

Circulant states with vanishing quantum discord

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We analyze a class of 2-qudit circulant states. They define generalization of well known 2-qubit X -states. We formulate necessary and sufficient criteria for vanishing quantum discord. We illustrate our analysis by an important subclass of circulant states – so called Bell diagonal states.

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I. INTRODUCTION

Quantum state of composite system contains both classical and quantum correlation. Usually by quantum correlations one means quantum entanglement which provides an essential resource for quantum information processing such as quantum cryptography, dense coding, quantum computing [1, 2].

However, a quantum state of a composed system may contain other types of nonclassical correlation even if it is separable (not entangled). For a recent “catalogue” of nonclassical correlations, see [3]. The most popular measure of such correlations – quantum discord – was introduced by Ollivier and Zurek [4] and independently by Henderson and Vedral [5].

Recently quantum discord has received increasing attention. It was analyzed in the context of broadcasting of quantum states [6, 7]. Interestingly, it turned out that quantum discord might be responsible for the quantum computational efficiency of some quantum computation tasks [8–11]. Moreover, the dynamics of discord, both Markovian and non-Markovian, [12–21] was analyzed. Quantum discord was generalized for continuous variables to study correlations in Gaussian states [22, 23]. A geometric measure for quantum discord was introduced in [24] and analyzed in [25]. Finally, an operational interpretation of quantum discord was provided in [26, 27] – quantum discord received clear information-theoretic operational meaning in terms of entanglement consumption in an extended quantum state merging protocol.

Remarkable progress in characterization of set of zero-discord states was done. Interestingly, it was shown [28] to have vanishing volume in the set of all states. Actually, this result holds true for any Hilbert space dimension. It shows that a generic state of composed quantum system does contain non-classical correlation. Necessary and sufficient conditions were provided [24, 29–31], to determine states with vanishing discord. Moreover, nonlinear witnesses of discord were introduced [32–34]. For very recent papers analyzing various aspects of quantum discord see also [35–41].

In the present paper we analyze a large class of

two-qudit states introduced in [42] called circulant states (see also [43]). Construction of these states is based on a certain decomposition of the total d^2 -dimensional Hilbert space into d mutually orthogonal d -dimensional subspaces. A density matrix ρ representing a circulant state is a convex combination of density matrices supported on different subspace. Interestingly, circulant states provide natural generalization of so called X -states of two qubits. In the present paper we address a question when a quantum discord for a circulant state does vanish.

The paper is organized as follows: in Section II we recall basis definitions related to quantum discord and formulate necessary and sufficient condition for the vanishing discord. Section III introduces the construction of circulant states in $\mathbb{C}^d \otimes \mathbb{C}^d$ and analyzes the case of discord zero. Section IV discusses several well known examples of circulant states invariant under then action of the symmetry group – a subgroup of the unitary group $U(d)$. In Section V we investigate special class of circulant states – so called Bell diagonal states [44, 45] – which play important role in the theory of quantum entanglement. To simplify our discussion we consider only the case when d is prime. Prime dimensions already appeared for example in the discussion of mutually unbiased basis [46]. It turns out that in this case the discussion considerably simplifies. The general case may we analyzed in the same way. However, the corresponding analysis is technically much more involved. We illustrate our discussion with several examples and conclude in the last Section.

II. QUANTUM DISCORD

Consider a density operator ρ of a composite quantum system in Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The total amount of correlations in a bipartite state ρ is quantified by quantum mutual information:

$$\mathcal{I}(\rho) = S(\rho_A) + S(\rho_B) - S(\rho) , \quad (1)$$

where ρ_A and ρ_B are reduced density matrices in \mathcal{H}_A and \mathcal{H}_B respectively and $S(\sigma) = -\text{tr}(\sigma \log \sigma)$ stands for the von Neumann entropy of the density operator σ . Note, that mutual information may be rewritten as follows

$$\mathcal{I}(\rho) = S(\rho_B) - S(\rho_{B|A}), \quad (2)$$

where $\rho_{B|A}$ denotes a state of subsystem B given measurement on subsystem A and $S(\rho_{B|A})$ is a quantum conditional entropy. Let us introduce a local measurement on A part defined by a collection of one-dimensional projectors $\{\Pi_k\}$ in \mathcal{H}_A satisfying $\Pi_1 + \Pi_2 + \dots = \mathbb{I}_A$. Different outcomes of this measurement are labeled by 'k'. The state of part B after the measurement on part A , when the outcome corresponding to Π_k has been detected, is given by

$$\rho_{B|k} = \text{tr}_A \left[\frac{1}{p_k} (\Pi_k \otimes \mathbb{I}_B) \rho (\Pi_k \otimes \mathbb{I}_B) \right], \quad (3)$$

where $p_k = \text{tr}[\rho_{B|k}(\Pi_k \otimes \mathbb{I}_B)]$. The entropies $S(\rho_{B|k})$ weighted by probabilities p_k yield to the conditional entropy of part B given the complete measurement $\{\Pi_k\}$ on the part A

$$S(\rho_B|\{\Pi_k\}) = \sum_k p_k S(\rho_{B|k}). \quad (4)$$

This means that the corresponding measurement-introduced mutual information is

$$\mathcal{I}(\rho_B|\{\Pi_k\}) = S(\rho_B) - S(\rho_B|\{\Pi_k\}). \quad (5)$$

By optimizing over all possible measurements $\{\Pi_k\}$ on part A one has

$$\mathcal{C}_A(\rho) = \sup_{\{\Pi_k\}} \mathcal{I}(\rho_B|\{\Pi_k\}). \quad (6)$$

This quantity has been given an interpretation as measure of classical correlations.

Although this two quantities, $\mathcal{I}(\rho)$ and $\mathcal{C}_A(\rho)$, are equivalent for classical systems, in quantum domain they, in general, do not coincide. The difference

$$\mathcal{D}_A(\rho) = \mathcal{I}(\rho) - \mathcal{C}_A(\rho) \quad (7)$$

defines a new quantity, quantum discord, which is a measure of quantum correlations in a quantum state ρ .

Evidently, the above definition is not symmetric with respect to parties A and B . One can swap the role of A and B , introducing a collection of one-dimensional Π_α^B projectors in \mathcal{H}_B satisfying $\Pi_1^B + \Pi_2^B + \dots = \mathbb{I}_B$. Then one gets an analogous definition for discord of a composite system when part B is measured

$$\mathcal{D}_B(\rho) = \mathcal{I}(\rho) - \mathcal{C}_B(\rho), \quad (8)$$

where

$$\mathcal{C}_B(\rho) = \sup_{\{\Pi_\alpha^B\}} \mathcal{I}(\rho|\{\Pi_\alpha^B\}). \quad (9)$$

Quantum discord $\mathcal{D}_A(\rho)$ and $\mathcal{D}_B(\rho)$, is always non-negative. Although for all states with the same reduced density matrices $\mathcal{D}_A(\rho) = \mathcal{D}_B(\rho)$, this in general is not the case. Moreover, on pure states, quantum discord coincides with the von Neumann entropy of entanglement $S(\rho_A) = S(\rho_B)$. One shows that $\mathcal{D}_A(\rho) = 0$ (so called classical-quantum states) if and only if there exists an orthonormal basis $|k\rangle$ in \mathcal{H}_A such that

$$\rho = \sum_k p_k |k\rangle\langle k| \otimes \rho_k^{(B)}, \quad (10)$$

where $\rho_k^{(B)}$ are density matrices in \mathcal{H}_B . Similarly, $\mathcal{D}_B(\rho) = 0$ (quantum-classical states), if and only if there exists an orthonormal basis $|\alpha\rangle$ in \mathcal{H}_B such that

$$\rho = \sum_\alpha q_\alpha \rho_\alpha^{(A)} \otimes |\alpha\rangle\langle\alpha|, \quad (11)$$

where $\rho_\alpha^{(A)}$ are density matrices in \mathcal{H}_A . It is clear that if $\mathcal{D}_A(\rho) = \mathcal{D}_B(\rho) = 0$, then ρ is diagonal in the product basis $|k\rangle \otimes |\alpha\rangle$ and hence

$$\rho = \sum_{k,\alpha} \lambda_{k\alpha} |k\rangle\langle k| \otimes |\alpha\rangle\langle\alpha|, \quad (12)$$

is entirely represented by the classical joint probability distribution $\lambda_{k\alpha}$. Such states are called completely classical.

States with a positive quantum discord do contain non-classical correlations even if they are separable. Hence nonvanishing quantum discord indicates a kind of quantumness encoded in a separable mixed state.

Consider now states with vanishing quantum discord [24, 29, 30]. Take two arbitrary orthonormal basis $|e_i^{(A)}\rangle$ and $|e_\alpha^{(B)}\rangle$ in \mathcal{H}_A and \mathcal{H}_B , respectively. An arbitrary state ρ_{AB} in $\mathcal{H}_A \otimes \mathcal{H}_B$ may be written as follows

$$\rho_{AB} = \sum_{i,j} e_{ij}^{(A)} \otimes \rho_{ij}^{(B)}, \quad (13)$$

or

$$\rho_{AB} = \sum_{\alpha,\beta} \rho_{\alpha\beta}^{(A)} \otimes e_{\alpha\beta}^{(B)}, \quad (14)$$

where

$$e_{ij}^{(A)} = |e_i^{(A)}\rangle\langle e_j^{(A)}|, \quad e_{\alpha\beta}^{(B)} = |e_\alpha^{(B)}\rangle\langle e_\beta^{(B)}|,$$

defines orthonormal basis in $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$, respectively, and

$$\rho_{ij}^{(B)} \in \mathcal{B}(\mathcal{H}_B), \quad \rho_{\alpha\beta}^{(A)} \in \mathcal{B}(\mathcal{H}_A).$$

Theorem 1 $D_A(\rho_{AB}) = 0$ iff $\rho_{\alpha\beta}^{(A)}$ are simultaneously diagonalizable. Similarly, $D_B(\rho_{AB}) = 0$ iff $\rho_{ij}^{(B)}$ are simultaneously diagonalizable.

Now it is well known, that if $\rho_{ij}^{(B)}$ are simultaneously diagonalizable then they mutually commute, i.e.

$$[\rho_{ij}^{(B)}, \rho_{kl}^{(B)}] = 0. \quad (15)$$

Note, that

$$\rho_{ij}^{(B)} = \rho_{ji}^{(B)\dagger}, \quad (16)$$

and hence $[\rho_{ij}^{(B)}, \rho_{ji}^{(B)}] = 0$ implies that all $\rho_{ij}^{(B)}$ are normal (clearly, the diagonal blocks are $\rho_{ii}^{(B)}$ are Hermitian and hence normal as well).

Corollary 1 If at least one off-diagonal block $\rho_{ij}^{(B)}$ ($\rho_{\alpha\beta}^{(A)}$) is not normal then $D_B(\rho_{AB}) > 0$ ($D_A(\rho_{AB}) > 0$).

Normal matrices are simultaneously diagonalizable if and only if they mutually commute and hence

Corollary 2 $D_A(\rho_{AB}) = 0$ iff $\rho_{\alpha\beta}^{(A)}$ mutually commute. Similarly, $D_B(\rho_{AB}) = 0$ iff $\rho_{ij}^{(B)}$ mutually commute.

III. CIRCULANT STATES IN $\mathbb{C}^d \otimes \mathbb{C}^d$

Let $\{|e_0\rangle, \dots, |e_{d-1}\rangle\}$ denotes an orthonormal basis in \mathbb{C}^d . One introduces shift operator $S : \mathbb{C}^d \rightarrow \mathbb{C}^d$ defined as follows

$$S|e_n\rangle = |e_{n+1}\rangle, \quad (\text{mod } d). \quad (17)$$

Now, let us define

$$\Sigma_0 = \text{span}\{|e_0\rangle \otimes |e_0\rangle, \dots, |e_{d-1}\rangle \otimes |e_{d-1}\rangle\} \quad (18)$$

and

$$\Sigma_n = (\mathbb{I} \otimes S^n)\Sigma_0, \quad (19)$$

for $n = 1, \dots, d-1$. It is clear that d -dimensional subspaces Σ_n and Σ_m are mutually orthogonal for $m \neq n$ and hence one has the following direct sum decomposition

$$\Sigma_0 \oplus \dots \oplus \Sigma_{d-1} = \mathbb{C}^d \otimes \mathbb{C}^d. \quad (20)$$

Now, consider a class of states living in $\mathbb{C}^d \otimes \mathbb{C}^d$ that may be written as a direct sum

$$\rho = \rho_0 \oplus \dots \oplus \rho_{d-1}, \quad (21)$$

where each ρ_n are supported on Σ_n , that is,

$$\rho_n = \sum_{i,j=1}^{d-1} a_{ij}^{(n)} e_{ij} \otimes S^n e_{ij} S^{\dagger n}, \quad (22)$$

where $[a_{ij}^{(n)}]$ is a $d \times d$ semi-positive matrix, for $n = 0, \dots, d-1$. Normalization of ρ implies following condition for matrices $a^{(n)}$

$$\text{Tr}(a^{(0)} + \dots + a^{(d-1)}) = 1. \quad (23)$$

These states were called circulant [42, 43] due to the cyclic structure of the shift operator S . The above construction defines natural generalization of the well known X -states. Indeed, for $d = 2$ one obtains

$$\rho = \left(\begin{array}{cc|cc} a_{00} & \cdot & \cdot & a_{01} \\ \cdot & b_{00} & b_{01} & \cdot \\ \cdot & b_{10} & b_{11} & \cdot \\ a_{10} & \cdot & \cdot & a_{11} \end{array} \right) \quad (24)$$

where to make the picture more transparent we replaced all zeros by dots and introduced two matrices $a := a^{(0)}$, $b := a^{(1)}$. For $d = 3$ the structure of a circulant state reads as follows

$$\rho = \left(\begin{array}{ccc|ccc|ccc} a_{00} & \cdot & \cdot & \cdot & a_{01} & \cdot & \cdot & \cdot & a_{02} \\ \cdot & b_{00} & \cdot & \cdot & \cdot & b_{01} & b_{02} & \cdot & \cdot \\ \cdot & \cdot & c_{00} & c_{01} & \cdot & \cdot & \cdot & c_{02} & \cdot \\ \hline \cdot & \cdot & c_{10} & c_{11} & \cdot & \cdot & \cdot & c_{12} & \cdot \\ a_{10} & \cdot & \cdot & \cdot & a_{11} & \cdot & \cdot & \cdot & a_{12} \\ \cdot & b_{10} & \cdot & \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\ \hline \cdot & b_{20} & \cdot & \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\ \cdot & \cdot & c_{20} & c_{21} & \cdot & \cdot & \cdot & c_{22} & \cdot \\ a_{20} & \cdot & \cdot & \cdot & a_{21} & \cdot & \cdot & \cdot & a_{22} \end{array} \right) \quad (25)$$

where $a := a^{(0)}$, $b := a^{(1)}$, $c := a^{(2)}$. Actually, it turns out that many well known examples of quantum states of composite systems belong to the class of circulant states: the most prominent are Werner state, isotropic state, states invariant under the local action of the unitary group $U(d)$ and many others [42]. In Section V we analyze an interesting subclass of circulant states – Bell diagonal states.

Let $\Pi_n^{(AB)}$ be an orthogonal projector onto Σ_n , that is

$$\Pi_n^{(AB)} = \sum_{i=0}^{d-1} e_{ii} \otimes e_{i+n, i+n}. \quad (26)$$

We add a superscript “AB” to emphasize that $\Pi_n^{(AB)}$ is a non-local (or rather non-separable) projector. Note, that ρ_{AB} is circulant if and only if

$$\rho = \sum_{n=0}^{d-1} \Pi_n^{(AB)} \rho \Pi_n^{(AB)}. \quad (27)$$

Now, let us look for a circulant states with vanishing quantum discord. One easily finds for the corresponding blocks:

$$\rho_{ij}^{(B)} = \sum_{n=0}^{d-1} a_{ij}^{(n)} e_{i+n, j+n}, \quad (28)$$

and

$$\rho_{ij}^{(A)} = \sum_{n=0}^{d-1} a_{i-n, j-n}^{(n)} e_{i-n, j-n}. \quad (29)$$

Due to the Corollary 1 the necessary condition for $D_B(\rho) = 0$ ($D_A(\rho) = 0$) is that $\rho_{ij}^{(B)}$ ($\rho_{ij}^{(A)}$) are normal. To simplify our analysis we shall consider only prime dimension d .

Proposition 1 *If d is prime, then off-diagonal blocks $\rho_{ij}^{(B)}$ are normal iff*

$$|a_{ij}^{(n)}| = |a_{ij}^{(0)}|, \quad (30)$$

for $i, j, n = 0, 1, 2$. The off-diagonal blocks $\rho_{ij}^{(A)}$ are normal iff

$$|a_{ij}^{(n)}| = |a_{i+n, j+n}^{(0)}|, \quad (31)$$

for $i, j, n = 0, 1, 2$.

If d is not prime then (30) and (31) are sufficient but not necessary to guarantee that $\rho_{ij}^{(B)}$ and $\rho_{ij}^{(A)}$ are normal. Note, that if d is not prime, then off-diagonal blocks may display sub-block structure corresponding to the decomposition $d = d_1^{p_1} \dots d_r^{p_r}$, with d_1, \dots, d_r prime numbers. We shall not analyze this situation in the present paper.

Note, that if all $a^{(n)}$ are diagonal, then $\rho_{ij}^{(A)} = \rho_{ij}^{(B)} = 0$ and the corresponding circulant state is purely classical. Suppose now that at least one matrix $a^{(n)}$ is not diagonal.

Proposition 2 *If the off-diagonal blocks $\rho_{ij}^{(B)}$ are normal, then $[\rho_{kk}^{(B)}, \rho_{ij}^{(B)}] = 0$, iff*

$$a_{kk}^{(n)} = a_{kk}^{(0)}. \quad (32)$$

Similarly, If the off-diagonal blocks $\rho_{ij}^{(A)}$ are normal, then $[\rho_{kk}^{(A)}, \rho_{ij}^{(A)}] = 0$, iff

$$a_{kk}^{(n)} = a_{k+n, k+n}^{(0)}. \quad (33)$$

It should be stressed that conditions (30) and (32) are necessary for $D_B(\rho) = 0$. Similarly, conditions (31) and (33) are necessary for $D_A(\rho) = 0$.

Now, let us formulate sufficient conditions. Let V be a unitary operator

$$V = \sum_{n=0}^{d-1} e_{nn} e^{i\phi_n}, \quad (34)$$

with $\phi_0 = 0$ (the global phase would play no role in what follows). The main result of this paper consists in the following

Theorem 2 *Assume that at least one matrix $a^{(n)}$ is not diagonal and d is prime.*

1. $D_A(\rho) = 0$, if and only if

$$a^{(k)} = (VS^\dagger)^k a^{(0)} (SV^\dagger)^k, \quad (35)$$

for $k = 1, \dots, d-1$.

2. $D_B(\rho) = 0$, if and only if

$$a^{(k)} = S^{\dagger(k-1)} (VS)^{(k-1)} V a^{(0)} V^\dagger (VS)^\dagger{}^{(k-1)} S^{k-1}, \quad (36)$$

for $k = 1, \dots, d-1$.

Hence, discord zero circulant state is fully characterized by a single matrix $a^{(0)} \geq 0$ and a unitary operator V . In particular taking $V = \mathbb{I}$ one obtains

$$a^{(k)} = S^{\dagger k} a^{(0)} S^k, \quad (37)$$

for $D_A(\rho) = 0$, and

$$a^{(k)} = a^{(0)}, \quad (38)$$

for $D_B(\rho) = 0$.

Example 1 In two-qubit case a circulant state is given by (24). Interestingly, if (24) is not diagonal, then X -state with vanishing discord is fully characterized by Propositions 1 and 2:

1. X -state has vanishing D_A , iff

$$a_{00} = b_{11}, \quad a_{11} = b_{00}, \quad (39)$$

and $|a_{01}| = |b_{01}|$.

2. X -state has vanishing D_B , iff

$$a_{00} = b_{00}, \quad a_{11} = b_{11}, \quad (40)$$

and $|a_{01}| = |b_{01}|$.

3. X -state is purely classical, i.e. $D_A(\rho) = D_B(\rho) = 0$, iff

$$a_{00} = b_{00} = a_{11} = b_{11} = \frac{1}{4}, \quad (41)$$

and $|a_{01}| = |b_{01}|$.

These results were already derived in [31, 47–49].

Example 2 Consider now a circulant state in $3 \otimes 3$ defined in (25). Assume that a matrix a_{ij} is not diagonal.

1. $D_A(\rho) = 0$, iff the matrices b_{ij} and c_{ij} are defined by

$$b = \begin{pmatrix} a_{11} & a_{12}e^{i\varphi_1} & a_{10}e^{i(\varphi_1+\varphi_2)} \\ a_{21}e^{-i\varphi_1} & a_{22} & a_{20}e^{i\varphi_2} \\ a_{01}e^{-i(\varphi_1+\varphi_2)} & a_{02}e^{-i\varphi_2} & a_{00} \end{pmatrix},$$

and

$$c = \begin{pmatrix} a_{22} & a_{20}e^{i(\varphi_1+\varphi_2)} & a_{21}e^{i\varphi_2} \\ a_{02}e^{-i(\varphi_1+\varphi_2)} & a_{00} & a_{01}e^{-i\varphi_1} \\ a_{12}e^{-i\varphi_2} & a_{10}e^{i\varphi_1} & a_{11} \end{pmatrix}.$$

2. $D_B(\rho) = 0$, iff the matrices b_{ij} and c_{ij} are defined by

$$b = \begin{pmatrix} a_{00} & a_{01}e^{-i\varphi_1} & a_{02}e^{-i(\varphi_1+\varphi_2)} \\ a_{10}e^{i\varphi_0} & a_{11} & a_{12}e^{-i\varphi_2} \\ a_{20}e^{i(\varphi_1+\varphi_2)} & a_{21}e^{i\varphi_1} & a_{22} \end{pmatrix},$$

and

$$c = \begin{pmatrix} a_{00} & a_{01}e^{-i(\varphi_1+\varphi_2)} & a_{02}e^{-i\varphi_2} \\ a_{10}e^{i(\varphi_1+\varphi_2)} & a_{11} & a_{12}e^{i\varphi_1} \\ a_{20}e^{i\varphi_2} & a_{21}e^{-i\varphi_1} & a_{22} \end{pmatrix}.$$

3. If $D_A(\rho) = D_B(\rho)$, then

$$a_{ii}^{(n)} = \frac{1}{9}, \quad (42)$$

and

$$|a_{ij}^{(n)}| = \text{const.} \quad (43)$$

for $i \neq j$.

More generally one has the following

Theorem 3 A two-qudit circulant state ρ living in $\mathbb{C}^d \otimes \mathbb{C}^d$, where d is a prime number, is completely classical, i.e. $D_A(\rho) = 0$ and $D_B(\rho) = 0$, if and only if

$$a_{ii}^{(0)} = \frac{1}{d^2}, \quad (44)$$

for $i = 0, \dots, d-1$, and the off-diagonal elements

$$|a_{ij}^{(0)}| = \text{const.} \quad (45)$$

The remaining matrices $a^{(n)}$ are defined as follows

$$a^{(k)} = (VS^\dagger)^k a^{(0)} (SV^\dagger)^k. \quad (46)$$

IV. CIRCULANT SYMMETRIC STATES

Interestingly several classes of symmetric states, like e.g. Werner or isotropic states, belong to the class of circulant states. Let G be a subgroup of the unitary group $U(d)$. A bipartite operator A living in $\mathbb{C}^d \otimes \mathbb{C}^d$ is $G \otimes G$ -invariant [51, 52] if

$$U \otimes U \rho = \rho U \otimes U, \quad (47)$$

for all $U \in G$. Note, that if A is $G \otimes G$ -invariant then its partial transposition A^Γ is $U \otimes \bar{U}$ -invariant, where \bar{U} denotes complex conjugation with respect to a fixed basis in \mathbb{C}^d .

If $G = U(d)$ then $G \otimes G$ -invariant state – Werner state – is given by

$$\rho_W = \frac{1-\lambda}{d^2} \mathbb{I} \otimes \mathbb{I} + \frac{\lambda}{d} \mathbb{F}, \quad (48)$$

where \mathbb{F} is the flip operator defined by $\mathbb{F} = \sum_{i,j=0}^{d-1} e_{ij} \otimes e_{ji}$. It is a circulant state and the corresponding matrices $a^{(n)}$ read as follows

$$a^{(0)} = \begin{cases} 0 & , i \neq j, \\ \frac{\lambda}{d} + \frac{1-\lambda}{d^2} & , i = j \end{cases}$$

$$a^{(k)} = \begin{cases} \frac{\lambda}{d} & , j = i + k, \\ \frac{1-\lambda}{d^2} & , i = j \end{cases}$$

Hence $D_A(\rho_W) = D_B(\rho_W) = 0$ only if $\lambda = 0$.

Similarly an isotropic states which is invariant under $G \otimes \bar{G}$ is defined by

$$\rho_I = \frac{1-\lambda}{d^2} \mathbb{I} \otimes \mathbb{I} + \lambda P_d^+, \quad (49)$$

where

$$P_d^+ = \frac{1}{d} \sum_{i,j=0}^{d-1} e_{ij} \otimes e_{ij}. \quad (50)$$

One finds

$$a^{(0)} = \begin{cases} \frac{\lambda}{d} & , i \neq j, \\ \frac{\lambda}{d} + \frac{1-\lambda}{d^2} & , i = j \end{cases}$$

$$a^{(k)} = \begin{cases} 0 & , i \neq j, \\ \frac{1-\lambda}{d^2} & , i = j, \end{cases}$$

for $k = 1, \dots, d-1$. Again $D_A(\rho_I) = D_B(\rho_I) = 0$ only if $\lambda = 0$. This results agree with [25] where geometric discord for two-qudit Werner and Isotropic states was calculated.

Consider now G consisting of real unitary operators from $U(d)$ (again in fixed basis in \mathbb{C}^d). It turns out that $G = O(d)$ [51, 52]. One shows that $O(d) \otimes O(d)$ -invariant state has the following form

$$\rho = a\tilde{\mathcal{P}}_0 + b\tilde{\mathcal{P}}_1 + c\tilde{\mathcal{P}}_2, \quad (51)$$

with $a + b + c = 1$ and $a, b, c \geq 0$. Normalized projectors $\tilde{\mathcal{P}}_0$ are defined as follows $\tilde{\mathcal{P}}_k = \mathcal{P}_k / \text{Tr} \mathcal{P}_k$, where

$$\mathcal{P}_0 = Q^+ - P^+, \quad \mathcal{P}_1 = Q^-, \quad \mathcal{P}_2 = P_d^+, \quad (52)$$

and Q^+, Q^- are projectors onto the symmetric and antisymmetric subspaces in $\mathbb{C}^d \otimes \mathbb{C}^d$, that is,

$$Q^\pm = \frac{1}{2}(\mathbb{I} \otimes \mathbb{I} \pm \mathbb{F}). \quad (53)$$

Again, one easily shows that for $d > 2$ a symmetric state ρ has vanishing discord if and only if ρ is maximally mixed. Interestingly for $d = 2$ a class of symmetric discord zero states is nontrivial. The density matrix has the following form

$$\rho = \frac{1}{4} \left(\begin{array}{cc|cc} a+2c & . & . & 2c-a \\ . & a+2b & a-2b & . \\ \hline . & a-2b & a+2b & . \\ 2a-c & . & . & a+2c \end{array} \right), \quad (54)$$

and hence it belongs to the class of X -states. It is well known that ρ is separable iff $b, c \leq 1/2$. Note that ρ has vanishing discord iff $b = c$ (see Fig. 1). Note, that the simplex in the bc -plane is defined by three vertices $\tilde{\mathcal{P}}_k$: separable $\tilde{\mathcal{P}}_0$ and entangled $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_2$. Interestingly, $\tilde{\mathcal{P}}_0$ is not only separable but even purely classical.

Finally, let G be a maximally commuting subgroup of $U(d)$ (again with respect to a fixed basis in \mathbb{C}^d). It was shown [53] that $G \otimes G$ -invariant state has the following form

$$\rho = \sum_{i,j=0}^{d-1} a_{ij} e_{ij} \otimes e_{ij} + \sum_{i \neq j} d_{ij} e_{ii} \otimes e_{jj}, \quad (55)$$

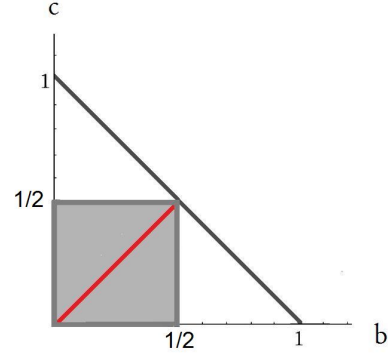


FIG. 1: A simplex of states with orthogonal symmetry for $d=2$. Separable states form a gray square while zero-discord states are represented by the red line.

where a_{ij} is a $d \times d$ positive matrix and d_{ij} are non-negative numbers. Evidently, ρ defines a circulant state with $a^{(0)} = a$ and $a^{(k)}$ are diagonal for $k > 0$. Interestingly, this class of symmetric states is characterized by a simple PPT condition, namely, ρ is PPT iff

$$|a_{ij}|^2 \leq d_{ij} d_{ji}, \quad i \neq j. \quad (56)$$

However, the general condition for separability is not known [53]. Note that ρ has vanishing discord iff $a_{ij} = 0$ for $i \neq j$, that is, ρ is diagonal.

V. GENERALIZED BELL DIAGONAL STATES

In this section we analyze an important subclass of circulant states. Let us introduce d^2 maximally entangled projectors defined by

$$P_{mn} = (\mathbb{I} \otimes U_{mn}) P_d^+ (\mathbb{I} \otimes U_{mn}^\dagger), \quad (57)$$

where U_{mn} are unitary matrices defined as follows:

$$U_{mn} |e_k\rangle = \lambda^{mk} S^n |e_k\rangle, \quad (58)$$

with S being the shift operator defined in (17), P_d^+ - projector on maximally entangled state, and $\lambda = e^{2\pi i/d}$.

Remark 1 Actually, one may define a more general class of states based on a class of ‘shift and multiply

basis' of unitary matrices in \mathbb{C}^d defined as follows [50]

$$U_{ij}|e_k\rangle = H_{ik}^j|e_{L(j,k)}\rangle, \quad (59)$$

where a set of complex numbers H_{ik}^j , and $L : I_d \times I_d \rightarrow I_d$ with $I_d := \{0, 1, \dots, d-1\}$, satisfy the following conditions:

- i) each H^j is a Hadamard matrix,
- ii) L is a Latin square, i.e., the maps $k \rightarrow L(k; \ell)$ and $\ell \rightarrow L(k; \ell)$ are injective for every ℓ . It is clear that (58) defines a special example of (59).

Consider simplex of Bell diagonal states defined by

$$\rho = \sum_{m,n=0}^{d-1} p_{mn} P_{mn}, \quad (60)$$

where $p_{mn} \geq 0$ and $\sum_{m,n=0}^{d-1} p_{mn} = 1$. It is evident from the construction that Bell diagonal states belong to the class of circulant states. One can easily check that corresponding matrices $a^{(n)}$ have the following form

$$a_{ij}^{(n)} = \frac{1}{d} \sum_{m=0}^{d-1} p_{mn} \lambda^{m(i-j)}, \quad (61)$$

hence defining Bell diagonal state is equivalent with determining d^2 coefficients p_{mn} . Let us notice that marginal density matrices of ρ are equal, $\rho_A = \rho_B = \mathbb{I}/d$ which means that $D_A(\rho) = 0$ if and only if $D_B(\rho) = 0$. Hence, whenever Bell diagonal states is classical with respect to one party it is already completely classical.

Consider now $\pi_k \geq 0$ ($k = 0, \dots, d-1$) such that

$$\sum_{k=0}^{d-1} \pi_k = \frac{1}{d}. \quad (62)$$

Using results of the previous section one proves

Theorem 4 *A Bell diagonal state (60), living in $\mathbb{C}^d \otimes \mathbb{C}^d$, where d is a prime number, is a zero-discord state if and only if*

$$p_{ik} = \pi_{i+k\alpha} \pmod{d}, \quad (63)$$

for some $\alpha \in \{0, 1, \dots, d-1\}$.

Hence, any Bell diagonal state is uniquely determined by a vector π_k and the number ' α '.

Example 3 Density operator for two-qubit case is defined by the following matrices $a^{(n)}$

$$a^{(n)} = \begin{pmatrix} x_n & y_n \\ y_n & x_n \end{pmatrix}, \quad (64)$$

for $n = 0, 1$, where

$$x_n = \frac{1}{2}(p_{0n} + p_{1n}), \quad y_n = \frac{1}{2}(p_{0n} - p_{1n}), \quad (65)$$

This state is classical if and only if $x_n = 1/4$ and $y_1 = \pm y_0$. In terms of the probability matrix

$$p_{ij} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix},$$

one has

$$\begin{pmatrix} \pi_0 & \pi_0 \\ \pi_1 & \pi_1 \end{pmatrix}, \quad \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_1 & \pi_0 \end{pmatrix}, \quad (66)$$

corresponding to $\alpha = 0$ and $\alpha = 1$, respectively.

Example 4 A two-qutrit Bell diagonal state is defined by matrices

$$a^{(n)} = \begin{pmatrix} x_n & z_n & \bar{z}_n \\ \bar{z}_n & x_n & z_n \\ z_n & \bar{z}_n & x_n \end{pmatrix}, \quad (67)$$

for $n = 0, 1, 2$, where

$$x_n = \frac{1}{3}(p_{0n} + p_{1n} + p_{2n}) \quad (68)$$

and

$$z_n = \frac{1}{3}(p_{0n} + \bar{\lambda}p_{1n} + \lambda p_{2n}). \quad (69)$$

This state is classical if and only if diagonal elements $x_n = 1/9$ and the off-diagonal elements fulfill one of the following conditions

$$z_n = \lambda^{n\alpha} z_0, \quad n = 1, 2, \quad (70)$$

where $\alpha \in \{0, 1, 2\}$. In terms of the probability matrix p_{ij} one has

$$\begin{pmatrix} \pi_0 & \pi_0 & \pi_0 \\ \pi_1 & \pi_1 & \pi_1 \\ \pi_2 & \pi_2 & \pi_2 \end{pmatrix}, \quad \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \\ \pi_1 & \pi_2 & \pi_0 \\ \pi_2 & \pi_0 & \pi_1 \end{pmatrix}, \quad \begin{pmatrix} \pi_0 & \pi_2 & \pi_1 \\ \pi_1 & \pi_0 & \pi_2 \\ \pi_2 & \pi_1 & \pi_0 \end{pmatrix},$$

for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$, respectively.

VI. CONCLUSIONS

We analyzed a large class of two-qudit circulant states which provide natural generalization of the celebrated X -states. For prime dimension d we formulated necessary and sufficient conditions for vanishing discord. It turns out that such states are fully

characterized by a density operator living on one particular subspace from the direct sum decomposition $\Sigma_0 \oplus \dots \oplus \Sigma_{d-1}$. Interestingly the class of circulant states contains several well known classes of symmetric states, i.e. states invariant under the local action of

$U(d)$ or its subgroups. Finally, we characterized Bell diagonal states (another important class of circulant states) with vanishing quantum discord. This analysis generalizes the well known characterization of Bell diagonal states of two qubits.

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