement witnesses). We shown that each entanglement witness detecting given bipartite entangled state in $\mathcal{H}_A \otimes \mathcal{H}_B$ provides an estimation of its concurrence. Hence, EWs define a universal tool not only for detecting quantum entanglement but also for estimating its measure. We compare estimation based on entanglement witnesses with other ones provided by the trace norm of partial transposition and realignment.

The paper is organized as follows: in the next Section we provide basic introduction to concurrence and its estimations. Section III presents our main result which is illustrated by the family of examples in Section IV. Final conclusions are collected in the last section.

II. CONCURRENCE – PRELIMINARIES

Let us recall that the concurrence for a pure bipartite state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is defined as follows

$$C(|\psi\rangle) \ = \ \sqrt{2(1 - \mathrm{Tr}\,\rho_A^2)}\,, \tag{1}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ is the reduced density matrix. In the following, we will use a Schmidt decomposition of the pure state

$$|\psi\rangle \ = \ \sum_{i=1}^{m} \sqrt{\mu_i}|a_i\rangle \otimes |b_i\rangle, \tag{2}$$

where $m = \min\{\dim\mathcal{H}_A, \dim\mathcal{H}_B\}$, and $\{|a_i\rangle\}$, $\{|b_i\rangle\}$ are orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The Schmidt coefficients $\mu_i \geq 0$ and satisfy the following normalization condition

$$\sum_{i=1}^{m} \mu_i \ = \ 1\,. \tag{3}$$

It is easy to check that concurrence $C(|\psi\rangle))$ is uniquely defined in terms of the Schmidt coefficients μ_i. One has

$$C(|\psi\rangle) \ = \ \sqrt{2\sum_{k,l\neq k} \mu_k\mu_l}\,. \tag{4}$$

For a mixed state $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ the concurrence is defined via a convex roof construction

$$C(\rho) \ = \ \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)\,. \tag{5}$$

It is well known [8] that for a two-qubit case one finds the following formula for the concurrence of the arbitrary mixed state

$$C(\rho) \ = \ \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{6}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are singular values of a matrix $T_{kl} = \langle v_k|\sigma_y \otimes \sigma_y|v_l^*\rangle$ with $|v_k\rangle$ denoting eigenvectors of ρ

and σ_y stands for the Pauli matrix. In general, however, one has only the following estimation [27]

$$C^2(\rho) \ \geq \ \sum_{k=1}^{D}\sum_{l=1}^{D} \Big(\max\{0, \lambda_{kl}^{(1)} - \lambda_{kl}^{(2)} - \lambda_{kl}^{(3)} - \lambda_{kl}^{(4)}\} \Big)^2 \ , \tag{7}$$

where now $\lambda_{kl}^{(i)}$ are singular values of $(T^{k,l})_{\alpha,\beta} = \langle v_\alpha | L_k \otimes L_l | v_\beta^* \rangle$, $D = m(m-1)/2$ and L_k are generators of the SO(m) group. It is also possible to carry out the optimalisation procedure involved in (5) for particular families of states possessing some symmetry properties (Werner states, isotropic states) [13–16].

Let us recall two basic results which enable estimation of concurrence for an arbitrary entangled mixed state ρ.

Theorem 1 (Chen, Albeverio, Fei [28]) *The following estimation is valid:*

$$C(\rho) \geq \sqrt{\frac{2}{m(m-1)}} \Big(\max\{||\rho^{T_A}||_1, ||\mathcal{R}(\rho)||_1\} - 1 \Big) \ , \tag{8}$$

where $||X||_1$ denotes the trace norm of X.

As usual ρ^{T_A} denotes a partial transposition of ρ and $\mathcal{R}(\rho)$ stands for the realigned matrix [29, 30]. For some generalizations see [31]. Note that although for a PPT state $||\rho^{T_A}||_1 = 1$, the norm of a realigned matrix $||\mathcal{R}(\rho)||_1$ can still be greater than 1 resulting in a nontrivial estimation.

Let us recall that a hermitian operator W is called an entanglement witness for a state ρ, if $\mathrm{Tr}(\rho W) < 0$ while $\mathrm{Tr}(\sigma W) \geq 0$ for all separable states σ. There are many examples [32] of entanglement measures $M(\rho)$ (concurrence, negativity, robustness, etc.) which can be related to the expectation value of some entanglement witness

$$M(\rho) \ = \ \max\Big\{0, - \inf_{W\in\mathcal{M}} \mathrm{Tr}(\rho W)\Big\} \ , \tag{9}$$

where the set $\mathcal{M}$ depends on the measure M in question. It is therefore clear that if W is an entanglement witness for ρ, i.e. $\mathrm{Tr}(\rho W) < 0$, and $W \in \mathcal{M}$, then one finds the following estimation

$$M(\rho) \ \geq \ |\mathrm{Tr}(\rho W)|$$

for the measure of entanglement of ρ. In the case of concurrence one has the following theorem

Theorem 2 (Breuer [33]) *Let W be an entanglement witness such that*

$$-\langle\psi|W|\psi\rangle \ \leq \ \sum_{i,j\neq i} \sqrt{\mu_i\mu_j} \tag{10}$$

for every pure state (2). Then for an arbitrary mixed state ρ detected by the witness W

$$C(\rho) \ \geq \ \sqrt{\frac{2}{m(m-1)}} \ |\mathrm{Tr}(\rho W)| \, . \tag{11}$$

III. MAIN RESULT

Theorem 2 distinguishes a class of witnesses satisfying condition (10). Suppose now that W does not satisfy this condition. Clearly, for any $\alpha > 0$ the rescaled operator $\alpha^{-1}W$ still defines an EW. Does $\alpha^{-1}W$ satisfy (10)? To answer this question let us observe that for $|\psi\rangle = \sum_{i=1}^m \sqrt{\mu_i}|a_i, b_i\rangle$ the expectation value of W reads as follows

$$\langle\psi|W|\psi\rangle \ = \ \sum_{k,l} \sqrt{\mu_k\mu_l} A_{kl}^{(W)}(\psi) \, , \tag{12}$$

where the ψ-dependent matrix $A_{kl}^{(W)}$ is defined by

$$A_{kl}^{(W)}(\psi) = \mathrm{Re}\, \langle a_k, b_k|W|a_l, b_l\rangle \ . \tag{13}$$

Note, that

$$A_{kk}^{(W)}(\psi) \geq 0 \ , \tag{14}$$

by the very definition of entanglement witness. It is clear that $A_{kl}^{(W)}(\psi)$ encodes the entire information about W. Moreover, the condition (10) is equivalent to

$$\sum_{k,l} \sqrt{\mu_k\mu_l}(A_{kl}^{(W)} + 1) \ \geq \ 1 \, . \tag{15}$$

Let us observe that the space of normalized vectors defines a compact set and hence one may define a positive number λ by the following procedure

$$-\lambda := \min_{\psi} \min_{k\neq l} A_{kl}^{(W)}(\psi) \ . \tag{16}$$

Now, comes the main result

Theorem 3 *For any $\alpha \geq \lambda$ the rescaled entanglement witness $\alpha^{-1}W$ does satisfy (10).*

The proof is almost trivial. One has

$$\begin{aligned}\sum_{k,l} \sqrt{\mu_k\mu_l}(A_{kl}^{(W)}(\psi) + 1) \ &= \ 1 + \sum_k \mu_k A_{kk}^{(W)}(\psi) \\ &+ \ \sum_{k,l\neq k} \sqrt{\mu_k\mu_l}(A_{kl}^{(W)}(\psi) + 1) \\ &\geq \ 1 + \sum_{k,l\neq k} \sqrt{\mu_k\mu_l}(A_{kl}^{(W)}(\psi) + 1) \ ,\end{aligned}$$

where we have used (14). Hence, if

$$A_{kl}^{(W)}(\psi) \geq -1 \ , \tag{17}$$

for every normalized ψ, then W does satisfy (10). Suppose now that the above condition is not satisfied. It is therefore clear that for the rescaled witness $W_\alpha := \alpha^{-1}W$ with $\alpha \geq \lambda$, one has

$$A_{kl}^{(W_\alpha)}(\psi) \geq -1 \ , \tag{18}$$

which proves our theorem. It should be stressed that the best estimation is provided by the witness corresponding to $\alpha = \lambda$.

IV. EXAMPLES

Example 1 Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^m$ and consider the flip operator

$$F \;=\; \sum_{i,j=1}^{m} |i\rangle\langle j| \otimes |j\rangle\langle i|$$

where $\{|i\rangle\}$ is the computional basis in $\mathbb{C}^m$. Simple calculation gives

$$A^{(F)}_{kl} \;=\; \langle a_k, b_k|F|a_l, b_l\rangle \;=\; \langle a_k|b_l\rangle\langle b_k|a_l\rangle\,.$$

Now, evidently $A^{(F)}_{kk} = |\langle a_k|b_k\rangle|^2 \geq 0$ and for $k \neq l$

$$A^{(F)}_{kl} \;=\; \mathrm{Re}\,(\langle a_k|b_l\rangle\langle b_k|a_l\rangle) \;\geq\; -1$$

according to orthonormality of both basis.

Example 2 Let us consider isotropic states in $\mathbb{C}^m \times \mathbb{C}^m$

$$\rho_f = \frac{1-f}{m^2-1}(\mathbb{1} - P^+_m) + fP^+_m\,, \qquad (19)$$

where P^+_m denotes the maximally entangled state and f is the fidelity defined by $f = \langle\psi^+_m|\rho_f|\psi^+_m\rangle$. Moreover, one introduces a family of k-EWs [34]

$$W^{\rm iso}_k = \frac{k}{m}\mathbb{1} - P^+_m\,, \qquad k = 1,\ldots,m-1\,, \qquad (20)$$

satisfying

$$\mathrm{Tr}\,[W^{\rm iso}_k\rho_f] = \frac{k}{m} - f\,,$$

that is, $W^{\rm iso}_k$ detects isotropic state with fidelity $f > k/m$. Such state has Schmidt number strictly greater than k. Since $P^+_m = \frac{1}{m}F^{T_A}$ the previous example implies for $i \neq j$

$$A^{(W^{\rm iso}_k)}_{ij} \geq -\frac{1}{m}\,,$$

which shows that $\lambda = 1/m$. As a consequence, the optimal $W^{\rm iso}_k$, in the sense of (18), is $\widetilde{W}^{\rm iso}_k = mW^{\rm iso}_k$. Now,

$$\mathrm{Tr}\,(\rho_f\widetilde{W}^{\rm iso}_k) = m\mathrm{Tr}\,(\rho_f W^{\rm iso}_k) = m\Big(\frac{k}{m} - f\Big) \qquad (21)$$

and the estimation (11) takes the form

$$\sqrt{\frac{2m}{m-1}}\Big(f - \frac{k}{m}\Big) \leq C(\rho_f)\,.$$

Note that although for $k \neq 1$, $W^{\rm iso}_k$ provides only the bound for concurrence, when $k = 1$, we obtain an *exact* result [13].

Let us note that a much more general (but also numerically more involved) method of estimating various entanglement measures was proposed in [18]. The method uses a concept of an entanglement witness and on the other hand provides a numerical procedure to calculate the Legendre transform of the measure in question. The method provided in this paper is much more restricted. However, being simpler it provides estimation of concurrence which can be very often computed analytically. The above examples show that it can leads not only to upper bounds for concurrence but also to exact results.

Example 3 In [35] we have investigated an ε-family ($\varepsilon > 0$) of states in $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\rho(\varepsilon) \;=\; N_\varepsilon\Big(P^+_3 + \frac{1}{3}\sum_{i\neq j=1}^{3} d_{ij}|ij\rangle\langle ij|\Big)\,, \qquad (22)$$

where P^+_3 denotes a maximally entangled state,

$$d_{i,i+1} = \varepsilon\,, \qquad d_{i,i+2} = \frac{1}{\varepsilon}\,, \quad (mod\ 3)$$

and the normalization factor

$$N_\varepsilon = \frac{1}{1+\varepsilon+\varepsilon^{-1}}\,.$$

It turns out that $\rho(\varepsilon)$ is entangled if and only if $\varepsilon \neq 1$. Moreover, its entanglement is detected by the entanglement witness

$$W_1 \;=\; \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1
\end{array}\right)\,, \qquad (23)$$

for $\varepsilon < 1$ and

$$W_2 \;=\; \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
-1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1
\end{array}\right) \qquad (24)$$

for $\varepsilon > 1$. To make pictures more transparent we replaced all zeros by dots. Interestingly, W_1 corresponds to the celebrated Choi positive indecomposable map and

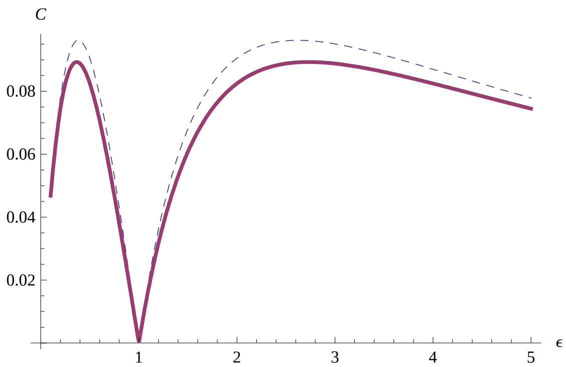

FIG. 1: Two estimations of concurrence as a function of ε. The dashed line is for the estimation based on $||\mathcal{R}(\rho(\varepsilon))||_1$ due to (8). The solid line is for the estimation based on (25).

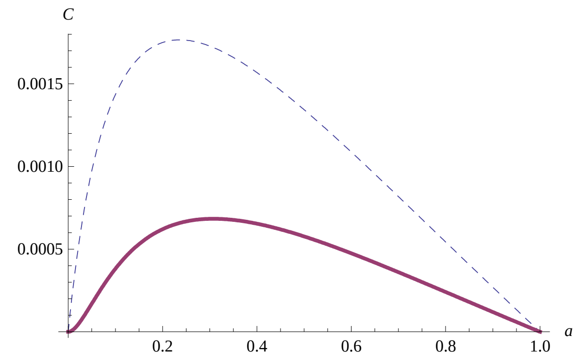

FIG. 2: Two estimations of concurrence as a function of a. The dashed line is for the estimation based on $||\mathcal{R}(\rho(a))||_1$ due to (8). The solid line is for the estimation based on (27).

W_2 to its dual. Numerical calculations show that indeed $A_{kl}^{(W_i)}(\psi) \geq -1$ for $i = 1, 2$. Hence one obtains the following estimation for concurrence based on the above EWs

$$C(\rho(\varepsilon)) \;\geq\; -\frac{1}{\sqrt{3}} \begin{cases} \dfrac{\varepsilon(\varepsilon-1)}{1+\varepsilon+\varepsilon^2} & 0 < \varepsilon < 1 \\ \dfrac{1-\varepsilon}{1+\varepsilon+\varepsilon^2} & \varepsilon > 1 \end{cases} . \tag{25}$$

We stress, however, that this estimation is weaker than the one obtained from the trace norm of realigned matrix (see Fig. 1).

Example 4 Sixia and Yu [36] constructed a family of entanglement witnesses $W(a)$ for the Horodecki states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ [37] $(0 < a < 1)$:

$$\rho(a) = \frac{1}{8a+1} \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2}(1+a) & \cdot & \frac{1}{2}\sqrt{1-a^2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \frac{1}{2}\sqrt{1-a^2} & \cdot & \frac{1}{2}(1+a) \end{array} \right) .$$

The witness $W(a)$ which detects entanglement of $\rho(a)$ has the following form

$$W(a) \;=\; \mathbb{1} - f(a)V(a)$$

where

$$f(a) \;=\; 2\sqrt{(a+2)[(1+8a)^2 + a^2(1-a)]}$$

and the real symmetric matrix $V(a)$ reads

$$V(a) = \begin{pmatrix} v_{11} & \cdot & v_{13} & \cdot & v_{15} & \cdot & v_{17} & \cdot & v_{19} \\ \cdot & v_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & v_{28} & \cdot \\ v_{13} & \cdot & v_{33} & \cdot & \cdot & \cdot & v_{37} & \cdot & v_{39} \\ \cdot & \cdot & \cdot & v_{44} & \cdot & v_{46} & \cdot & \cdot & \cdot \\ v_{15} & \cdot & \cdot & \cdot & v_{55} & \cdot & \cdot & \cdot & v_{59} \\ \cdot & \cdot & \cdot & v_{46} & \cdot & v_{66} & \cdot & \cdot & \cdot \\ v_{17} & \cdot & v_{37} & \cdot & \cdot & \cdot & v_{77} & \cdot & v_{79} \\ \cdot & v_{28} & \cdot & \cdot & \cdot & \cdot & \cdot & v_{88} & \cdot \\ v_{19} & \cdot & v_{39} & \cdot & v_{59} & \cdot & v_{79} & \cdot & v_{99} \end{pmatrix} \tag{26}$$

For the full list of entries v_{ij} see the Appendix. One has

$$\begin{aligned} \mathrm{Tr}\,[W(a)\rho(a)] \;&=\; 1 - f(a)\,\mathrm{Tr}\,(V(a)\rho(a)) \\ &=\; 1 - f(a)\,\frac{2(2+33a+145a^2+63a^3)}{1+8a} . \end{aligned}$$

Numerical calculations show again that $A_{kl}^{(W(a))} \geq -1$ for $k \neq l$. Hence, one obtains the following estimation for concurrence

$$C(\rho(a)) \;\geq\; -\mathrm{Tr}\,[W(a)\rho(a)]/\sqrt{3} . \tag{27}$$

Again this estimation is weaker than the one obtained from the realignment (see Fig. 2).

Example 5 Using the Tang map [38] one can construct the following family of entanglement witnesses

$$W(u) \;=\; \left[\begin{array}{cccc|cccc} 1-u^2/6 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -2 & \cdot \\ \cdot & \cdot & 2 & \cdot & u & \cdot & \cdot & -2 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & u & \cdot & u^2 & \cdot & \cdot & -u \\ -1 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & -2 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & -2 & \cdot & -u & \cdot & \cdot & 1 \end{array} \right] .$$

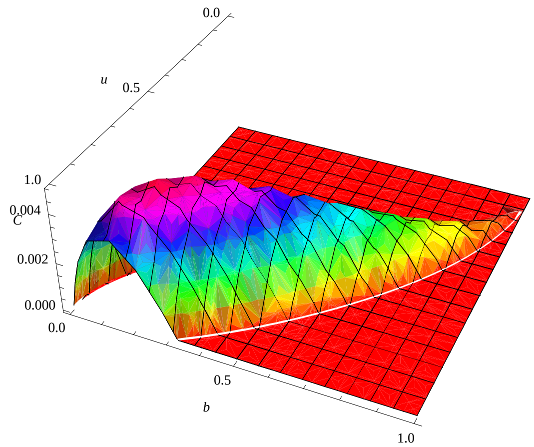

FIG. 3: Estimation of concurrence as a function of b and u.

This family detects Horodecki states $\rho(b)$ $(0 < b < 1)$ in $\mathbb{C}^2 \otimes \mathbb{C}^4$ [37]. Now,

$$\mathrm{Tr}\,[W(u)\rho(b)] \;=\; \frac{3-3b-6u\sqrt{1-b^2}+3u^2+2bu^2}{6+42b}$$

and $W(u)$ detects $\rho(b)$ if and only if $u_1 \le u \le u_2$, where

$$u_1 \;=\; \frac{3\sqrt{1-b^2}-\sqrt{3b(1-b)}}{3+2b}$$

$$u_2 \;=\; \begin{cases} 1 & b < \dfrac{12}{37} \\ \dfrac{3\sqrt{1-b^2}+\sqrt{3b(1-b)}}{3+2b} & b \ge \dfrac{12}{37} \end{cases}.$$

Numerical results show that $A_{kl}^{(W(u))} \ge -2$ for $k \neq l$ and hence we define rescaled witness by

$$\widetilde{W}(u) = \frac{1}{2} W(u)\ .$$

The estimation for concurrence

$$C(\rho(b)) \ge -\mathrm{Tr}\,[\rho(b)\widetilde{W}(u)]/\sqrt{3}$$

is shown in Fig. 3.

Example 6 Consider a family of states in $\mathbb{C}^d \otimes \mathbb{C}^d$ defined by [39, 40]

$$\rho_\gamma = \frac{1}{N_\gamma} \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes A_{ij}^\gamma\,, \qquad (28)$$

where

$$\begin{aligned} A_{11}^\gamma &= |1\rangle\langle 1| + a_\gamma |2\rangle\langle 2| + \sum_{\ell=3}^{d-1} |\ell\rangle\langle \ell| + b_\gamma |d\rangle\langle d| \\ A_{ij}^\gamma &= |i\rangle\langle j|\,, \qquad i \neq j\,, \\ A_{jj}^\gamma &= S^{j-1} A_{11}^\gamma S^{\dagger j-1}\,, \end{aligned}$$

with

$$a_\gamma = \frac{1}{d}(\gamma^2 + d - 1)\,, \qquad b_\gamma = \frac{1}{d}(\gamma^{-2} + d - 1)\,.$$

and the normalization factor reads

$$N_\gamma = d^2 - 2 + \gamma^2 + \gamma^{-2}\ .$$

The operator $S : \mathbb{C}^d \to \mathbb{C}^d$ is defined by $S|k\rangle = |k+1\rangle$ $(\mathrm{mod}\, d)$. Note that for $d = 3$ the state ρ_γ has a very similar structure to the states $\rho(\varepsilon)$ (22) considered in Example 2. Now, as was shown in [39], the states ρ_γ are detected by a family of entanglement witnesses

$$W_{d,k} = \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes X_{ij}^{d,k}$$

generalizing those described by (23) and (24) (which correspond to $d = 3$ and $k = 1$). The $d \times d$ matrices $X_{ii}^{d,k} = (d-k-1)|i\rangle\langle i| + \sum_{\ell=1}^{k} |i+\ell\rangle\langle i+\ell|$ and $X_{ij}^{d,k} = -|i\rangle\langle j|$ for $i \neq j$ (all additions mod d). Numerical calculations show that $A_{mn}^{(W_{d,k})} \ge -C_{d,k}$, where for coefficients $C_{d,k}$ we conjecture the following analytic formulae

$$C_{d,k} = \begin{cases} \dfrac{d-k}{2} & \text{for } d-2 \ge 2k\,, \\ \dfrac{d-k}{2} - \dfrac{1}{4} & \text{for } d-2 < 2k\,, \\ 1 & \text{for } k = d-2\,. \end{cases}$$

Now, for a rescaled witness $\widetilde{W}_{d,k} = W_{d,k}/C_{d,k}$ we obtain

$$\mathrm{Tr}\,(\widetilde{W}_{d,k}\rho_\gamma) = \frac{\gamma^2 - 1}{d^2 - 2 + \gamma^2 + \gamma^{-2}} \cdot \frac{1}{C_{d,k}}$$

and hence the estimation for concurrence of ρ_γ reads

$$C(\rho_\gamma) \ge \sqrt{\frac{2}{d(d-1)}} \cdot \frac{1-\gamma^2}{d^2 - 2 + \gamma^2 + \gamma^{-2}} \cdot \frac{1}{C_{d,k}}\,. \qquad (29)$$

In Fig. 4 we have shown the estimation of concurrence (29) of ρ_γ for different values of d and maximal $k = d-2$. The case corresponds therefore to the detection of entanglement by the Choi witness in $d = 3$ and its natural generalization in $d = 4, 5$. It is shown that the estimation of concurrence becomes weaker when the dimension d increases.

The influence of the parameter k on the estimation of concurrence for $d = 5$ is shown in Fig. 5. One can see that the best estimation gives the witness corresponding to the maximal available value of $k = d - 2$ – the one which generalizes the Choi witness.

V. CONCLUSIONS

We shown that each entanglement witness detecting given bipartite entangled state in $\mathcal{H}_A \otimes \mathcal{H}_B$ provides an

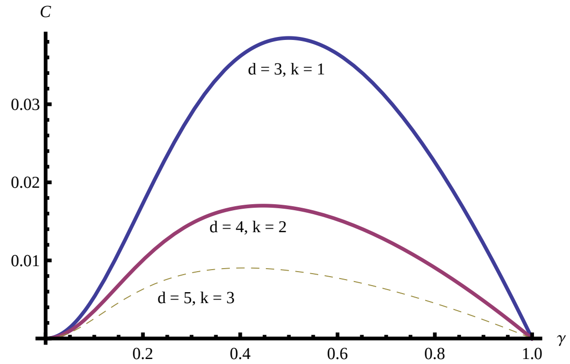

FIG. 4: Estimation of concurrence (29) as a function of γ for different values of d and $k = d - 2$, i.e., for the case corresponding to the Choi map for $d = 3$.

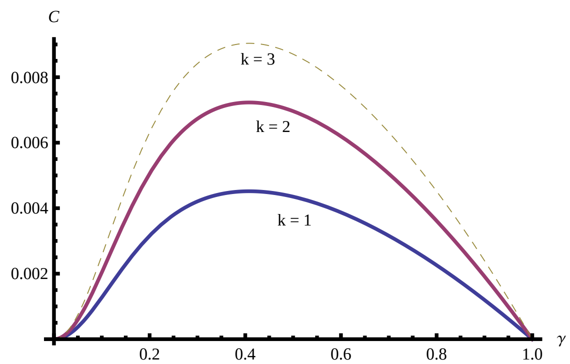

FIG. 5: Estimation of concurrence (29) as a function of γ for $d = 5$ and different values of k.

estimation of its concurrence. We analyzed an estimation for concurrence provided by an (properly rescaled) entanglement witness for different families of states in various dimensions and compared the corresponding estimation of concurrence with other estimations provided by the trace norm of partial transposition and realignment. It is shown that typically entanglement witnesses give weaker estimations than those obtained by realignment but formulae for estimations are analytic.

We introduced a quantity λ (cf. formula (16)) which does provide new characterization of an arbitrary entanglement witness. This quantity defines an *optimal* rescaling which gives rise to the best estimation of concurrence.

We analyzed a family of EWs $W_{d,k}$ in $\mathbb{C}^d \otimes \mathbb{C}^d$. It is shown that the best estimation is provided by EW corresponding to $k = d-2$. It turns out that it generalizes the witness based on the Choi map in $d = 3$.

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Appendix

The list of entries of the matrix $V(a)$:

$$\begin{aligned}
v_{11} = v_{66} &= -(1+a)(1+8a) \\
v_{13} = v_{46} &= -(1+7a)\sqrt{1-a^2} \\
v_{15} = v_{59} &= 2(2+a)(1+8a) \\
v_{17} = v_{39} &= -(1+9a)\sqrt{1-a^2} \\
v_{19} &= 3(1+a)(1+8a) \\
v_{22} = v_{55} &= 2+19a+15a^2 \\
v_{28} &= (2+15a)\sqrt{1-a^2}
\end{aligned}$$

$$\begin{aligned}
v_{33} = v_{44} &= (3+a)(1+8a) \\
v_{37} &= -(1-a)(1+8a) \\
v_{77} = v_{99} &= 2+17a+17a^2 \\
v_{79} &= (2+17a)\sqrt{1-a^2} \\
v_{88} &= -2a(1+8a)
\end{aligned}$$

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