A class of commutative dynamics of open quantum systems

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Abstract

We analyze a class of dynamics of open quantum systems which is governed by the dynamical map mutually commuting at different times. Such evolution may be effectively described via spectral analysis of the corresponding time dependent generators. We consider both Markovian and non-Markovian cases.

1 Introduction

The dynamics of open quantum systems attracts nowadays increasing attention [1]. It is very much connected to the growing interest in controlling quantum systems and applications in modern quantum technologies such as quantum communication, cryptography and computation [2]. The most popular approach is to use a Markovian approximation and to consider a master equation

$$\dot{A}_{t,t_0} = \mathcal{L}_t A_{t,t_0} \ , \qquad A_{t_0,t_0} = \mathrm{id} \ , \tag{1.1}$$

with time dependent generator $\mathcal{L}_t$. The above equation gives rise to a quantum dynamical map (completely positive and trace preserving) A_t which in turn produces the evolution of a quantum state $\rho_t = A_t \rho$. The corresponding generator $\mathcal{L}_t$ has to satisfy well known condition [3, 4] (see also [5] for the detail presentation) and the solution is given by the following formula

$$A_{t,t_0} = \mathbb{T}\,\exp\left(\int_{t_0}^{t} \mathcal{L}_u\, du\right) \ , \tag{1.2}$$

where $\mathbb{T}$ denotes a chronological product. We stress that the above formula has only a formal character since the evaluation of its r.h.s. is in general not feasible. If the generator does not depend on time $\mathcal{L}_t = \mathcal{L}$ then it simplifies to

$$A_{t,t_0} = \exp\left(\mathcal{L}\,(t - t_0)\right) \ . \tag{1.3}$$

Let us note that characteristic feature of (1.3) is that Markovian semigroup $A_t := A_{t+t_0,t_0}$ is commutative, that is

$$A_t A_s = A_s A_t \ , \qquad s, t \geq 0 \ . \tag{1.4}$$

It is no longer true for the general time dependent case (1.2). The general Markovian evolution does satisfy the inhomogeneous composition law

$$A_{t,u} A_{u,s} = A_{t,s} \ , \tag{1.5}$$

for $t \geq u \geq s \geq t_0$, however, it is in general noncommutative.

Non-Markovian evolution is much more difficult to analyze (see [6]–[17] for the recent papers). The local master equation is replaced by the following equation

$$\dot{A}_{t,t_0} = \int_{t_0}^{t} \mathcal{K}_{t-u} A_{u,t_0}\, du \ , \quad \rho(t_0) = \rho_0 \ , \tag{1.6}$$

in which quantum memory effects are taken into account through the introduction of the memory kernel $\mathcal{K}_t$: this simply means that the rate of change of the state $\rho(t)$ at time t depends on its history (starting at $t = t_0$). Recently, we proposed a different approach [18] which replaces the non-local equation (1.6) by the following local in time master equation

$$\dot{A}_{t,t_0} = \mathcal{L}_{t-t_0} A_{t,t_0} \ , \qquad A_{t_0,t_0} = \mathrm{id} \ . \tag{1.7}$$

The price one pays for the local approach is that the corresponding generator keeps the memory about the starting point 't_0'. This is the very essence of non-Markovianity. Interestingly, this generator might be highly singular, nevertheless, the corresponding dynamics is perfectly regular. Remarkably, singularities of generator may lead to interesting physical phenomena like revival of coherence or sudden death and revival of entanglement [18]. Now, the formal solution to (1.7) reads as follows

$$A_{t,t_0} = \mathbb{T}\,\exp\left(\int_0^{t-t_0} \mathcal{L}_u\, du\right) \ . \tag{1.8}$$

It resembles very much Markovian dynamical map (1.2) and again its r.h.s. has only formal character due to the presence of the chronological operator. Note, however, important difference between (1.2) and (1.2): the former does satisfy composition law. The latter is homogeneous in time (depends upon the difference $t - t_0$) but does not satisfy (4.26).

In the present paper we analyze a special case of commutative dynamics, i.e. we generalize (1.4) for time dependent Markovian and non-Markovian dynamics. In this case formulae (1.2) and (1.2) considerably simplify – the chronological product drops out and may compute the formula for the dynamical map via spectral analysis.

2 Preliminaries

Consider d-dimensional complex Hilbert space $\mathbb{C}^d$ and let $\{e_0, \ldots, e_{d-1}\}$ be a fixed orthonormal basis. For any $x, y \in \mathbb{C}^d$ denote by $\langle x, y\rangle$ the corresponding scalar product of x and y. Let $M_d = \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$ denote a space of linear operators in $\mathbb{C}^d$. Now, M_d is equipped with the Hilbert-Schmidt scalar product

$$(a, b) := \sum_{k=0}^{d-1} \langle ae_k, be_k\rangle = \mathrm{tr}(a^* b) \ , \tag{2.1}$$

where $a^* : \mathbb{C}^d \to \mathbb{C}^d$ is defined by

$$\langle a^* x, y\rangle = \langle x, ay\rangle \ , \tag{2.2}$$

for arbitrary $x, y \in \mathbb{C}^d$. Finally, let us introduce the space $\mathcal{L}(M_d, M_d)$ of linear maps $A : M_d \to M_d$. For any $A \in \mathcal{L}(M_d, M_d)$ one defines a dual map $A^\# \in \mathcal{L}(M_d, M_d)$ by

$$(A^\# a, b) = (a, Ab) \ , \tag{2.3}$$

for arbitrary $a, b \in M_d$. Note, that if the dual map $A^\#$ is unital, i.e. $A^\# \mathbb{I}_d = \mathbb{I}_d$, then A is trace preserving. It is clear that $\mathcal{L}(M_d, M_d)$ defines $d^2 \times d^2$ complex Hilbert space equipped with the following inner product

$$\langle\langle A, B\rangle\rangle = \sum_{\alpha=0}^{d^2-1} (Af_\alpha, Bf_\alpha) = \sum_{\alpha=0}^{d^2-1} \mathrm{tr}\left[(Af_\alpha)^*(Bf_\alpha)\right] \ , \tag{2.4}$$

for any $A, B \in \mathcal{L}(M_d, M_d)$. In the above formula f_α denote an orthonormal basis in M_d. Let us observe that in $\mathcal{L}(M_d, M_d)$ one constructs two natural orthonornal basis

$$F_{\alpha\beta} \ : \ M_d \longrightarrow M_d \ , \tag{2.5}$$

and

$$E_{\alpha\beta} \ : \ M_d \longrightarrow M_d \ , \tag{2.6}$$

defined as follows

$$F_{\alpha\beta} a = f_\alpha a f_\beta^* \ , \tag{2.7}$$

and

$$E_{\alpha\beta} a = f_\alpha (f_\beta, a) \ , \tag{2.8}$$

for any $a \in M_d$. One easily proves

$$\langle\langle F_{\alpha\beta}, F_{\mu\nu} \rangle\rangle = \langle\langle E_{\alpha\beta}, E_{\mu\nu} \rangle\rangle = \delta_{\alpha\mu}\delta_{\beta\nu} \ . \tag{2.9}$$

Moreover, the following relations are satisfied

$$\sum_{\alpha=0}^{d^2-1} F_{\alpha\alpha}\, a = \mathbb{I}_d \, \mathrm{tr} a \ , \tag{2.10}$$

and

$$\sum_{\alpha=0}^{d^2-1} E_{\alpha\alpha}\, a = a \ . \tag{2.11}$$

Remark 1 Note, that representing a linear map A in the basis $F_{\alpha\beta}$

$$A = \sum_{\alpha,\beta} a_{\alpha\beta}\, F_{\alpha\beta} \ , \tag{2.12}$$

with

$$a_{\alpha\beta} = \langle\langle A, F_{\alpha\beta} \rangle\rangle \ , \tag{2.13}$$

one has a simple criterion for complete positivity of A: a map A is complete positive if and only if the corresponding $d^2 \times d^2$ matrix $||a_{\alpha\beta}||$ is semipositive definite. On the other hand the E-representation

$$A = \sum_{\alpha,\beta} a'_{\alpha\beta}\, E_{\alpha\beta} \ , \tag{2.14}$$

with

$$a'_{\alpha\beta} = \langle\langle A, E_{\alpha\beta} \rangle\rangle \ , \tag{2.15}$$

does not give any simple criterion for complete positivity. Note, however, that E-representation is well suited for the composition of maps. If

$$B = \sum_{\alpha,\beta} b'_{\alpha\beta}\, E_{\alpha\beta} \ , \tag{2.16}$$

with

$$b'_{\alpha\beta} = \langle\langle B, E_{\alpha\beta} \rangle\rangle \ , \tag{2.17}$$

then the map $C = A \circ B$ gives rise to the following representation

$$C = \sum_{\alpha,\beta} c'_{\alpha\beta}\, E_{\alpha\beta} \ , \tag{2.18}$$

where the matrix $c' = a' \cdot b'$.

Consider now a linear map A from $\mathcal{L}(M_d, M_d)$ and let us assume that A is diagonalizable, that is, it gives rise to the Jordan representation with 1-dimensional Jordan blocks. One has

$$A = V\,D\,V^{-1}\ , \tag{2.19}$$

where D is diagonal. It means that there exists an orthonormal basis f_α in M_d such that

$$\langle\langle f_\alpha, Df_\beta\rangle\rangle = d_\alpha \delta_{\alpha\beta}\ , \tag{2.20}$$

with $d_\alpha \in \mathbb{C}$. It shows that

$$D = \sum_{\alpha=0}^{d^2-1} d_\alpha\, P_\alpha\ , \tag{2.21}$$

where

$$P_\alpha a = f_\alpha(f_\alpha, a)\ , \quad a \in M_d\ . \tag{2.22}$$

Note, that a set P_α defines a family of orthogonal projectors

$$P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha\ , \tag{2.23}$$

together with

$$\sum_{\alpha=0}^{d^2-1} P_\alpha = \mathrm{id}\ , \tag{2.24}$$

where id denotes an identity map in $\mathcal{L}(M_d, M_d)$. Hence, one obtains the following representation of A

$$\begin{aligned} Aa &= V\,D\,V^{-1}a = \sum_{\alpha=0}^{d^2-1} d_\alpha\, V P_\alpha V^{-1} a \\ &= \sum_{\alpha=0}^{d^2-1} d_\alpha\, V f_\alpha(f_\alpha, V^{-1}a) = \sum_{\alpha=0}^{d^2-1} d_\alpha\, V f_\alpha(V^{-1\#} f_\alpha, a)\ . \end{aligned} \tag{2.25}$$

Let us define new basis

$$g_\alpha := V f_\alpha\ , \qquad h_\alpha := V^{-1\#} f_\alpha\ . \tag{2.26}$$

Note g_α and h_α define a pair of bi-orthogonal (or damping [19]) basis in M_d

$$(g_\alpha, h_\beta) = (V f_\alpha, V^{-1\#} f_\beta) = (f_\alpha, f_\beta) = \delta_{\alpha\beta}\ . \tag{2.27}$$

Finally, one obtains the following spectral representation of the linear map A

$$A = \sum_{\alpha=0}^{d^2-1} d_\alpha\, \widetilde{P}_\alpha\ , \tag{2.28}$$

where

$$\widetilde{P}_\alpha a := g_\alpha(h_\alpha, a) \ , \qquad a \in M_d \ . \tag{2.29}$$

Note, that a set $\widetilde{P}_\alpha$ satsfies

$$\widetilde{P}_\alpha \widetilde{P}_\beta = \delta_{\alpha\beta} \widetilde{P}_\alpha \ , \tag{2.30}$$

together with

$$\sum_{\alpha=0}^{d^2-1} \widetilde{P}_\alpha = \mathrm{id} \ . \tag{2.31}$$

However, contrary to P_α operators $\widetilde{P}_\alpha$ are not Hermitian, i.e. $\widetilde{P}_\alpha^\# \neq \widetilde{P}_\alpha$

$$\widetilde{P}_\alpha^\# a := h_\alpha(g_\alpha, a) \ , \qquad a \in M_d \ . \tag{2.32}$$

It shows that $\widetilde{P}_\alpha$ are not projectors unless $g_\alpha = h_\alpha$. The corresponding spectral representation of the dual map $A^\#$ reads as follows

$$A^\# = \sum_{\alpha=0}^{d^2-1} \overline{d_\alpha}\, \widetilde{P}_\alpha^\# \ , \tag{2.33}$$

where $\overline{x}$ stands for the complex conjugation of the complex number x. Hence, one obtains the following family of eigenvectors

$$A g_\alpha = d_\alpha g_\alpha \ , \qquad A^\# h_\alpha = \overline{d_\alpha} h_\alpha \ . \tag{2.34}$$

Consider for example a special case with $V = U$ and U is a unitary operator in M_d. One has $U^\# = U^{-1}$ and hence $V^{-1\#} = U$. One obtains

$$g_\alpha \equiv h_\alpha = U f_\alpha \ , \tag{2.35}$$

and hence $\widetilde{P}_\alpha^\# = \widetilde{P}_\alpha$. Note that

$$A = \sum_{\alpha=0}^{d^2-1} d_\alpha\, \widetilde{P}_\alpha \ , \qquad A^\# = \sum_{\alpha=0}^{d^2-1} \overline{d_\alpha}\, \widetilde{P}_\alpha \ , \tag{2.36}$$

which implies that the super-operator A is normal

$$A A^\# = A^\# A \ . \tag{2.37}$$

3 How to generate commutative dynamics

Consider a family of Markovian semigroups $A_t^{(k)}$ defined by

$$A_t^{(k)} = e^{tL_k} \ , \quad k = 1, \ldots, n \ , \tag{3.1}$$

where L_k are the corresponding generators. Suppose that L_k are mutually commuting and define

$$A_{t,t_0} = \sum_{k=1}^{n} p_k(t-t_0)\, A^{(k)}_{t-t_0}\ , \tag{3.2}$$

where $p_k(t)$ denotes time dependent probability distribution: $p_k(t) \geq 0$ and $p_1(t)+\ldots+p_n(t) = 1$. Let us observe that A_{t,t_0} defines a commutative non-Markovian evolution satisfying local in time Master Equation [18]

$$\dot{A}_{t,t_0} = \mathcal{L}_{t-t_0} A_{t,t_0}\ , \qquad A_{t_0,t_0} = \mathrm{id}\ . \tag{3.3}$$

To find the non-Markovian generator $\mathcal{L}_t$ let us assume the following spectral representation of L_k

$$L_k\rho = \sum_\alpha \lambda^{(k)}_\alpha g_\alpha \mathrm{tr}(h^*_\alpha \rho)\ . \tag{3.4}$$

One obtains

$$\mathcal{L}_t\rho = \sum_\alpha \mu_\alpha(t) g_\alpha \mathrm{tr}(h^*_\alpha \rho)\ , \tag{3.5}$$

with

$$\mu_\alpha(t) = \frac{\sum_k p_k(t)\lambda^{(k)}_\alpha e^{\lambda^{(k)}_\alpha t}}{\sum_j p_j(t) e^{\lambda^{(j)}_\alpha t}}\ . \tag{3.6}$$

Hence, the solution A_t has the following form

$$A_{t,t_0}\rho = \sum_\alpha \exp\left(\int_0^{t-t_0} \mu_\alpha(u)\, du\right)\, g_\alpha \mathrm{tr}(h^*_\alpha \rho)\ . \tag{3.7}$$

Actually, one can easily generate a family of commuting generators $L_1,\ldots,L_n$. Suppose one is given a Markovian generator L of a unital semigroup $A_t = e^{Lt}$. Denote by $\widehat{A}_s$ the Laplace transform of A_t

$$\widehat{A}_s = \int_0^\infty e^{-st} A_t\, dt\ = \frac{1}{s-L}\ . \tag{3.8}$$

It is evident that for $s > 0$, $\widehat{A}_s$ is completely positive. Moreover,

$$\Phi^{(0)}_s := s\widehat{A}_s\ , \tag{3.9}$$

is unital. Indeed, one has

$$\Phi^{(0)}_s\, \mathbb{I} = s\int_0^\infty e^{-st} dt\, \mathbb{I} = \mathbb{I}\ . \tag{3.10}$$

Now, let us define

$$\Phi^{(k)}_s := \frac{s^{k+1}}{k!}(-1)^k\, \frac{d^k}{ds^k}\widehat{A}_s\ . \tag{3.11}$$

One gets

$$\Phi^{(k)}_s = \frac{s^{k+1}}{k!}\, \int_0^\infty e^{-st} t^k A_t\, dt\ = \frac{s^{k+1}}{(s-L)^{k+1}}\ . \tag{3.12}$$

It is clear that for $s > 0$, $\Phi_s^{(k)}$ is completely positive and unital. Therefore, for any integer k and $s > 0$ one obtains the following Markovian generator

$$L_s^{(k)} = \Phi_s^{(k)} - \mathrm{id} \ . \tag{3.13}$$

Hence, fixing s, one arrives at $L_k := L_s^{(k)}$.

Let us observe that the construction of the commutative (s,k)–family $L_k^{(s)}$ may be used to construct a huge family of commuting time dependent generators. Note, that taking a discrete family of function f_k

$$f_k \ : \ \mathbb{R}_+ \times \mathbb{R}_+ \ \longrightarrow \ \mathbb{R}_+ \ ,$$

one may define

$$\mathcal{L}_t[\mathbf{f}] := \sum_k \int_0^\infty f_k(t,s) L_s^{(k)}\, ds \ , \tag{3.14}$$

where we used a compact notation $\mathbf{f} = (f_1, f_2, \ldots)$. It is clear from the construction that $[\mathcal{L}_t[\mathbf{f}], \mathcal{L}_s[\mathbf{f}]] = 0$, and hence one easily find for the evolution

$$A_{t,t_0}[\mathbf{f}] = \exp\left(\int_0^{t-t_0} \mathcal{L}_u[\mathbf{f}]\, du \right) \ . \tag{3.15}$$

defines a family of commuting time dependent Markovian generators.

4 A class of commutative dynamics of stochastic classical systems

I Markovian classical dynamics

Consider the dynamics of a stochastic d-level system described by a probability distribution $p = (p(0), \ldots, p(d-1))$. Its time evolution is defined by

$$p_t(m) = \sum_{n=0}^{d-1} T_{t,t_0}(m,n) p_0(n) \ , \tag{4.1}$$

where $T_{t,t_0}(n,m)$ is a stochastic matrix satisfying the following time-dependent master equation

$$\dot{T}_{t,t_0} = L_t\, T_{t,t_0} \ , \quad T_{t_0,t_0} = \mathbb{I}_d \ , \tag{4.2}$$

that is

$$\dot{T}_{t,t_0}(m,n) = \sum_{k=0}^{d-1} L_t(m,k)\, T_{t,t_0}(k,n) \ , \qquad T_{t_0,t_0}(m,n) = \delta(n,m) \ . \tag{4.3}$$

Let us assume that L_t defines a commuting family of $d \times d$ matrices, i.e.

$$\sum_{k=0}^{d-1} L_t(m,k) L_u(k,n) = \sum_{k=0}^{d-1} L_u(m,k) L_t(k,n) \ , \tag{4.4}$$

for any $t, u \geq t_0$. A particular example of commutative dynamics is provided by circulant generators. Let us recall that a $d \times d$ matrix $L(m,n)$ is circulant [20] if

$$L(m,n) = a(m-n) \ , \quad \mod d \ , \tag{4.5}$$

that is L is defined in terms of a single vector $a = (a(0), \ldots, a(d-1))$.

Proposition 1 *Circulant matrices define a commutative subalgebra of M_d. Hence, if L and L' are circulant then $L'' = LL' = L'L$ is circulant. Moreover, if*

$$L(i,j) = a(i-j) \ , \quad L'(i,j) = a'(i-j) \ , \quad L''(i,j) = a''(i-j) \ , \quad \mod d \ ,$$

then

$$a'' = a * a' \ , \tag{4.6}$$

where $a * a'$ denotes a discrete convolution in $\mathbb{Z}_d$, i.e.

$$a''(n) = \sum_{k=0}^{d-1} a(n-k)\, a'(k) \ . \tag{4.7}$$

Therefore, multiplication of circulant matrices induces convolution of defining d-vectors. Interestingly, spectral properties of circulant matrices are governed by the following

Proposition 2 *The eigenvalues l_m and eigenvectors ψ_m of a circulant matrix*

$$L\, \psi_m = l_m\, \psi_m \ , \tag{4.8}$$

read as follows:

$$l_m = \sum_{k=0}^{d-1} a_k \lambda^{mk} \ , \tag{4.9}$$

and

$$(\psi_m)_n = \frac{1}{\sqrt{d}}\, \lambda^{mn} \ , \tag{4.10}$$

where

$$\lambda = e^{2\pi i/d} \ . \tag{4.11}$$

Let us observe that the Kolmogorov conditions for the stochastic circulant generator L_t give rise to the following condition upon the time dependent vector $a_t(m)$:

1. $a_t(m) \geq \ 0$, for $\ m \neq 0$
2. $a_t(0) < 0$,
3. $\sum_m a_t(m) = 0$,

for $t \geq t_0$. Now, it is clear that the solution to (4.3)

$$T_{t,t_0} = \exp\left(\int_{t_0}^{t} L_u du\right) \ , \tag{4.12}$$

defines a circulant stochastic matrix. Hence

$$T_{t,t_0}(m,n) =: P_{t,t_0}(m-n) \ , \tag{4.13}$$

defines a time-dependent stochastic vector $P_{t,t_0}(m)$. Note that

$$p_t(m) = \sum_{n=0}^{d-1} T_{t,t_0}(m,n) p_0(n) = \sum_{n=0}^{d-1} P_{t,t_0}(m-n) p_0(n) \ , \tag{4.14}$$

and hence

$$p_t = P_{t,t_0} * p_0 \ . \tag{4.15}$$

One obtains from (4.3)

$$\frac{dP_{t,t_0}(m)}{dt} = \sum_{k=0}^{d-1} a_t(m-k)\, P_{t,t_0}(k) \ , \qquad P_{t_0,t_0}(m) = \delta_{m0} \ , \tag{4.16}$$

which can be rewritten in terms of discrete convolution

$$\dot{P}_{t,t_0} = a_t * P_{t,t_0} \ , \qquad P_{t_0,t_0} = e \ , \tag{4.17}$$

where 'e' corresponds to the distribution concentrated at 0, i.e. $e(m) = \delta_{m0}$.

Proposition 3 *A convex set $\mathcal{P}_d$ of probabilistic d-vectors defines a semigroup with respect to the discrete convolution. The unit element $e = (1, 0, \ldots, 0)$ satisfies*

$$P * e = e * P = P \ ,$$

for all $P \in \mathcal{P}_d$.

To solve (4.17) one transform it via discrete Fourier transform to get

$$\frac{d\widetilde{P}_{t,t_0}(m)}{dt} = \widetilde{a}_t(m)\, \widetilde{P}_{t,t_0}(m) \ , \qquad \widetilde{P}_{t_0,t_0}(m) = 1 \ , \tag{4.18}$$

where

$$\widetilde{x}(n) = \sum_{k=0}^{d-1} \lambda^{nk}\, x(k) \ , \tag{4.19}$$

and the inverse transform reads

$$x(k) = \frac{1}{d} \sum_{n=0}^{d-1} \lambda^{-nk} \, \widetilde{x}(n) \ . \tag{4.20}$$

The solution of (4.18) reads as follows

$$\widetilde{P}_{t,t_0}(m) = \exp\left(\int_{t_0}^{t} \widetilde{a}_u(m) du \right) \ , \tag{4.21}$$

and hence one obtains for the stochastic vector $P_{t,t_0}(m)$

$$P_{t,t_0}(m) = \frac{1}{d} \sum_{k=0}^{d-1} \lambda^{-mk} \, \exp\left(\int_{t_0}^{t} \widetilde{a}_u(m) du \right) \ . \tag{4.22}$$

It is clear that P_{t,t_0} satisfies the following composition law

$$P_{t,s} * P_{s,u} = P_{t,u} \ , \tag{4.23}$$

or equivalently

$$\widetilde{P}_{t,s} \cdot \widetilde{P}_{s,u} = \widetilde{P}_{t,u} \ , \tag{4.24}$$

for all $t \geq s \geq u$. In particular when $a(n)$ does not depend on time then (4.21) simplifies to

$$\widetilde{P}_{t,t_0}(m) = \exp\left(\widetilde{a}(m)[t - t_0] \right) \ , \tag{4.25}$$

and hence 1-parameter semigroup $P_{t-t_0} := P_{t,t_0}$ satisfies homogeneous composition law

$$P_t * P_s = P_{t+s} \ , \tag{4.26}$$

or equivalently

$$\widetilde{P}_t \cdot \widetilde{P}_s = \widetilde{P}_{t+s} \ , \tag{4.27}$$

for all $t \geq s \geq t_0$.

II Non–Markovian classical dynamics

Consider now the non-Markovian case governed by the following local in time master equation

$$\dot{P}_{t,t_0} = a_{t-t_0} * P_{t,t_0} \ , \qquad P_{t_0,t_0} = e \ . \tag{4.28}$$

One easily obtain for the solution

$$P_{t,t_0}(m) = \frac{1}{d} \sum_{k=0}^{d-1} \lambda^{-mk} \, \exp\left(\int_{0}^{t-t_0} \widetilde{a}_u(m) du \right) \ . \tag{4.29}$$

Note the crucial difference between (4.22) and (4.29). The former defines inhomogeneous semigroup whereas the latter is homogeneous in time (depends upon the difference '$t - t_0$') but does not define a semigroup, i.e. does not satisfy the composition law (4.23).

Let us analyze conditions for a_τ which do guarantee that P_{t,t_0} defined in (4.29) is a probability vector, that is,

$$P_{t,t_0}(m) \geq 0 \ , \qquad \sum_{m=0}^{d-1} P_{t,t_0}(m) = 1 \ ,$$

for all $t \geq t_0$. It is clear from (4.28) that $a(\tau)$ has to satisfy

$$\int_0^\tau a_u(m)du \geq 0 \ , \tag{4.30}$$

for $m > 0$, and

$$\sum_{m=0}^{d-1} \int_0^\tau a_u(m)du = 0 \ , \tag{4.31}$$

which implies that

$$\int_0^\tau a_u(0)du < 0 \ , \tag{4.32}$$

for all $\tau \geq 0$. These conditions generalize Kolmogorov conditions in the inhomogeneous Markovian case. We stress, that $a_u(m)$ needs not be positive (for $m > 0$). One has a weaker condition (4.30). Note, that if $a_t(m) \geq 0$ for $m > 0$, then $\int_0^\tau a_u(m)du$ defines a monotonic function of time and hence the non-Markovian relaxation $\exp(\int_0^\tau a_u(m)du)$ is monotonic in time as well.

Finally, let us consider the corresponding nonlocal equation

$$\dot{P}_{t,t_0} = \int_{t_0}^t K_{t-u} * P_{u,t_0}\, du \ , \qquad P_{t_0,t_0} = e \ , \tag{4.33}$$

with the memory kernel K_{t-u}. Note, that we already know solution represented by (4.29) but still do not know the memory kernel K. Performing discrete Fourier transform one gets from (4.34)

$$\dot{\widetilde{P}}_{t,t_0}(m) = \int_{t_0}^t \widetilde{K}_{t-u}(m)\widetilde{P}_{u,t_0}(m)\, du \ , \qquad \widetilde{P}_{t_0,t_0}(m) = 1 \ . \tag{4.34}$$

Define the time-dependent vector

$$f_t(m) = \widetilde{a}_t(m) \, \exp\left(\int_0^t \widetilde{a}_u(m)du \right) \ , \tag{4.35}$$

then following [14] one obtains

$$\widehat{\widetilde{K}}_s(m) = \frac{s\widehat{f}_s(m)}{1 + \widehat{f}_s(m)} \ , \tag{4.36}$$

where $\widehat{x}_s$ denote the Laplace transform of x_t. Clearly, the problem of performing the inverse Laplace transform $\widehat{\widetilde{K}}_s(m) \longrightarrow \widetilde{K}_t(m)$ is in general not feasible. Hence, the memory kernel remains unknown. Nevertheless, the solution is perfectly known.

Remark 2 Note that a stochastic map $p_0 \to p_t = T_{t,t_0} p_0$ may be rewritten in a 'quantum fashion' as follows. Any probability distribution $p = (p(0), \ldots, p(d-1))$ gives rise to a diagonal density matrix

$$\rho = \sum_{n=0}^{d-1} p(n) e_{nn} \ , \tag{4.37}$$

and the map $\rho_0 \to \rho_t$ reads as follows

$$\rho_t = \sum_{m,n=0}^{d-1} T_{t,t_0}(m,n)\, e_{mm}\, \rho_0\, e_{nn} \ . \tag{4.38}$$

III Dynamics of composite systems

Consider now dynamics of N-partite system living in $\mathbb{Z}_d^N = \mathbb{Z}_d \times \ldots \times \mathbb{Z}_d$. Let $\mathbf{n} = (n_1, \ldots, n_N)$, with $n_k \in \mathbb{Z}_d$ and let

$$P_{t,t_0} \ : \ \mathbb{Z}_d^N \ \longrightarrow \ [0,1] \ , \tag{4.39}$$

be a probability vector living on $\mathbb{Z}_d^N$ satisfying the following Markovian master equation

$$\dot{P}_{t,t_0} = a_t * P_{t,t_0} \ , \qquad P_{t_0,t_0} = e \ , \tag{4.40}$$

where 'e' is defined by

$$e(\mathbf{n}) = \delta_{\mathbf{n}\mathbf{0}} := \delta_{n_1 0} \ldots \delta_{n_N 0} \ . \tag{4.41}$$

Now, performing the discrete Fourier transform one gets

$$\frac{d\widetilde{P}_{t,t_0}(\mathbf{m})}{dt} = \widetilde{a}_t(\mathbf{m})\, \widetilde{P}_{t,t_0}(\mathbf{m}) \ , \qquad \widetilde{P}_{t_0,t_0}(\mathbf{m}) = 1 \ , \tag{4.42}$$

where

$$\widetilde{x}(\mathbf{m}) = \sum_{\mathbf{k}} \lambda^{\mathbf{mk}}\, x(\mathbf{k}) \ , \tag{4.43}$$

and the inverse transform reads

$$x(\mathbf{k}) = \frac{1}{d^N} \sum_{\mathbf{m}} \lambda^{-\mathbf{mk}}\, \widetilde{x}(\mathbf{n}) \ . \tag{4.44}$$

The solution of (4.42) reads as follows

$$\widetilde{P}_{t,t_0}(\mathbf{m}) = \exp\left(\int_{t_0}^t \widetilde{a}_u(\mathbf{m}) du \right) \ , \tag{4.45}$$

and hence one obtains for the stochastic vector $P_{t,t_0}(\mathbf{m})$

$$P_{t,t_0}(\mathbf{m}) = \frac{1}{d^N} \sum_{\mathbf{k}} \lambda^{-\mathbf{mk}} \exp\left(\int_{t_0}^t \widetilde{a}_u(\mathbf{m}) du \right) \ . \tag{4.46}$$

It is clear that P_{t,t_0} satisfies the inhomogeneous composition law (4.23). If $a_{\mathbf{m}}$ is time independent then P_{t,t_0} defines 1-parameter semigroup $P_\tau := P_{\tau+t_0,t_0}$ satisfying homogeneous composition law (4.26).

Note, that in the case of non-Markovian dynamics one has

$$\dot{P}_{t,t_0} = a_{t-t_0} * P_{t,t_0} \ , \qquad P_{t_0,t_0} = e \ , \tag{4.47}$$

giving rise to the following solution

$$P_{t,t_0}(\mathbf{m}) = \frac{1}{d^N} \sum_{\mathbf{k}} \lambda^{-\mathbf{mk}} \exp\left(\int_0^{t-t_0} \widetilde{a}_{\mathbf{m}}(u) du \right) \ . \tag{4.48}$$

The non-Markovian dynamics is time homogeneous but does not satisfy (4.23).

5 A class of commutative quantum dynamics

Consider now an abelian group $\mathbb{Z}_d \times \mathbb{Z}_d$. Equivalently, one may consider a cyclic toroidal lattice $\mathbb{T}_d \times \mathbb{T}_d$, where

$$\mathbb{T}_d = \{ \lambda^m \ , \ m = 0, 1, \ldots, d-1 \} \ , \tag{5.1}$$

which is an abelian multiplicative group. Let us define the following representation of $\mathbb{T}_d \times \mathbb{T}_d$ in M_d:

$$\mathbb{Z}_d \times \mathbb{Z}_d \ni (m,n) \ \longrightarrow \ u_{mn} \in M_d \ , \tag{5.2}$$

where u_{mn} are unitary matrices defined as follows

$$u_{mn} e_k = \lambda^{mk} e_{n+k} \ , \tag{5.3}$$

where $\{e_0, \ldots, e_{d-1}\}$ denotes an orthonormal basis in $\mathbb{C}^d$, and λ stands for dth root of identity (see formula (4.11)).

Proposition 4 *Matrices u_{mn} satisfy*

$$u_{mn} u_{rs} = \lambda^{ms} \, u_{m+r,n+s} \ , \tag{5.4}$$
$$u^*_{mn} = \lambda^{mn} \, u_{-m,-n} \ , \tag{5.5}$$

and the following orthogonality relations

$$\mathrm{tr}(u^*_{mn} u_{kl}) = d \, \delta_{mk} \delta_{nl} \ . \tag{5.6}$$

Hence, formula (5.2) defines a projective representation of the abelian group $\mathbb{T}_d \times \mathbb{T}_d$. It is therefore clear that

$$\mathbb{Z}_d \times \mathbb{Z}_d \ni (m,n) \ \longrightarrow \ U_{mn} \in \mathcal{L}(M_d, M_d) \ , \tag{5.7}$$

with

$$U_{mn} a := u_{mn} a \, u^*_{mn} \ , \qquad a \in M_d \ , \tag{5.8}$$

defines the representation of $\mathbb{T}_d \times \mathbb{T}_d$ in the space of superoperators $\mathcal{L}(M_d, M_d)$.

Now, for any

$$a \ : \ \mathbb{Z}_d \otimes \mathbb{Z}_d \ \longrightarrow \ \mathbb{C} \ , \tag{5.9}$$

let us define a linear map $A \in \mathcal{L}(M_d, M_d)$

$$A = \sum_{m,n=0}^{d-1} a(m,n) U_{n,-m} \ , \tag{5.10}$$

that is, we define a representation of M_d in $\mathcal{L}(M_d, M_d)$.

Proposition 5 *If $a(m,n) \in \mathbb{R}$, then A is self-adjoint, that is*

$$A\, x^* = (A\, x)^* \ , \qquad x \in M_d \ . \tag{5.11}$$

If $a(m,n) \geq 0$, then A is completely positive. If moreover $\sum_{m,n} a(m,n) = 1$, then A is trace preserving and unital.

One proves the following

Proposition 6 *Let $a, b, c \in M_d$ be represented by $A, B, C \in \mathcal{L}(M_d, M_d)$, respectively, that is*

$$A = \sum_{m,n=0}^{d-1} a(m,n) U_{n,-m} \ , \qquad B = \sum_{m,n=0}^{d-1} b(m,n) U_{n,-m} \ , \qquad C = \sum_{m,n=0}^{d-1} c(m,n) U_{n,-m} \ .$$

Then $A \circ B = C$ if and only if $c = a * b$.

Hence, the set of maps constructed via (5.10) defines a commutative subalgebra in $\mathcal{L}(M_d, M_d)$.

Proposition 7 *The spectral properties of the linear map (5.10) are characterized by*

$$A\, u_{kl} \ = \ \widetilde{a}_{kl}\, u_{kl} \ , \tag{5.12}$$

$$A^{\#}\, u_{kl}^* \ = \ \widetilde{\widetilde{a}}_{kl}\, u_{kl}^* \ , \tag{5.13}$$

and hence its spectral decomposition reads as follows

$$A \ = \sum_{m,n=0}^{d-1} \widetilde{a}(m,n)\, P_{mn} \ , \tag{5.14}$$

where P_{mn} is a projector defined by

$$P_{mn}\, x = \frac{1}{d}\, u_{mn}\, \mathrm{tr}(u_{mn}^* x) \ , \tag{5.15}$$

for any $x \in M_d$.

In particular, if $a(m,n)$ is real, i.e. A is self-adjoint, then one has

$$\begin{aligned} A\, u_{kl} &= \widetilde{a}_{kl}\, u_{kl} \ , && (5.16) \\ A^{\#}\, u^*_{kl} &= \widetilde{a}_{kl}\, u^*_{kl} \ . && (5.17) \end{aligned}$$

Note, that the action of A upon the basis e_{ij} is given by

$$A\, e_{ij} = \sum_{m,n=0}^{d-1} a(m,n) \lambda^{n(i-j)}\, e_{i-m,j-m} \ . \tag{5.18}$$

Hence, diagonal elements satisfy define an invariant subspace in M_d

$$A\, e_{ii} = \sum_{m,n=0}^{d-1} a(m,n)\, e_{i-m,i-m} \ . \tag{5.19}$$

Let $P_{t,t_0} : \mathbb{Z}_d \times \mathbb{Z}_d \to [0,1]$ satisfy the following inhomogeneous master equation

$$\dot{P}_{t,t_0} = a_t * P_{t,t_0} \ , \qquad P_{t_0,t_0} = e \ . \tag{5.20}$$

Now, following (5.10), let us define

$$A_{t,t_0} = \sum_{m,n=0}^{d-1} P_{t,t_0}(m,n)\, U_{n,-m} \ , \tag{5.21}$$

and

$$\mathcal{L}_t = \sum_{m,n=0}^{d-1} a_t(m,n)\, U_{n,-m} \ . \tag{5.22}$$

Then, Proposition 6 implies the following local master equation for the dynamical map A_{t,t_0}:

$$\dot{A}_{t,t_0} = \mathcal{L}_t A_{t,t_0} \ , \qquad A_{t_0,t_0} = \mathrm{id} \ . \tag{5.23}$$

Note, that the time dependent Markovian generator may be rewrite as follows

$$\mathcal{L}_t\, \rho \ = \frac{1}{2} \sum_{m,n}{}' a_t(m,n) \Big([u_{n,-m}, \rho u^*_{n,-m}] + [u_{n,-m}\rho, u^*_{n,-m}] \Big) \ , \tag{5.24}$$

where ${\sum}'_{m,n} X_{mn} := \sum_{m,n} X_{mn} - X_{00}$. Hence, recalling that $a_t(m,n) \geq 0$ for $(m,n) \neq (0,0)$, the above formula provides the Lindblad form of $\mathcal{L}_t$. The corresponding spectral representation of the generator reads as follows

$$\mathcal{L}_t = \sum_{m,n=0}^{d-1} \widetilde{a}_t(m,n)\, P_{mn} \ . \tag{5.25}$$

Note, that due to $\widetilde{a}_t(0,0) = 0$, one has $\mathcal{L}_t \mathbb{I}_d = 0$. The corresponding solution of (6.11) is therefore given by

$$A_{t,t_0} = \sum_{m,n=0}^{d-1} \exp\left(\int_{t_0}^{t} \widetilde{a}_u(m,n)\, du \right) P_{mn} \ . \tag{5.26}$$

If P_{t,t_0} satisfies non-Markovian classical master equation

$$\dot{P}_{t,t_0} = a_{t-t_0} * P_{t,t_0} \ , \qquad P_{t_0,t_0} = e \ , \tag{5.27}$$

then the quantum dynamical map A_{t,t_0} satisfies non-Markovian equation

$$\dot{A}_{t,t_0} = \mathcal{L}_{t-t_0} A_{t,t_0} \ , \qquad A_{t_0,t_0} = \mathrm{id} \ , \tag{5.28}$$

with the solution given by the following formula

$$A_{t,t_0} = \sum_{m,n=0}^{d-1} \exp\left(\int_{0}^{t-t_0} \widetilde{a}_u(m,n)\, du \right) P_{mn} \ . \tag{5.29}$$

This spectral representation of $A_\tau := A_{t_0+\tau,t_0}$ enables one to construct the corresponding memory kernel $\mathcal{K}_\tau$. Using the following representation [15]

$$A_\tau = \mathrm{id} + \int_0^\tau F_s ds \ , \tag{5.30}$$

where

$$F_s = \mathcal{L}_s A_s \ , \tag{5.31}$$

one finds the spectral representation for the super-operator function F_s:

$$F_t = \sum_{m,n} f_t(m,n) P_{mn} \ , \tag{5.32}$$

with

$$f_t(m,n) = \widetilde{a}_t(m,n) \exp\left(\int_0^t \widetilde{a}_u(m,n) du \right) \ . \tag{5.33}$$

Therefore, one may write the corresponding non-local equation

$$\dot{A}_t = \int_0^t \mathcal{K}_{t-u} A_u du \ , \tag{5.34}$$

with the memory kernel is defined in terms of its Laplace transform as follows

$$\widehat{\mathcal{K}}_s = \sum_{m,n} \frac{s \widehat{f}_s(m,n)}{1 + \widehat{f}_s(m,n)} P_{mn} \ , \tag{5.35}$$

where $\widehat{f}_s(m,n)$ denotes the Laplace transform of $f_t(m,n)$. Note, that in general one is not able to invert the Laplace transform $\widehat{\mathcal{K}}_s$ and hence the above formula in general does not have any practical meaning.

6 Dynamics of composite quantum systems

Consider now a quantum dynamics of N-partite d-level quantum systems defined by

$$A_{t,t_0} = \sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}_d^N} P_{t,t_0}(\mathbf{m},\mathbf{n})\, U_{\mathbf{n},-\mathbf{m}} \ , \tag{6.1}$$

where

$$U_{\mathbf{k},\mathbf{l}}\, x = u_{\mathbf{k},\mathbf{l}}\, x\, u^*_{\mathbf{k},\mathbf{l}} \ , \tag{6.2}$$

for $x \in M_d^{\otimes N}$, and

$$u_{\mathbf{k},\mathbf{l}} = u_{k_1,l_1} \otimes \ldots \otimes u_{k_N,l_N} \ . \tag{6.3}$$

Proposition 8 *Matrices $u_{\mathbf{m},\mathbf{n}}$ satisfy*

$$u_{\mathbf{m},\mathbf{n}} u_{\mathbf{r},\mathbf{s}} = \lambda^{\mathbf{m}\,\mathbf{s}}\, u_{\mathbf{m}+\mathbf{r},\mathbf{n}+\mathbf{s}} \ , \tag{6.4}$$

$$u^*_{\mathbf{m},\mathbf{n}} = \lambda^{\mathbf{m}\,\mathbf{n}}\, u_{-\mathbf{m},-\mathbf{n}} \ , \tag{6.5}$$

and the following orthogonality relations

$$\mathrm{tr}(u^*_{\mathbf{m},\mathbf{n}} u_{\mathbf{k},\mathbf{l}}) = d^N\, \delta_{\mathbf{m},\mathbf{k}} \delta_{\mathbf{n},\mathbf{l}} \ . \tag{6.6}$$

The spectral representation of A_{t,t_0} has the following form

$$A_{t,t_0} = \sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}_d^N} \widetilde{P}_{t,t_0}(\mathbf{m},\mathbf{n})\, P_{\mathbf{m},\mathbf{n}} \ , \tag{6.7}$$

where $P_{\mathbf{m},\mathbf{n}}$ is a projector defined by

$$P_{\mathbf{m},\mathbf{n}}\, x = \frac{1}{d^N}\, u_{\mathbf{m},\mathbf{n}}\, \mathrm{tr}(u^*_{\mathbf{m},\mathbf{n}} x) \ , \tag{6.8}$$

for any $x \in M_d^{\otimes N}$. Assuming that

$$P_{t,t_0} \ : \ \mathbb{Z}_d^N \times \mathbb{Z}_d^N \ \longrightarrow \ [0,1] \ , \tag{6.9}$$

satisfies classical Markovian inhomogeneous master equation

$$\dot{P}_{t,t_0} = a_t * P_{t,t_0} \ , \qquad P_{t_0,t_0} = e \ , \tag{6.10}$$

one obtains

$$\dot{A}_{t,t_0} = \mathcal{L}_t A_{t,t_0} \ , \qquad A_{t_0,t_0} = \mathrm{id} \ , \tag{6.11}$$

where the time dependent Markovian generator is defined by

$$\mathcal{L}_t = \sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}_d^N} \widetilde{a}_t(\mathbf{m},\mathbf{n})\, P_{\mathbf{m},\mathbf{n}} \ . \tag{6.12}$$

Hence, the corresponding solution reads as follows

$$A_{t,t_0} = \sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}_d^N} \exp\left(\int_{t_0}^t \widetilde{a}_u(\mathbf{m},\mathbf{n})\, du \right)\, P_{\mathbf{m},\mathbf{n}} \ . \tag{6.13}$$

7 Commutative dynamics of 2-level system

Consider the time dependent generator for a 2-level system defined by

$$\mathcal{L}_t\rho = -\frac{i}{2}\varepsilon(t)[\sigma_3,\rho] + \gamma(t)\Big(\mu\mathcal{L}_1 + (1-\mu)\mathcal{L}_2\Big)\rho + \frac{1}{2}\sum_{\alpha,\beta=0}^{1} c_{\alpha\beta}(t)\Big([\pi_\alpha,\rho\pi_\beta] + [\pi_\alpha\rho,\pi_\beta]\Big)\ , \quad (7.1)$$

where the time independent Markovian generators $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined as follows

$$\begin{aligned} \mathcal{L}_1\rho &= \sigma^+\rho\sigma^- - \frac{1}{2}\{\sigma^-\sigma^+,\rho\}\ , \\ \mathcal{L}_2\rho &= \sigma^-\rho\sigma^+ - \frac{1}{2}\{\sigma^+\sigma^-,\rho\}\ . \end{aligned}$$

One easily shows that

$$[\mathcal{L}_t,\mathcal{L}_s] = 0\ , \quad (7.2)$$

and hence $\mathcal{L}_t$ does generate a commutative quantum dynamics. In (7.1) the 'mixing' parameter $\mu \in [0,1]$, and projectors π_α are defined by

$$\pi_0 = \sigma^-\sigma^+\ , \qquad \pi_1 = \sigma^+\sigma^-\ . \quad (7.3)$$

Note, that if $\gamma(t) > 0$ and the time dependent matrix $||c_{\alpha\beta}(t)||$ is semi-positive definite, than $\mathcal{L}_t$ defines time dependent Markovian generator. If

$$\int_0^t \gamma(u)du > 0\ , \quad (7.4)$$

and the matrix

$$||\int_0^t c_{\alpha\beta}(u)du\ || \geq 0\ , \quad (7.5)$$

for all $t \geq 0$, then $\mathcal{L}_t$ generates non-Markovian dynamics.

One easily solves the corresponding spectral problem for $\mathcal{L}_t$

$$\begin{aligned} \mathcal{L}_t\,\omega &= 0\ , \\ \mathcal{L}_t\,\sigma^+ &= \Gamma(t)\,\sigma^+\ , \\ \mathcal{L}_t\,\sigma^- &= \overline{\Gamma(t)}\,\sigma^-\ , \\ \mathcal{L}_t\,\sigma_3 &= -\gamma(t)\,\sigma_3\ , \end{aligned}$$

where the invariant state ω reads as follows

$$\omega = \mu\pi_1 + (1-\mu)\pi_0\ , \quad (7.6)$$

and the complex eigenvalue $\Gamma(t)$ is defined by

$$\Gamma(t) = -\frac{1}{2}\Big[\gamma(t) + c_{00}(t) + c_{11}(t) - 2c_{10}(t) + 2i\varepsilon(t)\Big] \ . \tag{7.7}$$

Similarly, one solves for the dual generator

$$\begin{aligned} \mathcal{L}_t^{\#}\mathbb{I}_2 &= 0 \ , \\ \mathcal{L}_t^{\#}\sigma^+ &= \overline{\Gamma(t)}\sigma^+ \ , \\ \mathcal{L}_t^{\#}\sigma^- &= \Gamma(t)\sigma^- \ , \\ \mathcal{L}_t^{\#}\sigma &= -\gamma(t)\,\sigma \ , \end{aligned}$$

where

$$\sigma = (1-\mu)\pi_1 - \mu\pi_0 = \frac{1}{2}\Big(\sigma_3 - \mathbb{I}_2\mathrm{tr}(\omega\sigma_3)\Big) \ . \tag{7.8}$$

Hence, introducing a bi-orthogonal basis

$$\begin{aligned} g_0 &= \omega \ , & h_0 &= \mathbb{I}_2 \ , \\ g_1 &= \sigma^+, & h_1 &= \sigma^+, \\ g_2 &= \sigma^-, & h_2 &= \sigma^-, \\ g_3 &= \sigma_3 \ , & h_3 &= \sigma \ , \end{aligned}$$

such that

$$(g_\alpha, h_\beta) = \mathrm{tr}(g_\alpha^* h_\beta) = \delta_{\alpha\beta} \ , \tag{7.9}$$

one has

$$\mathcal{L}_t\rho = \sum_{\alpha=0}^{3} \lambda_\alpha(t)\, g_\alpha\, \mathrm{tr}(h_\alpha^*\,\rho) \ , \tag{7.10}$$

with

$$\lambda_0(t) = 0 \ , \qquad \lambda_1(t) = \overline{\lambda_2(t)} = \Gamma(t) \ , \qquad \lambda_3(t) = -\gamma(t) \ . \tag{7.11}$$

Hence, the solution to the Markovian master equation

$$\dot{A}_{t,t_0} = \mathcal{L}_t A_{t,t_0} \ , \quad A_{t_0,t_0} = \mathrm{id} \ , \tag{7.12}$$

reads

$$A_{t,t_0}\,\rho = \sum_{\alpha=0}^{3} \exp\left(\int_{t_0}^{t} \lambda_\alpha(u)\,du\right)\, g_\alpha\, \mathrm{tr}(h_\alpha^*\,\rho) \ . \tag{7.13}$$

Consider now

$$V \ : \ M_2 \ \longrightarrow \ M_2 \ , \tag{7.14}$$

defined by

$$\begin{aligned} V\,a \ &= \ e_{00}\Big(\mu\,\mathrm{tr}(e_{11}a) + \mathrm{tr}(e_{00}a)\Big) + e_{11}\Big((1-\mu)\,\mathrm{tr}(e_{11}a) - \mathrm{tr}(e_{00}a)\Big) \\ &+ \ e_{10}\mathrm{tr}(e_{01}a) + e_{01}\mathrm{tr}(e_{10}a) \ . \end{aligned} \tag{7.15}$$

One easily finds for the inverse

$$\begin{aligned} V^{-1}a &= e_{00}\Big(-\mu\,\mathrm{tr}(e_{11}a) + (1-\mu)\,\mathrm{tr}(e_{00}a)\Big) + e_{11}\Big(\mathrm{tr}(e_{11}a) + \mathrm{tr}(e_{00}a)\Big) \\ &+ e_{10}\mathrm{tr}(e_{01}a) + e_{01}\mathrm{tr}(e_{10}a)\ , \end{aligned} \tag{7.16}$$

and hence

$$\begin{aligned} V^{-1\#}a &= e_{00}\Big(\mathrm{tr}(e_{11}a) + (1-\mu)\,\mathrm{tr}(e_{00}a)\Big) + e_{11}\Big(\mathrm{tr}(e_{11}a) - \mu\,\mathrm{tr}(e_{00}a)\Big) \\ &+ e_{10}\mathrm{tr}(e_{01}a) + e_{01}\mathrm{tr}(e_{10}a)\ . \end{aligned} \tag{7.17}$$

One finds

$$V\,e_{00} = \sigma_3\ , \quad V\,e_{11} = \omega\ , \quad V\,\sigma^{\pm} = \sigma^{\pm}\ , \tag{7.18}$$

and

$$V^{-1\#}e_{00} = \sigma\ , \quad V^{-1\#}e_{11} = \mathbb{I}_2\ , \quad V^{-1\#}\sigma^{\pm} = \sigma^{\pm}\ . \tag{7.19}$$

Hence, defining

$$f_0 = e_{11}\ , \quad f_1 = \sigma^{+}\ , \quad f_2 = \sigma^{-}\ , \quad f_3 = e_{00}\ , \tag{7.20}$$

one has

$$g_\alpha = V\,f_\alpha\ , \qquad h_\alpha = V^{-1\#}f_\alpha\ , \tag{7.21}$$

which shows that V diagonalizes $\mathcal{L}_t$ and A_{t,t_0}, that is,

$$\mathcal{L}_t = \sum_{\alpha=0}^{3} \lambda_\alpha(t) V P_\alpha V^{-1}\ , \tag{7.22}$$

and

$$A_{t,t_0} = \sum_{\alpha=0}^{3} \exp\left(\int_{t_0}^{t} \lambda_\alpha(u)\,du\right) V P_\alpha V^{-1}\ , \tag{7.23}$$

where

$$P_\alpha \rho = f_\alpha \mathrm{tr}(f_\alpha^* \rho)\ . \tag{7.24}$$

8 Conclusions

In this paper we analyzed a class of commutative dynamics of quantum open systems. It is shown that such evolution may be effectively described via spectral analysis of the corresponding time dependent generators. The characteristic feature of the corresponding time-dependent dynamical map is that all its eigenvectors do not depend on time (only its eigenvalues do). Actually, majority of examples studied in the literature (see e.g. [1]) belong to this class. If eigenvectors vary in time then the solution is formally defined by the time ordered exponential but the problem of finding an explicit solution is rather untractable. We stress that both Markovian and non-Markovian dynamics were studied. Our analysis shows that the local approach to non-Markovian dynamics proposed in [18] is much more suitable in practice than the corresponding non-local approach based on the memory kernel.

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