

On measures of non-Markovianity: divisibility vs. backflow of information

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We analyze two recently proposed measures of non-Markovianity: one based on the concept of divisibility of the dynamical map and the other one based on distinguishability of quantum states. We provide a toy model to show that these two measures need not agree. In addition, we discuss possible generalizations and intricate relations between these measures.

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I. INTRODUCTION

The dynamics of open quantum systems attracts nowadays increasing attention [1–3]. It is relevant not only for the better understanding of quantum theory but it is fundamental in various modern applications of quantum mechanics. Since the system-environment interaction causes dissipation, decay and decoherence it is clear that dynamic of open systems is fundamental in modern quantum technologies, such as quantum communication, cryptography and computation [4]. The usual approach to the dynamics of an open quantum system consists in applying an appropriate Born-Markov approximation leading to the celebrated quantum Markov semigroup [5, 6] which neglects all memory effects. However, recent theoretical studies and technological progress call for more refine approach based on non-Markovian evolution.

Non-Markovian systems appear in many branches of physics, such as quantum optics [1, 7], solid state physics [8], quantum chemistry [9], and quantum information processing [10]. Since non-Markovian dynamics modifies monotonic decay of quantum coherence it turns out that when applied to composite systems it may protect quantum entanglement for longer time than standard Markovian evolution [11]. In particular it may protect the system against the sudden death of entanglement [12]. It is therefore not surprising that non-Markovian dynamics was intensively studied during last years [13].

Surprisingly, it turns out the concept of (non)Markovianity is not uniquely defined. One approach is based on the idea of the composition law which is essentially equivalent to the idea of divisibility [14]. This approach was used recently by Rivas, Huelga and Plenio (RHP) [16] to construct the corresponding measure of non-Markovianity, that is, RHP measure the deviation from divisibility. A different approach is advocated by Breuer, Laine and Piilo (BLP) in Ref. [15]. BLP define non-Markovian dynamics as a time evolution for the open system characterized by a temporary flow of information from the environment

back into the system and manifests itself as an increase in the distinguishability of pairs of evolving quantum states. It is clear that RHP characterize a mathematical property of the dynamical map whereas the idea of BLP is based on physical features of the system-reservoir interaction, rather than the mathematical properties of the dynamical map of the open system. In a recent paper [17] Haikka, Cresser and Maniscalco performed detailed analysis of these approaches studying the dynamics of the driven qubit in a structured environment. It is indicated [17] that the concepts of RHP and BLP need not agree, as was also conjectured in [18] and checked in [19]. BLP measure was recently analyzed for the dynamics of a qubit coupled to a spin environment via an energy-exchange mechanism [20].

In the present paper we perform further analysis of this problem. In particular we provide a simple toy model showing that Markovian evolution à la BLP may be indivisible and hence non-Markovian according to RHP. Actually, one may feel the relation BLP vs. RHP quite analog to the relation between separable and PPT states in entanglement theory, where separable states define the proper subset of PPT states. States which are PPT but entangled are bound entangled. Using this analogy one may call Markovian evolution à la BLP but indivisible (non-Markovian according to RHP) – ‘bound non-Markovian’. Finally we discuss possible generalizations of BLP using tensor product structures, and show that a slight modification of BLP reduces their concept of Markovianity to divisibility (the bound non-Markovianity can be washed out). The main motivations of this paper are to expose the relation between two different concepts on a simple example, and to present the way to unify them, which we hope clarify the issue.

II. DIVISIBILITY VS. BACKFLOW OF INFORMATION

The measure for non-Markovianity proposed by Rivas, Huelga and Plenio (RHP) [16] is based on a notion of

divisibility: a trace preserving completely positive map $\Lambda(t, 0)$ is divisible if it can be written as a

$$\Lambda(t + \tau, 0) = \Lambda(t + \tau, t)\Lambda(t, 0) , \quad (1)$$

and $\Lambda(t + \tau, t)$ is completely positive for any $t, \tau > 0$. RHP define a map to be Markovian exactly when it is divisible. Note that

$$\Lambda(t + \tau, t) = \Lambda(t + \tau, 0)\Lambda^{-1}(t, 0) , \quad (2)$$

satisfies composition law

$$\Lambda(s, t) = \Lambda(s, u)\Lambda(u, t) , \quad (3)$$

for any $s \geq u \geq t$, which is usually attributed for Markovian evolution. It is shown [16] that the quantity

$$g(t) = \lim_{\epsilon \rightarrow 0^+} \frac{\|\mathbb{1}_d \otimes \Lambda(t + \epsilon, t)P_d^+\|_1 - 1}{\epsilon} , \quad (4)$$

enjoys $g(t) > 0$ if and only if the original map $\Lambda(t, 0)$ is indivisible ($\|\cdot\|_1$ is a trace-norm). As usual $\mathbb{1}_d$ denotes an identity map in $M_d(\mathbb{C})$, and P_d^+ denotes the maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$ (we consider the evolution of a d -level system). Now, using the fact that any divisible (and differentiable) completely positive map satisfies a local in time master equation

$$\frac{d}{dt}\Lambda(t, 0) = L(t)\Lambda(t, 0) , \quad \Lambda(0, 0) = \mathbb{1}_d , \quad (5)$$

where the local generator $L(t)$ is a legitimate Markovian generator for any $t \geq 0$. The formula (4) may be equivalently rewritten in terms of $L(t)$:

$$g(t) = \lim_{\epsilon \rightarrow 0^+} \frac{\|[\mathbb{1}_d \otimes (\mathbb{1}_d - \epsilon L(t))]P_d^+\|_1 - 1}{\epsilon} . \quad (6)$$

A second criterion of non-Markovianity was proposed by Breuer, Laine and Piilo (BLP) in Ref. [15]. The BLP criterion identifies non-Markovian dynamics with certain physical features of the system-reservoir interaction. They define non-Markovian dynamics as a time evolution for the open system characterized by a temporary flow of information from the environment back into the system. This backflow of information may manifest itself as an increase in the distinguishability of pairs of evolving quantum states. Hence, according to BLP the dynamical map $\Lambda(t, 0)$ is non-Markovian if there exists a pair of initial states ρ_1 and ρ_2 such that for some time $t > 0$ the distinguishability of ρ_1 and ρ_2 increases, that is,

$$\sigma(\rho_1, \rho_2; t) = \frac{d}{dt} D[\rho_1(t), \rho_2(t)] , \quad (7)$$

where $D(\rho_1, \rho_2) = \frac{1}{2}\|\rho_1 - \rho_2\|_1$ is the distinguishability of ρ_1 and ρ_2 , and $\rho_k(t) = \Lambda(t, 0)\rho_k$.

Using these two criteria one easily defines the corresponding non-Markovianity measures of the dynamical map Λ :

$$\mathcal{N}_{\text{RHP}}(\Lambda) = \frac{\mathcal{I}}{\mathcal{I} + 1} , \quad (8)$$

where $\mathcal{I} = \int_0^\infty g(t)dt$. Similarly, one has

$$\mathcal{N}_{\text{BLP}}(\Lambda) = \sup_{\rho_1, \rho_2} \int_{\sigma > 0} \sigma(\rho_1, \rho_2; t)dt . \quad (9)$$

It is clear [15] that $\mathcal{N}_{\text{RHP}}(\Lambda) = 0$ implies $\mathcal{N}_{\text{BLP}}(\Lambda) = 0$. Hence, all divisible maps are Markovian according to BLP. The converse is in general not true. This problem was carefully analyzed in the paper [19] for phenomenological integro-differential master equations and in the recent paper [17], where the authors have studied the non-Markovian character of a driven qubit in a structured reservoir in different dynamical regime using $\mathcal{N}_{\text{RHP}}(\Lambda)$ and $\mathcal{N}_{\text{BLP}}(\Lambda)$.

III. TOY MODEL – CLASSICAL STOCHASTIC DYNAMICS

It should be clear that the above discussion applies for a classical stochastic systems as well. Consider stochastic dynamics of d -state classical system described by the probability vector $\mathbf{p} = (p_1, \dots, p_d)$. Its evolution $\mathbf{p}(0) \rightarrow \mathbf{p}(t) = \Lambda(t) \cdot \mathbf{p}(0)$ is defined by the family of stochastic matrices $\Lambda(t)$, that is, $\Lambda_{ij}(t) \geq 0$ for all $i, j = 1, \dots, d$ and

$$\sum_{i=1}^d \Lambda_{ij}(t) = 1 , \quad (10)$$

for each $j = 1, \dots, d$ and $t \geq 0$. These conditions guarantee that $\mathbf{p}(t)$ is a legitimate probability vector for all $t \geq 0$. The corresponding local-in-time master equation

$$\dot{p}_i(t) = \sum_{j=1}^d L_{ij}(t)p_j(t) , \quad (11)$$

is defined in terms of the local generator $L(t)$ which is defined in by the classical dynamical map $\Lambda(t)$ as follows

$$L(t) = \dot{\Lambda}(t) \cdot \Lambda^{-1}(t) . \quad (12)$$

In particular if L does not depend on time $\Lambda(t)$ defines a one-parameter semigroup of stochastic matrices $\Lambda(t) := \Lambda(t, 0) = e^{tL}$. Let us recall that Λ is stochastic if and only if it satisfies well known Kolmogorov conditions:

$$L_{ij} \geq 0 , \quad i \neq j , \quad \text{and} \quad \sum_{i=1}^d L_{ij} = 0 , \quad (13)$$

for each $j = 1, \dots, d$. Now, a stochastic evolution $\Lambda(t, 0)$ is divisible if it can be written as a composition of two stochastic maps $\Lambda(t + \tau, t)$ and the original $\Lambda(t, 0)$

$$\Lambda(t + \tau, 0) = \Lambda(t + \tau, t) \cdot \Lambda(t, 0) , \quad (14)$$

for any $\tau > 0$. Clearly, $\Lambda(t, 0)$ is divisible if and only if the corresponding local generator $L(t)$ satisfies Kolmogorov conditions for any $t \geq 0$. Now, we would like to compare two non-Markovianity measures due to RHP and BLP.

It is clear that classical evolution may be rewritten in the quantum framework as follows: any probability vector \mathbf{p} gives rise to the diagonal density matrix $\rho = \sum_k p_k |k\rangle\langle k|$, and hence the stochastic map – classical channel – Λ gives rise to the following Kraus representation

$$\Lambda \rho = \sum_{i,j=1}^d \Lambda_{ji} |i\rangle\langle j| \rho |j\rangle\langle i| . \quad (15)$$

Now, to apply (4) one needs the classical analog of $(\mathbb{1} \otimes \Lambda)P^+$. Let us define $(\mathbb{1}_{\text{cl}} \otimes \Lambda)P_{\text{cl}}^+$, where the classical identity map is defined by

$$\mathbb{1}_{\text{cl}} \rho = \sum_{i=1}^d |i\rangle\langle i| \rho |i\rangle\langle i| , \quad (16)$$

and the classical analog of the maximally entangled states reads as follows

$$P_{\text{cl}}^+ = \frac{1}{d} \sum_{i=1}^d |i\rangle\langle i| \otimes |i\rangle\langle i| . \quad (17)$$

One finds therefore

$$(\mathbb{1}_{\text{cl}} \otimes \Lambda)P_{\text{cl}}^+ = \frac{1}{d} \sum_{i,j=1}^d \Lambda_{ji} |i\rangle\langle i| \otimes |j\rangle\langle j| . \quad (18)$$

Now, the ‘classical’ quantity

$$g(t) = \lim_{\epsilon \rightarrow 0^+} \frac{\|(\mathbb{1}_{\text{cl}} \otimes \Lambda(t + \epsilon, t)P_{\text{cl}}^+)\|_1 - 1}{\epsilon} , \quad (19)$$

is strictly positive if and only if the original stochastic map $\Lambda(t, 0)$ is indivisible and hence $g(t)$ identifies non-Markovianity of the classical stochastic process. Equivalently, it may be rewritten using local generator $L(t)$

$$g(t) = \lim_{\epsilon \rightarrow 0} \frac{\|(\mathbb{1}_{\text{cl}} \otimes (\mathbb{1}_{\text{cl}} - \epsilon L(t)))P_{\text{cl}}^+\|_1 - 1}{\epsilon} . \quad (20)$$

Consider now the following toy model of stochastic dynamics of 2-state system

$$\Lambda(t, 0) = \begin{pmatrix} 1 - x_0(t) & x_1(t) \\ x_0(t) & 1 - x_1(t) \end{pmatrix} , \quad (21)$$

where $x_0(t), x_1(t) \in [0, 1]$ for all $t \geq 0$. One finds for the local generator

$$L(t) = \begin{pmatrix} -a_0(t) & a_1(t) \\ a_0(t) & -a_1(t) \end{pmatrix} , \quad (22)$$

where

$$a_0 = \frac{\dot{x}_0(1 - x_1) + \dot{x}_1 x_0}{1 - x_0 - x_1} , \quad (23)$$

$$a_1 = \frac{\dot{x}_0 x_1 + \dot{x}_1(1 - x_0)}{1 - x_0 - x_1} . \quad (24)$$

Now, $\Lambda(t, 0)$ is divisible if and only if $a_0(t), a_1(t) \geq 0$ for all $t \geq 0$. On the other hand one easily finds

$$\sigma(t) = -2[a_0(t) + a_1(t)]|\Delta_0| , \quad (25)$$

where $\Delta_k = (\mathbf{p}_1 - \mathbf{p}_2)_k$. Hence, contrary to $g(t)$, the quantity $\sigma(t)$ controls only the sum of a_0 and a_1 . In principle one may have $\sigma(t) \leq 0$ even if $a_0(t) < 0$ or $a_1(t) < 0$. Indeed, let us define

$$x_0(t) = \int_0^t f_0(\tau) d\tau , \quad x_1(t) = \int_0^t f_1(\tau) d\tau , \quad (26)$$

such that $0 \leq \int_0^t f_k(\tau) d\tau \leq 1$, for all $t \geq 0$. Hence, our toy model is fully controlled by functions f_0 and f_1 . Let

$$f_0(t) = \kappa \sin t , \quad t \geq 0 , \quad (27)$$

and $f_1(t) = 0$ for $t \in [0, \pi]$ together with

$$f_1(t) = -\kappa \sin t , \quad t \geq \pi , \quad (28)$$

where $0 < \kappa < 1/2$. One finds

$$a_0(t) + a_1(t) = \frac{\kappa \sin t}{1 - \kappa + \kappa \cos t} , \quad t \in [0, \pi] , \quad (29)$$

and $a_0(t) + a_1(t) = 0$ for $t \geq \pi$. Note, that for $t \geq \pi$ one has

$$a_0(t) = -a_1(t) = \kappa \sin t , \quad (30)$$

which proves that $\Lambda(t, 0)$ is not divisible. However, this map is ‘classically’ Markovian according to BLP [15] due to $a_0(t) + a_1(t) \geq 0$ for all $t \geq 0$.

IV. QUBIT DYNAMICS

Consider now the following dynamics of a qubit

$$\begin{aligned} \rho_{00}(t) &= \rho_{00} x_0(t) + \rho_{11} [1 - x_1(t)] , \\ \rho_{11}(t) &= \rho_{00} [1 - x_0(t)] + \rho_{11} x_1(t) , \\ \rho_{01}(t) &= \rho_{01} \gamma(t) , \end{aligned} \quad (31)$$

where $x_0(t), x_1(t) \in [0, 1]$, and

$$|\gamma(t)|^2 \leq x_0(t)x_1(t) . \quad (32)$$

The above conditions for $x_k(t)$ and $\gamma(t)$ guarantee that the dynamics is completely positive. One easily finds for the corresponding local generator

$$L \rho = -i \frac{\Omega}{2} [\sigma_z, \rho] + \sum_{k=0}^1 a_k L_k \rho + \frac{\Gamma}{2} L_z \rho , \quad (33)$$

where

$$\begin{aligned} L_0\rho &= \sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}, \\ L_1\rho &= \sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}, \\ L_z\rho &= \sigma_z\rho\sigma_z - \rho. \end{aligned} \quad (34)$$

The time-dependent coefficients a_0 and a_1 are defined in (23) and (24), whereas $\Gamma(t)$ and $\Omega(t)$ read as follows

$$\Gamma(t) = -\frac{a_0(t) + a_1(t)}{2} - \operatorname{Re} \frac{\dot{\gamma}(t)}{\gamma(t)}, \quad (35)$$

$$\Omega(t) = \operatorname{Im} \frac{\dot{\gamma}(t)}{\gamma(t)}. \quad (36)$$

The corresponding dynamical map $\Lambda(t, 0)$ is divisible if and only if $a_k(t), \Gamma(t) \geq 0$ for all $t \geq 0$. On the other hand one finds the following formula

$$\sigma(t) = -\frac{2A(t)\Delta_{00}^2 + [A(t) + 4\Gamma(t)]|\Delta_{01}|^2}{\sqrt{\Delta_{00}^2 + |\Delta_{01}|^2}}, \quad (37)$$

where $A(t) = a_0(t) + a_1(t)$, and a 2×2 matrix Δ reads

$$\Delta = \rho_1(0) - \rho_2(0). \quad (38)$$

It is therefore clear that using the same arguments as in the classical toy model we may have $\sigma(t) \leq 0$ but the dynamical map is not divisible.

V. GENERALIZED CONDITIONS

We have studied so far the non-Markovian character of simple classical and quantum (toy) models and compare two non-Markovianity measures due to RHP and BLP. We performed explicit construction showing that in general these two measures do not agree supporting [17–19].

Let us observe that both approaches may be analyzed from a slightly more general perspective. Consider a family $\Lambda(t, 0)$ of trace preserving positive maps (not necessarily completely positive). Actually, the approach of BLP needs only positivity of $\Lambda(t, 0)$. It is clear that $\Lambda(t, 0)$ is contractive (one has $\|\Lambda(t, 0)\rho\|_1 = \|\rho\|_1$). Now, $\Lambda(t, 0)$ is divisible if

$$\Lambda(t + \tau, 0) = \Lambda(t + \tau, t) \cdot \Lambda(t, 0), \quad (39)$$

and $\Lambda(t + \tau, t)$ is a contraction for all $\tau \geq 0$. It implies

$$\|\Lambda(t + \tau, t)\| = 1, \quad (40)$$

where the norm of the map is defined via

$$\|\Lambda\| = \sup_{\|\rho\|_1=1} \|\Lambda\rho\|_1. \quad (41)$$

Note, that $\Lambda(t, 0)$ is divisible if and only if

$$\frac{d}{dt}\Lambda(t, 0) = L(t)\Lambda(t, 0), \quad (42)$$

where $L(t)$ is a local generator satisfying the following dissipativity condition [21] (see also [5]): for any set of mutually orthogonal rank-1 projectors P_i such that $\sum_i P_i = \mathbb{I}$, one has

$$\begin{aligned} \operatorname{Tr}[P_i L(t) P_j] &\geq 0, \quad i \neq j, \\ \sum_i \operatorname{Tr}[P_i L(t) P_j] &= 0. \end{aligned} \quad (43)$$

The second condition guaranties that $\Lambda(t, 0)$ is trace preserving. Hence, formula (40) is equivalent to

$$\lim_{\epsilon \rightarrow 0^+} \frac{\|\mathbb{I} - \epsilon L(t)\| - 1}{\epsilon} = 0. \quad (44)$$

We call $\Lambda(t, 0)$ completely divisible iff $\Lambda(t + \tau, t)$ is completely contractive. Recall that $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is completely contractive [22] iff $\mathbb{1}_k \otimes \Phi$ is contractive for all $k = 2, 3, \dots$. If $\dim \mathcal{H} = d < \infty$, then, as in the case of complete positivity, it is enough to check that $\mathbb{1}_d \otimes \Phi$ is contractive. Note, that due to the fact that $\Lambda(t + \tau, t)$ is trace preserving complete contractivity is equivalent to complete positivity. Hence, complete divisibility is the standard divisibility considered by RHP [16]. Finally, recall that Φ is completely bounded [22] iff $\mathbb{1}_k \otimes \Phi$ is bounded all $k = 2, 3, \dots$. One defines

$$\|\Phi\|_{(k)} = \|\mathbb{1}_k \otimes \Phi\|, \quad (45)$$

and

$$\|\Phi\|_{\text{cb}} = \sup_k \|\Phi\|_{(k)}. \quad (46)$$

Φ is completely contractive iff $\|\Phi\|_{\text{cb}} \leq 1$. If $\dim \mathcal{H} = d < \infty$, then necessarily Φ is completely bounded. Moreover

$$\|\Phi\|_{\text{cb}} = \|\Phi\|_{(d)}. \quad (47)$$

Hence, $\Lambda(t, 0)$ is completely divisible iff

$$\|\Lambda(t + \tau, t)\|_{\text{cb}} = 1, \quad (48)$$

which is much more restrictive than (40). Interestingly, for completely positive Φ one has

$$\|\Phi\|_{\text{cb}} = \|(\mathbb{1}_d \otimes \Phi)P_d^+\|_1, \quad (49)$$

which reproduces result of RHP. Actually, one may generalize the whole analysis to (completely) contractive dynamics on the Banach space generalizing old results for contractive semigroups [23].

Now, let us observe that the approach of BLP which is valid for positive maps $\Lambda(t, 0)$ may be easily generalized if we assume that $\Lambda(t, 0)$ is completely positive. One may introduce

$$\tilde{\sigma}(\tilde{\rho}_1, \tilde{\rho}_2; t) = \frac{d}{dt} D[\tilde{\rho}_1(t), \tilde{\rho}_2(t)], \quad (50)$$

where

$$\tilde{\rho}_k(t) = [\mathbb{1}_d \otimes \Lambda(t, 0)]\tilde{\rho}_k, \quad k = 1, 2. \quad (51)$$

It is clear that if $\tilde{\sigma}(\tilde{\rho}_1, \tilde{\rho}_2; t) \leq 0$ for all initial states $\tilde{\rho}_k$, then $\sigma(\rho_1, \rho_2; t) \leq 0$ for all initial states ρ_k . Note, however, that the converse needs not be true. Again, one may show that there exists indivisible maps $\Lambda(t, 0)$ for which $\tilde{\sigma}(\tilde{\rho}_1, \tilde{\rho}_2; t) \leq 0$ for all initial states $\tilde{\rho}_k$. Note, that condition $\tilde{\sigma}(\tilde{\rho}_1, \tilde{\rho}_2; t) \leq 0$ may be reformulated as

$$\tilde{\sigma}(\Delta; t) = \frac{d}{dt} \|\Delta(t)\|_1 \leq 0, \quad (52)$$

where $\Delta(t) = [\mathbb{1}_d \otimes \Lambda(t, 0)]\Delta$, and $\Delta^\dagger = \Delta$ together with $\text{Tr}\Delta = 0$, which follows from $\Delta = (\tilde{\rho}_1 - \tilde{\rho}_2)/2$ and $\tilde{\rho}_k$ are true states. It should be stressed that condition $\text{Tr}\Delta = 0$ is very restrictive. Let us recall that if Φ is trace preserving then Φ is positive iff

$$\|\Phi a\|_1 \leq \|a\|_1, \quad (53)$$

for all $a^\dagger = a$. Note however that if one restricts to a enjoying $\text{Tra} = 0$, then $\|\Phi a\|_1 \leq \|a\|_1$ does not imply positivity of Φ . If we relax $\text{Tr}\Delta = 0$, then the condition $\tilde{\sigma}(\Delta; t) \leq 0$ for all Hermitian Δ imply that $\Lambda(t, 0)$ is (completely) divisible and hence definition of Markovianity due to BLP reduces to divisibility. More explicitly this fact can be formulated as:

Theorem. *For a bijective evolution $\Lambda(t, 0)$, $\tilde{\sigma}(\Delta; t) \leq 0$ for all $\Delta^\dagger = \Delta$, if and only if it is divisible.*

Proof. The ‘if’ part is straightforward because the trace norm is monotonically decreasing given the completely contracting property of the $\Lambda(t + \tau, t)$,

$$\|\Lambda(t + \tau)\|_1 = \|[\mathbb{1}_d \otimes \Lambda(t + \tau, t)]\Delta(t)\|_1 \leq \|\Delta(t)\|_1,$$

for every t and $\tau > 0$. Conversely, if the evolution $\Lambda(t, 0)$ is bijective the partitions $\Lambda(t + \tau, t) = \Lambda(t + \tau, 0)\Lambda^{-1}(t, 0)$ are well-defined. Then $\tilde{\sigma}(\Delta; t) > 0$ for some t implies that for some small τ

$$\|\Delta(t + \tau)\|_1 = \|[\mathbb{1}_d \otimes \Lambda(t + \tau, t)]\Delta(t)\|_1 > \|\Delta(t)\|_1.$$

So there exists some partition $\Lambda(t + \tau, t)$ which is not completely contractive, and therefore the dynamics is not divisible. \square

VI. OPERATIONAL MEANING OF $\tilde{\sigma}(\Delta; t)$

One may think that by relaxing the condition $\text{Tr}\Delta = 0$ the interpretation of an increase of $\tilde{\sigma}(\Delta; t)$ in terms of backflow of information from environment to system is lost, however that is not the case. To explain this more carefully we have to recall here some results from quantum hypothesis testing, particularly the one-shot two-state discrimination problem [24].

Consider a quantum system whose state is represented by the density matrix ρ_1 with probability p , and ρ_2 with

probability $(1 - p)$. We want to determine the density matrix that describes the true state of the quantum system by performing a measurement. If we consider some general POVM $\{M_j\}$, where $j \in \Omega$ is the set of possible outcomes, we may split this set in two cases. If the outcome of the measurement is inside of some subset $A \subset \Omega$, then we say that the state is ρ_1 . Conversely if the result of the measurement belongs to the complementary set A^c such that $A \cup A^c = \Omega$, we say that the state is ρ_2 . Let us group the results of this measurement in another POVM given by the operator $T = \sum_{j \in A} M_j$.

Thus, when the true state is ρ_1 (which happens with probability p) we erroneously conclude that the state is ρ_2 with probability

$$\begin{aligned} (1 - p) \sum_{j \in A^c} \text{Tr}[\rho_1 M_j] &= (1 - p) \text{Tr} \left[\rho_1 \left(\sum_{j \in A^c} M_j \right) \right] \\ &= (1 - p) \text{Tr} [\rho_1 (\mathbb{I} - T)]. \end{aligned}$$

On the other hand, when the true state is ρ_2 , we erroneously conclude that the state is ρ_1 with probability

$$p \sum_{j \in A} \text{Tr}[\rho_2 M_j] = p \text{Tr} \left[\rho_2 \left(\sum_{j \in A} M_j \right) \right] = p \text{Tr} [\rho_2 T].$$

Note that when $p = 0$ or $p = 1$ we immediately obtain zero probability to identify wrongly the true state. The problem in one-shot two-state discrimination is to examine the tradeoff between the two error probabilities $p \text{Tr} [\rho_2 T]$ and $(1 - p) \text{Tr} [\rho_1 (\mathbb{I} - T)]$. Thus, consider the best choice of T that minimizes the total error probability

$$\begin{aligned} &\min_{0 \leq T \leq \mathbb{I}} \{p \text{Tr} [\rho_2 T] + (1 - p) \text{Tr} [\rho_1 (\mathbb{I} - T)]\} \\ &= \min_{0 \leq T \leq \mathbb{I}} \{(1 - p) + \text{Tr} [p \rho_2 T - (1 - p) \rho_1 T]\} \\ &= (1 - p) - \max_{0 \leq T \leq \mathbb{I}} [\text{Tr} (\Delta T)], \end{aligned}$$

where $\Delta = (1 - p)\rho_1 - p\rho_2$ is a self-adjoint operator (sometimes called Helstrom matrix [25]) with trace $\text{Tr}\Delta = 1 - 2p$, which vanishes only for the unbiased case $p = 1/2$. By using analogous arguments that for $p = 1/2$ (see [4, 24]) the result of the optimization process turns out to be

$$\min_{0 \leq T \leq \mathbb{I}} \{p \text{Tr} [\rho_2 T] + (1 - p) \text{Tr} [\rho_1 (\mathbb{I} - T)]\} = \frac{1 - \|\Delta\|_1}{2}. \quad (54)$$

Thus the trace norm of $\Delta = (1 - p)\rho_1 - p\rho_2$ gives our capability to distinguish correctly between ρ_1 and ρ_2 in the one-shot two-state discrimination problem. Since that is only a function of the information we gather by the prior probability p and by the measurement T , $\|\Delta\|_1$ is a measure of the information we have.

Consider again a trace preserving map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$. If it increases the trace norm of Δ , $\|\Phi(\Delta)\|_1 >$

$\|\Delta\|_1$, we can assume that Φ carries information about the correct state of the system. Otherwise it cannot decrease the probability to identify wrongly the true state!. For that reason, if we process data before making a measurement, which means to apply some Φ over the states, we cannot increase the trace norm of Δ ; since we cannot gain more information about some data just by processing it!. This is the so-called data processing inequality [24], and implies that Φ is contractive, i.e. positive. In addition, we can never dismiss that the states ρ_1 and ρ_2 are part of a larger system ‘‘SA’’. In such a way that they are the result of tracing out the ancillary systems $\rho_{1,2} = \text{Tr}_A[\rho_{1A,2A}]$. By making the same analysis as before, in order not to gain information we then require the map $\mathbb{1}_d \otimes \Phi$ to be contractive, i.e. Φ completely contractive and thus completely positive.

Going back to the problem of Markovianity, one may now realize that $\tilde{\sigma}(\Delta; t)$ is a measure of the information gained by the system for some initial Helstrom matrix Δ . This interpretation puts together the two concepts of Markovianity based upon divisibility and backflows of information. Particularly if one ignores the potential existence of an ancilla, under the case of unbiased discrimination problem, $\|\Delta\|_1 = \frac{1}{2}\|\rho_1 - \rho_2\|_1$, which is the definition of the trace distance between ρ_1 and ρ_2 , recovering the approach of BLP. Nevertheless in principle there is no reason to presuppose the fulfillment of these particular assumptions.

One can define a measure of non-Markovianity as (9), with $\tilde{\sigma}(\Delta; t)$ in the place of $\sigma(\rho_1, \rho_2; t)$. That will be zero iff (8) is, however (8) is easier to compute because it avoids the complicated optimization procedure in (9) now upon every Helstrom matrix Δ .

We have already proposed examples of non-divisible dynamics with $\mathcal{N}_{BLP}(\Lambda) = 0$, but for completeness we have computed $\tilde{\sigma}(\Delta; t)$ for the phenomenological integro-differential master equation [26]:

$$\frac{d\rho(t)}{dt} = \int_0^t k(t')L\rho(t-t')dt', \quad (55)$$

where $k(t) = \gamma e^{-\gamma t}$ and

$$L\rho = \gamma_0(\bar{n} + 1)L_0\rho + \gamma_0\bar{n}L_1\rho, \quad (56)$$

with $\gamma, \gamma_0, \bar{n} \geq 0$ and L_0 and L_1 were given in (34). Despite this kind of equations is used as a simple model of non-Markovian dynamics, it was showed in [19] that $\mathcal{N}_{BLP}(\Lambda) = 0$. However $\tilde{\sigma}(\Delta; t)$ is not always decreasing for any Helstrom matrix. For example by taking $p = 0.07$ and

$$\rho_1 = \begin{pmatrix} 0.5 & 0 & 0 & 0.48 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.019 & 0 \\ 0.48 & 0 & 0 & 0.48 \end{pmatrix}, \quad (57)$$

$$\rho_2 = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (58)$$

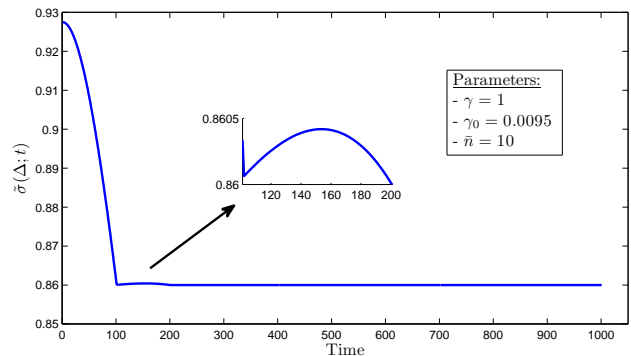


FIG. 1: Behavior of $\tilde{\sigma}(\Delta; t)$ for the integro-differential model (55). The initial Helstrom matrix is $\Delta = 0.93\rho_1 - 0.07\rho_2$, where ρ_1 and ρ_2 are given in (57) and (58). There is a time period between 100 and 150 where $\tilde{\sigma}(\Delta; t)$ increases.

there is a period where $\tilde{\sigma}(\Delta; t)$ grows as we have represented in Fig 1. This denotes the existence of a backflow of information which was expected from the non-divisible character of this dynamics.

VII. CONCLUSIONS

We have analyzed two concepts of Markovianity, one based on the divisibility property of the dynamical map and the other based upon the distinguishability of quantum states. We have given very simple examples where these two criteria do not coincide. Furthermore we have proposed a way to make them equivalent, in the sense that Markovianity would be identified by divisibility, but keeping the interpretation in terms of flows of information. For that we resort to the results in the one-shot two-state discrimination problem; and point out that an increase of the trace norm of any Hermitian matrix during the dynamics can also be associated with a backflow of information from the environment to the system.

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