A family of generalized Horodecki-like entangled states

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Abstract

We provide a multi-parameter family of 2-qudit PPT entangled states which generalizes the celebrated Horodecki state in $3 \otimes 3$. The entanglement of this family is identified via semidefinite programming based on"PPT symmetric extensions" by Doherty et al.

1 Introduction

The problem to determine whether a given quantum state is separable or entangled, is one of the most fundamental problems in Entanglement Theory [1]. Starting from the famous Peres-Horodecki PPT (Positive Partial Transpose) criterion [2], nowadays there are enormous number of different separability criteria (see e.g. [3, 4, 5] and [1, 6] for the recent reviews). It turn out that among known separability criteria, those based on "symmetric extensions and "PPT symmetric extensions, developed by Doherty et al. [7, 8] are considered to be the most effective. It turns out that both NPT and PPT symmetrically extendable states can be characterized by semidefinite programming, a well-known optimization problem for which many free solvers are available (like the MATLAB toolbox SeDuMi [9]). For the recent approach to symmetric extensions see also [10]. In the present Letter we use these criteria to identify entanglement of the new class of PPT states in $\mathbb{C}^d \otimes \mathbb{C}^d$. This family provide the multi-parameter generalization of the seminal Horodecki state in $\mathbb{C}^3 \otimes \mathbb{C}^3$ defined as follows [11]

$$\rho_a = \frac{1}{8a+1} \left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & c \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & c & \cdot & b \end{array} \right) , \tag{1}$$

with

$$b = \frac{1+a}{2} , \quad c = \frac{\sqrt{1-a^2}}{2} , \tag{2}$$

where $a \in [0, 1]$. The above matrix representation corresponds to the standard computational basis $|ij\rangle = |i\rangle \otimes |j\rangle$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$ and to make the picture more transparent we replaced all zeros by dots.

Since the partial transposition $\rho_a^\Gamma \geq 0$ the state is PPT for all $a \in [0,1]$. It is easy to show that for $a=0$ and $a=1$ the state is separable and it was shown [11] that for $a \in (0,1)$ the state is entangled. The entanglement of (1) was identified using so called range criterion [11]. However, one may easily show that Horodecki state may be detected also by the popular realignment criterion [13, 14]. Actually, the family (1) provides one of the first examples of bound entanglement.

Recently, Horodecki state was generalized for $\mathbb{C}^d \otimes \mathbb{C}^d$ [12]. Let us introduce 3×3 positive matrix

$$X = b\left(|1\rangle\langle 1| + |d\rangle\langle d|\right) + c\left(|1\rangle\langle d| + |d\rangle\langle 1|\right) + a\sum_{k=2}^{d-1}|k\rangle\langle k| \ , \tag{3}$$

and define ρ_a as follows

$$\rho_a = \frac{1}{[d^2-1]a+1}\sum_{i,j=1}^{d}|i\rangle\langle j| \otimes \rho_{ij} \ , \tag{4}$$

where

$$\rho_{ii} = a\mathbb{I}_d \ , \quad (i<d) \ , \quad \rho_{dd} = X \ , \quad \rho_{ij} = a|i\rangle\langle j| \ , \quad (i \neq j) \ . \tag{5}$$

Clearly, for $d=3$ one recovers (1). It was shown [12] that (4) defines 1-parameter family of PPT states. Moreover, for $0<a<1$ these state are entangled. Again it may be easily shown using e.g. realignment criterion.

The aim of this Letter is to provide a huge generalization of (4). Actually, we provide d-parameter family of PPT states and perform full separability/entanglement analysis. For pedagogical reason we start with $d=3$ in the next section and postpone the general construction for Section 3. Final conclusions are collected in the last section.

2 Generalized Horodecki-like states in $3 \otimes 3$

Consider the following 3-parameter family of states

$$\rho_3 \ = \ N_3 \left(\begin{array}{ccc|ccc|ccc} b_1 & c_1 & \cdot & \cdot & a & \cdot & \cdot & \cdot & a \\ c_1 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & b_2 & c_2 & \cdot & \cdot & a \\ \cdot & \cdot & \cdot & \cdot & c_2 & b_2 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & c \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot \\ a & \cdot & \cdot & \cdot & a & \cdot & c & \cdot & b \end{array} \right) \ , \tag{6}$$

where b and c are defined in (2), and

$$b_k = a + \lambda_k(b-a) \ , \quad c_k = \lambda_k c \ , \tag{7}$$

with $\lambda_1, \lambda_2 \in [0,1]$ for $k=1,2$. Finally, the normalization factor N_3 reads as follows

$$N_3^{-1} = 8a + 1 + (1-a)(\lambda_1+\lambda_2) \ . \tag{8}$$

It is clear that for $\lambda_1 = \lambda_2 = 0$ it reduces to the Horodecki state (1). Let us observe that ρ_3 gives rise to the direct sum decomposition

$$\mathbb{C}^3 \otimes \mathbb{C}^3 = \mathcal{H}_0 \oplus \mathcal{H}_{13} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{32} \ , \tag{9}$$

where

$$\mathcal{H}_0 = \mathrm{span}_{\mathbb{C}}\{\, |11\rangle,\ |12\rangle,\ |22\rangle,\ |23\rangle,\ |33\rangle,\ |31\rangle\} \ , \tag{10}$$

and the remaining three 1-dimensional subspaces are defined as follows

$$\mathcal{H}_{13} = \mathrm{span}_{\mathbb{C}}\{\, |13\rangle\,\} \ , \quad \mathcal{H}_{21} = \mathrm{span}_{\mathbb{C}}\{\, |21\rangle\,\} \ , \quad \mathcal{H}_{32} = \mathrm{span}_{\mathbb{C}}\{\, |32\rangle\,\} \ . \tag{11}$$

Hence the positivity of ρ_3 is governed by the positivity of 6×6 matrix M_3 written in the block form as follows

$$M_3 = \left(\begin{array}{ccc} B_1 & A & A' \\ A^T & B_2 & A' \\ A'^T & A'^T & B_3 \end{array} \right) \ , \tag{12}$$

with 2×2 blocks given by

$$B_k = \left(\begin{array}{cc} b_k & c_k \\ c_k & b_k \end{array} \right) \ , \quad A = \left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) \ , \quad A' = \left(\begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right) \ , \tag{13}$$

where $b_3 := b$ and $c_3 := c$. Note, that $M_3 = M_3' + a|\phi_3\rangle\langle\phi_3|$, where $|\phi_3\rangle = |101001\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^3$ and M_3' is block-diagonal with diagonal blocks

$$\widetilde{B}_k = \lambda_k \left(\begin{array}{cc} b-a & c \\ c & b \end{array} \right) \ , \tag{14}$$

where $\lambda_3 := 1$. It is therefore clear that $M_3 \geq 0$ and hence $\rho_3 \geq 0$ as well. Interestingly, its partial transposition

$$\rho_3^\Gamma \ = N_3 \left(\begin{array}{ccc|ccc|ccc} b_1 & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1 & b_1 & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & a & \cdot & \cdot \\ \hline \cdot & a & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & b_2 & c_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & c_2 & b_2 & \cdot & a & \cdot \\ \hline \cdot & \cdot & a & \cdot & \cdot & \cdot & b & \cdot & c \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & b \end{array} \right) \ , \tag{15}$$

gives rise to another direct sum decomposition

$$\mathbb{C}^3 \otimes \mathbb{C}^3 = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2 \oplus \widetilde{\mathcal{H}}_3 \ , \tag{16}$$

where

$$\begin{array}{rcl} \widetilde{\mathcal{H}}_1 & = & \mathrm{span}_{\mathbb{C}}\{\, |11\rangle,\ |12\rangle,\ |21\rangle\ \} \ , \\ \widetilde{\mathcal{H}}_2 & = & \mathrm{span}_{\mathbb{C}}\{\, |22\rangle,\ |23\rangle,\ |32\rangle\ \} \ , \\ \widetilde{\mathcal{H}}_3 & = & \mathrm{span}_{\mathbb{C}}\{\, |33\rangle,\ |31\rangle,\ |13\rangle\ \} \ . \end{array} \tag{17}$$

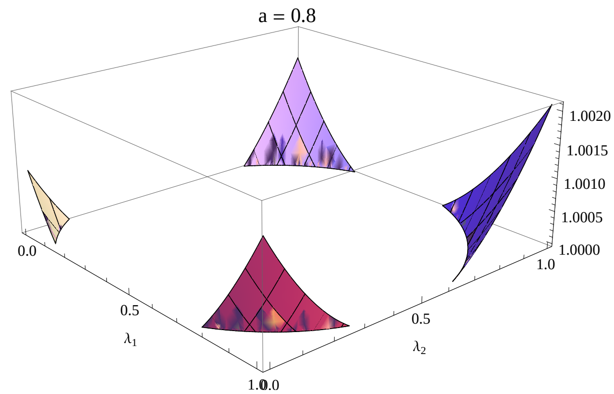

Figure 1: Realignment of ρ_3 for $a = 0.8$. Note that only "corners" of the parameter square $[0,1] \times [0,1]$ are detected.

Note that $\rho_3^\Gamma \geq 0$ due to the positivity of three 3×3 matrices

$$\widetilde{M}_k = \left(\begin{array}{ccc} b_k & c_k & 0 \\ c_k & b_k & a \\ 0 & a & a \end{array} \right) \ , \qquad k = 1,2,3 \ , \tag{18}$$

where as before $b_3 = b$ and $c_3 = c$. Therefore, ρ_3 defines a family of PPT states parameterized by $a, \lambda_1, \lambda_2 \in [0,1]$. Note, that for $a = 0$ it reduces to the block-diagonal and hence separable operator. For $a = 1$ one has $b_k = a = 1$ and $c_k = 0$ and hence it reduces to the standard Horodecki state with $a = 1$ which is known to be separable [11]. It turns out that ρ_3 is entangled for $0 < a < 1$. This result is based on the "PPT-symmetric extensions" by Doherty et al. [7, 8]. Interestingly, the entanglement of ρ_3 is only partially detected by the simple realignment criterion [13, 14] (see the Fig. 1.) Note, that the standard Horodecki state corresponding to $\lambda_1 = \lambda_2 = 0$ is detected by realignment. Other corners of the parameter square $[0,1] \times [0,1]$ are detected as well.

3 Generalized Horodecki-like states in $d \otimes d$

The above construction in $3 \otimes 3$ may be easily generalized for $d \otimes d$ with arbitrary (but finite) d. Define $d \times d$ positive matrix

$$X(\lambda) = b(\lambda)\,(|1\rangle\langle 1| + |d\rangle\langle d|) + c(\lambda)\,(|1\rangle\langle d| + |d\rangle\langle 1|) + a \sum_{k=2}^{d-1} |k\rangle\langle k| \ , \tag{19}$$

with $b(\lambda)$ and $c(\lambda)$ being the following linear functions of the parameter $\lambda \in [0,1]$

$$b(\lambda) = a + \lambda(b-a) \ , \quad c(\lambda) = \lambda\, c \ . \tag{20}$$

Note, that $X(1) = X$, where X was already defined in (3). Let

$$X_k = S^k X(\lambda_k) S^{k\dagger} \ , \tag{21}$$

where S is the shift operator defined by

$$S|k\rangle = |k+1\rangle \ , \quad (\mathrm{mod}\ d) \ , \tag{22}$$

and $\lambda_k \in [0,1]$ for $k = 1,\ldots,d$. Finally, let us introduce

$$\rho_d = N_d \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \rho_{ij} \ , \tag{23}$$

where

$$\rho_{ii} = X_i \ , \qquad \rho_{ij} = a\,|i\rangle\langle j| \ , \quad (i \neq j) \ . \tag{24}$$

Fixing $\lambda_d = 1$ one finds for the normalization factor

$$N_d^{-1} = [(d^2-1)a + 1] + (1-a)\sum_{k=1}^{d-1} \lambda_k \ . \tag{25}$$

Clearly, for $d = 3$ this construction reproduces the previous one. Note, that for $\lambda_1 = \ldots = \lambda_{d-1} = 0$ it reproduces generalized Horodecki state from [12].

In analogy to (9) ρ_d gives rise to the direct sum decomposition

$$\mathbb{C}^d \otimes \mathbb{C}^d = \mathcal{H}_0 \oplus \bigoplus_{k,l} \mathcal{H}_{kl} \ , \tag{26}$$

where

$$\mathcal{H}_0 = \mathrm{span}_{\mathbb{C}}\{\, |ii\rangle,\ |i,i+1\rangle \,\} \ , \quad (i = 1,\ldots,d \mod d) \ , \tag{27}$$

is $2d$-dimensional, and $d(d-2)$ 1-dimensional subspaces

$$\mathcal{H}_{kl} = \mathrm{span}_{\mathbb{C}}\{\, |kl\rangle \,\} \ , \tag{28}$$

where the indices k,l satisfy

$$k \neq l \ , \quad l \neq k+1 \ . \tag{29}$$

Therefore, the positivity of ρ_d reduces to the positivity of $2d \times 2d$ matrix

$$M_d = \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes M_{ij} \ , \tag{30}$$

with

$$M_{ii} = B_i \ , \quad M_{ij} = A \ , \ (i < j < d) \ , \quad M_{id} = A' \ , \ (i < d) \ , \tag{31}$$

where the 2×2 matrices B_i, A and A' are defined in (13) (clearly, i runs from 1 up to d and $b_d := b$ and $c_d := c$). Note, that for $d = 3$ one reproduces formula (12) for M_3. Now, the positivity of M_d follows from the following observation

$$M_d = M_d' + a|\phi_d\rangle\langle\phi_d| \ , \tag{32}$$

where M'_d is block-diagonal with diagonal blocks $\widetilde{B}_i$ defined in (14) (with $\lambda_d = 1$) and $|\phi_d\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^d$ is defined by

$$|\phi_d\rangle = (|10\rangle \oplus \ldots \oplus |10\rangle) \oplus |01\rangle \ , \tag{33}$$

where we have used $\mathbb{C}^2 \otimes \mathbb{C}^d = \mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2$ (d terms).

Interestingly, the partial transposition ρ_d^Γ is given by

$$\rho_d^\Gamma = N_d \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \widetilde{\rho}_{ij} \ , \tag{34}$$

where

$$\widetilde{\rho}_{ii} = \rho_{ii} = X_i \ , \quad \widetilde{\rho}_{ij} = \rho_{ij}^T = a\,|j\rangle\langle i| \ , \quad (i \neq j) \ , \tag{35}$$

gives rise to another direct sum decomposition

$$\mathbb{C}^d \otimes \mathbb{C}^d = \bigoplus_i \widetilde{\mathcal{H}}_i \oplus \bigoplus_{k,l} \widetilde{\mathcal{H}}_{kl} \ , \tag{36}$$

where there are d subspaces which are 3-dimensional

$$\widetilde{\mathcal{H}}_i = \mathrm{span}_{\mathbb{C}}\{\, |ii\rangle,\ |i,i+1\rangle,\ |i+1,i\rangle\} \ , \quad (i = 1, \ldots, d \mod d) \ , \tag{37}$$

and $d(d-3)/2$ subspaces $\widetilde{\mathcal{H}}_{kl}$ which are 2-dimensional

$$\widetilde{\mathcal{H}}_{kl} = \mathrm{span}_{\mathbb{C}}\{\, |kl\rangle,\ |lk\rangle \ \} \ , \tag{38}$$

where the indices k,l satisfy

$$k < l \ , \quad l \neq k+1 \ , \quad k \neq l+1 \ , \qquad (\mathrm{mod}\ d) \ . \tag{39}$$

Equivalently, this condition may be formulated as follows: given $k \in \{1, \ldots, d-2\}$, one has the following bound for l

$$l = \left\{ \begin{array}{ll} k+2, \ldots, d-1 \ , & \text{for} \ \ k = 1 \\ k+2, \ldots, d \ , & \text{for} \ \ k = 2, \ldots, d-2 \end{array} \right. \ . \tag{40}$$

Note, that condition (39) is more restrictive that (29). For $d = 3$ one has only 3-dimensional subspaces (the set of indices k,l satisfying (39) is empty) and hence (36) reduces to (16). Now, positivity of ρ_d^Γ is governed by the collection of d 3×3 matrices and '$d(d-3)/2$' 2×2 matrices. It is easy to see that all 2×2 matrices are equal to $a|11\rangle\langle 11|$ which is evidently positive, whereas 3×3 matrices are nothing but $\widetilde{M}_k$ defined by (18) (where i runs from 1 up to d and $b_d = b$, $c_d = c$). Therefore, ρ_d defines a family of PPT states parameterized by d parameters: $a, \lambda_1, \lambda_2, \ldots, \lambda_{d-1} \in [0,1]$. Note, that for $a = 0$ it reduces to the block-diagonal and hence separable operator. For $a = 1$ one has $b_k = a = 1$ and $c_k = 0$ and hence it reduces to the generalized Horodecki state with $a = 1$ which is known to be separable [12].

Let us introduce d product vectors

$$|\psi_k\rangle = |k\rangle \otimes \left(\sqrt{\frac{1-a}{2}}\,|k\rangle + \sqrt{\frac{1+a}{2}}\,|k+1\rangle \right) \ , \quad k = 1, \ldots, d \ . \tag{41}$$

One finds the following decomposition

$$\rho_d = N_d \left(X_{\rm ent} + X_{\rm sep} \right) \ , \tag{42}$$

where

$$X_{\rm sep} = \sum_{k=1}^{d} \lambda_k \, |\psi_k\rangle\langle\psi_k| \ , \tag{43}$$

with $\lambda_d = 1$, and

$$X_{\rm ent} = a(dP_d^+ + Q_d) \ , \tag{44}$$

where P_d^+ denotes maximally entangled state and

$$Q_d = \mathbb{I}_d \otimes \mathbb{I}_d - \sum_{k=1}^{d} (P_k \otimes P_k + \lambda_k P_k \otimes P_{k+1}) \ , \tag{45}$$

with $P_k := |k\rangle\langle k|$. It is clear that $X_{\rm sep}$ is separable and $X_{\rm ent}$ is entangled being an NPT operator. Hence, ρ_d is a convex combination of entangled and separable states. Note that for $a = 0$ the entangled part drops out and $\rho_d = N_d X_{\rm sep}$ with $N_d^{-1} = \sum_{k=1}^{d} \lambda_k$. Again, using semi-definite programming based on the "PPT-symmetric extensions" by Doherty et al. [7, 8] we show that for $0 < a < 1$ the state ρ_d is entangled.

4 Conclusions

We constructed a rich d-parameter family of PPT sates in $\mathbb{C}^d \otimes \mathbb{C}^d$ and performed full separability/entanglement analysis. These states generalize Horodecki state in $\mathbb{C}^3 \otimes \mathbb{C}^3$ [11] and $\mathbb{C}^d \otimes \mathbb{C}^d$ constructed recently in [12]. Interestingly, generalized Horodecki-like are invariant under the action of unitaries of the following form

$$U = \Pi_0 + \sum_{k,l} e^{i\alpha_{kl}} \, \Pi_{kl} \ , \tag{46}$$

where the indices k, l satisfy (29), and the projectors Π_0, Π_{kl} are defined as follows

$$\Pi_0 = \sum_{k=1}^{d} P_k \otimes (P_k + P_{k+1}) \ , \qquad \Pi_{kl} = P_k \otimes P_l \ . \tag{47}$$

Note, that (49) defines $d(d-2)$-dimensional commutative subgroup of $U(d^2)$. The characteristic feature of (49) is that U is nonlocal, that is, it cannot be written as $U_1 \otimes U_2$ with $U_1, U_2 \in U(d)$. Therefore, the symmetry group of the generalized Horodecki-like states have different symmetry than states defined by (4). It was shown [12] that (4) is invariant under $U_{\mathbf{x}} \otimes \overline{U}_{\mathbf{x}}$, where

$$U_{\mathbf{x}} = \sum_{k=1}^{d} e^{ix_k} \, P_k \ , \tag{48}$$

with $x_1 = x_d$. Hence, in our generalized multi-parameter family the local symmetry $U_{\mathbf{x}} \otimes \overline{U}_{\mathbf{x}}$ is changed to the nonlocal symmetry defined by (49). The crucial difference between local and nonlocal symmetries is related to he properties of PPT states. Note, that if ρ is invariant under $U_1 \otimes U_2$, that is $U_1 \otimes U_2 \rho = \rho U_1 \otimes U_2$, then ρ^Γ is invariant under $U_1 \otimes \overline{U}_2$. No such simple relation exists for nonlocal symmetries. In general even if $U\rho = \rho U$ there is no universal way to find the symmetry of ρ^Γ. It turns out that in the case of generalized Horodecki-like states one has $\widetilde{U}\rho^\Gamma = \rho^\Gamma \widetilde{U}$, where $\widetilde{U}$ are initaries defined by

$$\widetilde{U} = \sum_{m=1}^{d} e^{i\beta_m}\, \widetilde{\Pi}_m + \sum_{k,l} e^{i\gamma_{kl}}\, \widetilde{\Pi}_{kl} \ , \tag{49}$$

where the indices k, l satisfy (39), and the projectors $\widetilde{\Pi}_m$, $\widetilde{\Pi}_{kl}$ are defined as follows

$$\widetilde{\Pi}_m = P_m \otimes P_m + P_m \otimes P_{m+1} + P_{m+1} \otimes P_m \ , \quad \widetilde{\Pi}_{kl} = P_k \otimes P_l + P_l \otimes P_k \ . \tag{50}$$

Interestingly, generalized Horodecki-like entangled states with local symmetry are detected by realignment criterion. In general it is no longer the case for the states with nonlocal symmetry. These states are detected in the full parameters range by semi-definite programming methods.

It would be interesting to construct a family of (indecomposable) entanglement witnesses detecting the entanglement of generalized Horodecki-like states in $\mathbb{C}^d \otimes \mathbb{C}^d$.

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