Exposed positive maps: a sufficient condition

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Abstract

Exposed positive maps in matrix algebras define a dense subset of extremal maps. We provide a sufficient condition for a positive map to be exposed. This is an analog of a spanning property which guaranties that a positive map is optimal. We analyze a class of decomposable maps for which this condition is also necessary.

1 Introduction

Positive maps in $\mathbb{C}^*$-algebras play an important role both in mathematics, in connection with the operator theory [1], and in modern quantum physics. Normalized positive maps provide an affine mapping between sets of states of $\mathbb{C}^*$-algebras. In recent years positive maps found important application in entanglement theory defining basic tool for detecting quantum entangled states (see e.g. [2] for the recent review).

Let $\mathfrak{U}$ be a unital $\mathbb{C}^*$-algebra. A linear map $\Phi : \mathfrak{U} \to \mathcal{B}(\mathcal{H})$ is positive if $\Phi(\mathfrak{U}_+) \subset \mathcal{B}_+(\mathcal{H})$, where $\mathfrak{U}_+$ denotes positive elements in $\mathfrak{U}$. Denote by $M_k(\mathfrak{U}) = M_k(\mathbb{C}) \otimes \mathfrak{U}$ a space of $k \times k$ matrices with entries from $\mathfrak{U}$. One says that Φ is k-positive if a linear map $\Phi_k := \mathbb{1}_k \otimes \Phi : M_k(\mathfrak{U}) \to M_k(\mathcal{B}(\mathcal{H}))$ is positive. Finally, Φ is completely positive if it is k-positive for $k = 1, 2, \ldots$. Due to the Stinespring theorem [3] the structure of completely positive maps is perfectly known: any completely positive map Φ may be represented in the following form

$$\Phi(a) = V^\dagger \pi(a) V \ , \tag{1}$$

where $V : \mathcal{H} \to \mathcal{K}$, and π is a representation of $\mathfrak{U}$ in the Hilbert space $\mathcal{K}$. Unfortunately, in spite of the considerable effort, the structure of positive maps is rather poorly understood [4]–[23].

Denote by $\mathcal{P}$ a convex cone of positive maps $\Phi : \mathfrak{U} \to \mathcal{B}(\mathcal{H})$. Note, that a space $\mathcal{L}(\mathfrak{U}, \mathcal{B}(\mathcal{H}))$ of linear maps from $\mathfrak{U}$ to $\mathcal{B}(\mathcal{H})$ is isomorphic to $\mathcal{B}(\mathcal{H}) \otimes \mathfrak{U}$. The natural pairing between these two spaces in defined as follows [13]: taking an orthonormal basis in $\mathcal{H}$ ($m = \dim \mathcal{H} < \infty$) one identifies $\mathcal{B}(\mathcal{H})$ with $M_m(\mathbb{C})$ and defines

$$\langle X \otimes a, \Phi \rangle := \mathrm{tr}(X^{\mathrm{t}} \Phi(a)) \ , \tag{2}$$

where $X \in M_m(\mathbb{C})$, $a \in \mathfrak{U}$, and X^{t} denotes transposition of X with respect to a given basis in $\mathcal{H}$. Let $\mathcal{P}^\circ$ denote a dual cone [13, 24]

$$\mathcal{P}^\circ = \{ \ A \in M_m(C) \otimes \mathfrak{U} \ : \ \langle A, \Phi \rangle \geq 0 \ , \ \Phi \in \mathcal{P} \ \} \ . \tag{3}$$

Note that the definition of $\mathcal{P}^\circ$ may be reformulated as follows

$$\mathcal{P}^\circ = \mathrm{conv}\{ \ X \otimes a \in M_m(\mathbb{C}) \otimes \mathfrak{U} \ : \ \langle X \otimes a, \Phi \rangle \geq 0 \ , \ \Phi \in \mathcal{P} \ \} \ . \tag{4}$$

One finds $\mathcal{P}^\circ = M_m^+(\mathbb{C}) \otimes \mathfrak{U}_+$, where $M_m^+(\mathbb{C})$ denotes positive matrices from $M_m(\mathbb{C})$. It shows that $\mathcal{P}^\circ$ defines a convex cone of separable elements in $M_m(\mathbb{C}) \otimes \mathfrak{U}$.

Recall that a face of $\mathcal{P}$ is a convex subset $F \subset \mathcal{P}$ such that if the convex combination $\Phi = \lambda\Phi_1 + (1-\lambda)\Phi_2$ of $\Phi_1, \Phi_2 \in \mathcal{P}$ belongs to F, then both $\Phi_1, \Phi_2 \in F$. If a ray $\{\lambda\Phi \ : \ \lambda > 0\}$ is a face of $\mathcal{P}$ then it is called an extreme ray, and we say that Φ generates an extreme ray. For simplicity we call such Φ an extremal positive map. A face F is exposed if there exists a supporting hyperplane H for a convex cone $\mathcal{P}$ such that $F = H \cap \mathcal{P}$. The property of 'being an exposed face' may be reformulated as follows: A face F of $\mathcal{P}$ is exposed iff there exists $a \in \mathfrak{U}_+$ and $|h\rangle \in \mathcal{H}$ such that

$$F = \{ \ \Phi \in \mathcal{P} \ ; \ \Phi(a)|h\rangle = 0 \ \} \ .$$

A positive map $\Phi \in \mathcal{P}$ is exposed if it generates 1-dimensional exposed face. Let us denote by $\mathrm{Ext}(\mathcal{P})$ the set of extremal points and $\mathrm{Exp}(\mathcal{P})$ the set of exposed points of $\mathcal{P}$. Due to Straszewicz theorem [24] $\mathrm{Exp}(\mathcal{P})$ is a dense subset of $\mathrm{Ext}(\mathcal{P})$. Thus every extreme map is the limit of some sequence of exposed maps meaning that each entangled state may be detected by some exposed positive map. Hence, a knowledge of exposed maps is crucial for the full characterization of separable/entangled states of bi-partite quantum systems. For recent papers on exposed maps see e.g. [13, 21, 22, 23].

Now, if F is a face of $\mathcal{P}$ then

$$F' = \mathrm{conv}\{ \ a \otimes |h\rangle\langle h| \in \mathcal{P}^\circ \ : \ \Phi(a)|h\rangle = 0 \ , \ \Phi \in F \ \} \ . \tag{5}$$

defines a face of $\mathcal{P}^\circ$ (one calls F' a dual face of F). Actually, F' is an exposed face. One proves [13] that a face F is exposed iff $F'' = F$.

In this paper we analyze linear positive maps $\Phi : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$, where both $\mathcal{K}$ and $\mathcal{H}$ are finite dimensional Hilbert spaces. We provide a sufficient condition for the map to be exposed. We call it *strong spanning property* in analogy to well known spanning property which is sufficient for the map to be optimal [26]. Interestingly, this condition is also necessary if Φ is decomposable and $\dim \mathcal{K} = 2$. Finally, we characterize the property of exposedness in terms of entanglement witnesses.

2 Preliminaries

Consider a positive map $\Phi : \mathfrak{U} \rightarrow \mathcal{B}(\mathcal{H})$, where $\mathfrak{U}$ is a unital $\mathbb{C}^*$-algebra and $\mathcal{B}(\mathcal{H})$ denotes a set of bounded operators on the finite dimensional Hilbert space $\mathcal{H}$.

Proposition 2.1 *If $a \in \mathfrak{U}$ is strictly positive, i.e. $a \in \mathrm{int}\, \mathfrak{U}_+$, then $\mathrm{Im}\, \Phi(b) \subset \mathrm{Im}\, \Phi(a)$ for all $b \in \mathfrak{U}_+$.*

Proof: Let us observe that

$$\mathrm{Ker}\, \Phi(a) \subset \mathrm{Ker}\, \Phi(b) \ . \tag{6}$$

Indeed, suppose that there exists $x \in \mathcal{H}$ such that $x \in \mathrm{Ker}\,\Phi(a)$ and $x \notin \mathrm{Ker}\,\Phi(b)$. One has $\langle x|\Phi(b)|x\rangle > 0$ and $\langle x|\Phi(a)|x\rangle = 0$. Now, since $a \in \mathrm{int}\,\mathfrak{U}_+$ there exists $\epsilon > 0$ such that an open ball $B(a,\epsilon) \subset \mathfrak{U}_+$. It is therefore clear that

$$a' = a - \frac{\epsilon}{2}\frac{u-a}{||u-a||}$$

belongs to $\mathfrak{U}_+$. One has

$$\langle x|\Phi(a')|x\rangle = -\frac{\epsilon}{2||u-a||}\,\langle x|\Phi(u)|x\rangle < 0 \ , \tag{7}$$

which contradicts that Φ is a positive map. Hence, if $a \in \mathrm{int}\,\mathfrak{U}_+$, then $\mathrm{Ker}\,\Phi(a) \subset \mathrm{Ker}\,\Phi(b)$ for any $b \in \mathfrak{U}_+$ which implies $\mathrm{Im}\,\Phi(b) \subset \mathrm{Im}\,\Phi(a)$. $\Box$.

Corollary 2.1 *If* $a, b \in \mathrm{int}\,\mathfrak{U}_+$ *, then* $\mathrm{Im}\,\Phi(a) = \mathrm{Im}\,\Phi(b)$.

Corollary 2.2 *In particular for* $a \in \mathfrak{U}_+$ $(a \in \mathrm{int}\,\mathfrak{U}_+)$*, one has* $\mathrm{Im}\,\Phi(a) \subset \mathrm{Im}\,\Phi(\mathbb{1})$ $(\mathrm{Im}\,\Phi(a) = \mathrm{Im}\,\Phi(\mathbb{1}))$.

Let $A := \Phi(\mathbb{1})$. If $A > 0$, that is, A is of full rank, then one has

$$\Phi(a) = A^{1/2}\widetilde{\Phi}(a)A^{1/2} \ , \tag{8}$$

where $\widetilde{\Phi}(a) = A^{-1/2}\Phi(a)A^{-1/2}$ is a unital positive map from $\mathfrak{U}$ into $\mathcal{B}(\mathcal{H})$. If A is not strictly positive, that is, $A \in \partial\mathcal{B}_+(\mathcal{H})$, then denote by $\mathcal{H}_\Phi$ the range of A. A is invertible on its image and denote by $\widetilde{A}^{-1}$ the generalized inverse of A. Now, one has

$$\Phi(a) = A^{1/2}\widetilde{A}^{-1/2}\Phi(a)\widetilde{A}^{-1/2}A^{1/2} \ . \tag{9}$$

Note, that $\mathrm{Im}\,\Phi(a) \subset \mathcal{H}_\Phi$. Following [8] let us introduce the following

Definition 2.1 *Consider a positive map* $\phi : \mathfrak{U} \to \mathcal{B}(\mathcal{H})$*. A map* $\phi' : \mathfrak{U} \to \mathcal{B}(\mathcal{H}')$ *is called an* extension *of* ϕ *iff* $\mathcal{H}' \supset \mathcal{H}$ *and for any* $a \in \mathfrak{U}$

$$\phi(a) = \mathcal{P}\phi'(a)\mathcal{P} \ , \tag{10}$$

where $\mathcal{P}$ *denotes orthogonal projection* $\mathcal{H}' \to \mathcal{H}$*.*

Note that $\mathcal{H}' = \mathcal{H} \oplus \mathcal{H}^\perp$ and hence for any $|h'\rangle \in \mathcal{H}'$ one has $|h'\rangle = |h\rangle \oplus |h^\perp\rangle$, where $|h\rangle = \mathcal{P}|h'\rangle$ which implies $\phi(a)|h\rangle = \mathcal{P}\phi'(a)|h\rangle$, and an extension Φ' is trivial if

$$\phi'(a)|h\rangle = \phi(a)|h\rangle \ , \tag{11}$$

for all $a \in \mathfrak{U}$ and $|h\rangle \in \mathcal{H}$. According to this definition a positive map $\widetilde{\Phi}(a) := \widetilde{A}^{-1/2}\Phi(a)\widetilde{A}^{-1/2}$ is a trivial extension of the unital map $\Phi_1 : \mathfrak{U} \to \mathcal{B}(\mathcal{H}_\Phi)$

$$\Phi_1 = \mathcal{P}_\Phi\,\widetilde{\Phi}\,\mathcal{P}_\Phi \ , \tag{12}$$

where $\mathcal{P}_\Phi$ is a projector $\mathcal{H} \to \mathcal{H}_\Phi$. This way we proved the following

Proposition 2.2 *Any linear positive map $\Phi : \mathfrak{U} \to \mathcal{B}(\mathcal{H})$ can be written as follows*

$$\Phi(a) = V^\dagger \Phi_1(a) V \ , \tag{13}$$

where $V : \mathcal{H} \to \mathcal{H}_\Phi$ and $\Phi_1 : \mathfrak{U} \to \mathcal{B}(\mathcal{H}_\Phi)$ is unital.

Let us recall

Definition 2.2 *A linear map Φ is irreducible if $[\Phi(a), X] = 0$ for all $a \in \mathfrak{U}$ implies that $X = \lambda \mathbb{I}_\mathcal{H}$. Φ is irreducible on its image if $[\Phi(a), X] = 0$ for all $a \in \mathfrak{U}$ implies that $\mathcal{P}_\Phi X \mathcal{P}_\Phi = \lambda \mathbb{I}_{\mathcal{H}_\Phi}$.*

Remark 2.1 *Note, that one may restrict oneself to self-adjoint elements $\mathfrak{U}_{\rm sa}$ only. Indeed, suppose that Φ is irreducible and $[\Phi(a), X] = 0$ for all $a \in \mathfrak{U}_{\rm sa}$. Any element $x \in \mathfrak{U}$ may be decomposed as $x = x_1 + ix_2$, with $x_1, x_2 \in \mathfrak{U}_{\rm sa}$. One has*

$$[\Phi(x), X] = [\Phi(x_1), X] + i[\Phi(x_2), X] = 0 \ ,$$

and irreducibility of Φ implies therefore $X = \lambda \mathbb{I}_{\mathcal{H}_\Phi}$.

Proposition 2.3 *Let a positive map Φ be irreducible. If $X\Phi(a) = \Phi(a)X^\dagger$ for all $a \in \mathfrak{U}$, then $X = \lambda \mathbb{I}_\mathcal{H}$.*

Proof: Irreducibility implies that $A = \Phi(\mathbb{1}) > 0$ and hence

$$\Phi(a) = A^{1/2} \Phi_1(a) A^{1/2} \ , \tag{14}$$

where Φ_1 is unital. One has

$$X A^{1/2} \Phi_1(a) A^{1/2} = A^{1/2} \Phi_1(a) A^{1/2} X^\dagger \ ,$$

and hence

$$Y \Phi_1(a) = \Phi_1(a) Y^\dagger \ , \tag{15}$$

with $Y = A^{-1/2} X A^{1/2}$. Using $\Phi_1(\mathbb{1}) = \mathbb{I}_\mathcal{H}$ one finds $Y^\dagger = Y$. Let us observe that Φ_1 is irreducible as well and hence $Y = \lambda \mathbb{I}_\mathcal{H}$ which implies $X = \lambda \mathbb{I}_\mathcal{H}$. □

3 Exposed maps – sufficient condition

In this section we formulate a sufficient condition for a map $\Phi : \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{H})$ to be exposed. Recall that a linear operator $W \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H})$ is block-positive iff $\langle x \otimes y | W | x \otimes y \rangle \geq 0$ for all product vectors $|x \otimes y\rangle \in \mathcal{K} \otimes \mathcal{H}$. Now, due to the Choi-Jamiołkowski isomorphism, W is block-positive iff there exists a positive map $\Phi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ such that

$$W = (\mathbb{1}_\mathcal{K} \otimes \Phi) P^+_\mathcal{K} \ ,$$

where $\mathbb{1}_\mathcal{K}$ is an identity map in $\mathcal{B}(\mathcal{K})$, and $P^+_\mathcal{K}$ is a maximally entangled state in $\mathcal{K} \otimes \mathcal{K}$. Any block-positive but not positive W is called an entanglement witness. It is therefore clear that any property of a map Φ may be formulated in terms of W and vice versa. Now, let us define

$$P_W = \{ \, x \otimes y \ : \ \langle x \otimes y | W | x \otimes y \rangle = 0 \, \} \ . \tag{16}$$

Note, that

$$\langle x\otimes y|W|x\otimes y\rangle = \langle y|\Phi(|\overline{x}\rangle\langle\overline{x}|)|y\rangle \ ,$$

and hence one may equivalently introduce $P_\Phi \equiv P_W = \{\, x\otimes y \ : \ \Phi(|\overline{x}\rangle\langle\overline{x}|)|y\rangle = 0\,\}$. One says that Φ has *spanning property* iff $\mathrm{span}_{\mathbb{C}} P_\Phi = \mathcal{K}\otimes\mathcal{H}$. Denoting $d_{\mathcal{K}} = \dim\mathcal{K}$ and $d_{\mathcal{H}} = \dim\mathcal{H}$, one proves [26]

Theorem 3.1 *If a positive map Φ satisfies spanning property, then it is optimal.*

In analogy we have the following

Theorem 3.2 *Let $\Phi : \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a positive map irreducible on its image and*

$$N_\Phi = \mathrm{span}_{\mathbb{C}}\{\, a\otimes|h\rangle \in \mathcal{B}_+(\mathcal{K})\otimes\mathcal{H} \ : \ \Phi(a)|h\rangle = 0\,\} \ . \tag{17}$$

If the subspace $N_\Phi \subset \mathcal{B}(\mathcal{K})\otimes\mathcal{H}$ satisfies

$$\dim N_\Phi = d_{\mathcal{K}}^2 d_{\mathcal{H}} - \mathrm{rank}\,\Phi(\mathbb{I}_{\mathcal{K}}) \ , \tag{18}$$

then Φ is exposed.

Proof: The idea of the proof comes from [8] (see Theorem 3.3). Consider a map [25]

$$\tilde{\Phi} \ : \ \mathcal{B}(\mathcal{K})\otimes\mathcal{H} \ \longrightarrow \ \mathcal{H}$$

defined by

$$\tilde{\Phi}(a\otimes|h\rangle) := \Phi(a)|h\rangle \ . \tag{19}$$

Note, that $\dim(\mathrm{Im}\,\tilde{\Phi}) = \mathrm{rank}\,\Phi(\mathbb{I}_{\mathcal{K}})$ and hence N_Φ defines the kernel of $\tilde{\Phi}$. To show that Φ is exposed let us introduce a linear functional f on the space of positive maps $\mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ defined as follows

$$f(\Psi) = \sum_{i=1}^{d_N} \langle h_i|\Psi(a_i)|h_i\rangle \ , \tag{20}$$

where d_N vectors $a_i\otimes|h_i\rangle$ span N_Φ. Note that $f(\Psi)\geq 0$ for all positive maps Ψ and $f(\Phi)=0$. As a result f defines a supporting hyperplane to the cone of positive maps $\mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ passing through a map Φ. Note that Φ is exposed iff $f(\Psi)=0$ implies $\Psi = \lambda\Phi$, with λ being a positive number. Let us observe that $f(\Psi)=0$ if and only if $\tilde{\Psi}(a_i\otimes|h_i\rangle) = \Psi(a_i)|h_i\rangle = 0$, for all $i=1,\ldots,d_N$, and hence the kernel of $\tilde{\Psi}$ contains N_Φ. To complete the proof we use the following

Lemma 3.1 *Consider two linear operators $A, B : V \to W$, where V and W are finite dimensional vector spaces over $\mathbb{C}$. If $\mathrm{ker}A \supset \mathrm{ker}B$, then there exists $X : W \to W$ such that $A = XB$ and* $\mathrm{rank}\,X = \mathrm{rank}\,A$.

Proof: let

$$A = U_A\Sigma_A V_A^\dagger \ , \quad B = U_B\Sigma_B V_B^\dagger \ ,$$

denote the corresponding singular value decompositions of A and B. Let $\{v_\alpha(A)\}$, $\{w_\alpha(A)\}$, $\{v_\alpha(B)\}$ and $\{w_\alpha(B)\}$ denote the orthonormal basis made from columns of V_A, U_A, V_B, U_B respectively. One has

$$\Sigma_A = \sum_{\alpha=1}^{r_A} \sigma_\alpha(A)|w_\alpha(A)\rangle\langle v_\alpha(A)| \ , \quad \Sigma_B = \sum_{\alpha=1}^{r_B} \sigma_\alpha(B)|w_\alpha(B)\rangle\langle v_\alpha(B)| \ , \tag{21}$$

where $\sigma_\alpha(A)$ and $\sigma_\alpha(B)$ are strictly positive singular values of A and B, respectively. Note, that condition $\ker A \supset \ker B$, is equivalent to $r_B \geq r_A$. One finds $A = XB$, where

$$X = AV_B^\dagger \widetilde{\Sigma}_B U_B^\dagger \ , \tag{22}$$

with

$$\widetilde{\Sigma}_B = \sum_{\alpha=1}^{r_B} \sigma_\alpha(B)^{-1}|w_\alpha\rangle\langle v_\alpha| \ . \tag{23}$$

Indeed, one has

$$XB = (AV_B\widetilde{\Sigma}_B U_B^\dagger)(U_B\Sigma_B V_B^\dagger) = AV_B\widetilde{\Sigma}_B\Sigma_B V_B^\dagger = A\sum_{\alpha=1}^{r_B} |v_\alpha(B)\rangle\langle v_\alpha(B)| \ .$$

Now, since $\ker A \supset \ker B$, one has

$$A\sum_{\alpha=1}^{r_B} |v_\alpha(B)\rangle\langle v_\alpha(B)| = A \ ,$$

which ends the proof. □

One has, therefore, $\tilde{\Psi} = X\tilde{\Phi}$, for some operator X acting on the image of $\tilde{\Phi}$, meaning that

$$\Psi(a)|h\rangle = X\Phi(a)|h\rangle \ ,$$

for all $a \in \mathcal{B}(\mathcal{K})$ and $|h\rangle \in \mathcal{H}$. Note that for any $a \in \mathcal{B}_{\rm sa}(\mathcal{K})$ one has $\Psi(a) = \Psi(a)^\dagger$ and hence

$$X\Phi(a) = \Phi(a)X^\dagger \ . \tag{24}$$

Proposition 2.3 implies, therefore, that $X \sim \mathbb{I}$ on the image of Φ. Hence $\Psi = \lambda\Phi$ with $\lambda > 0$ due to the fact that both Φ and Ψ are positive maps. □

Corollary 3.1 *Let $\Phi : \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a positive, unital irreducible map. If*

$$\dim N_\Phi = (d_{\mathcal{K}}^2 - 1)d_{\mathcal{H}} \ , \tag{25}$$

then Φ is exposed.

We propose to call (18) *strong spanning property* in analogy to spanning poperty

$$\dim \mathrm{span}_{\mathbb{C}}\{|x\rangle\otimes|h\rangle \in \mathcal{K}\otimes\mathcal{H} \ : \ \Phi(|\overline{x}\rangle\langle\overline{x}|)|h\rangle = 0\,\} = d_{\mathcal{K}}d_{\mathcal{H}} \ , \tag{26}$$

which is sufficient for optimality.

4 A class of exposed decomposable maps $\mathcal{B}(\mathbb{C}^2) \longrightarrow \mathcal{B}(\mathbb{C}^m)$

In this section we provide a class of positive exposed maps for which strong spanning property (18) is also necessary.

Theorem 4.1 *Let $\Phi : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^m)$ be a decomposable positive but not completely positive map. Then the following conditions are equivalent:*

1. *Φ is exposed.*
2. *$\Phi(\rho) = V^\dagger \rho^{\rm t} V$, where $V : \mathbb{C}^n \to \mathbb{C}^2$ is a linear map of rank two.*
3. *There are $4m-2$ linearly independent vectors in the set $\{ a \otimes |h\rangle \in \mathcal{B}_+(\mathbb{C}^2) \otimes \mathbb{C}^m \, : \, \Phi(a)|h\rangle = 0\}$.*

Proof: ($1 \Rightarrow 2$) Any exposed map is extremal and hence being a decomposable map Φ is given by $\Phi(a) = V^\dagger a V$ or $\Phi(a) = V^\dagger a^{\rm t} V$. The former is evidently CP and the latter in not CP iff $\mathrm{rank}(V) = 2$.

($2 \Rightarrow 3$) Note, that using linear transformation one can transform V to the following form $V = \sum_{i=1}^2 |e_i\rangle\langle f_i|$, where $\{e_i\}_{i=1}^2, \{f_j\}_{j=1}^m$ are orthonormal bases in $\mathbb{C}^2$ and $\mathbb{C}^m$, respectively. One finds $4(m-2)$ independent vectors taking $a \in \mathcal{B}_+(\mathbb{C}^2)$ arbitrary and $|h\rangle = \sum_{j=3}^m h_j f_j$. Now, we look for the remaining vectors $a \otimes |h\rangle$, with $|h\rangle = h_1 f_1 + h_2 f_2$. It is clear that it is enough to consider $a \in \mathcal{B}_+(\mathbb{C}^n)$ being rank-1 projector, i.e. $a = |x\rangle\langle x|$. One has

$$\Phi(|x\rangle\langle x|)|h\rangle = V^\dagger |\overline{x}\rangle\langle \overline{x}|V|h\rangle = \left(\sum_{i=1}^2 x_i h_i \right) \sum_{j=1}^2 \overline{x}_j |f_j\rangle \ . \tag{27}$$

Note that $\Phi(|x\rangle\langle x|)|h\rangle = 0$ for $|x\rangle \neq 0$ if and only if $\sum_{i=1}^2 x_i h_i = 0$, and hence (up to trivial scaling) $x_1 = h_2$ and $x_2 = -h_1$. The family of vectors $|x\rangle\langle x| \otimes |h\rangle \in \mathcal{B}(\mathbb{C}^2) \otimes \mathbb{C}^m$ is linearly independent iff the corresponding vectors $|\overline{x}\rangle \otimes |x\rangle \otimes |h\rangle$ are linearly independent in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^m$. Note that coordinates of $\bar{x} \otimes x \otimes h$ are polynomial functions of h_k and $\bar{h}_k$, namely:

$$h_1 h_2 \overline{h}_2 \, , \ h_1 h_2 \overline{h}_1 \, , \ h_1^2 \overline{h}_2 \, , \ h_1 h_2 \overline{h}_2 \, , \ h_2^2 \overline{h}_2 \, , \ h_2^2 \overline{h}_1 \, , \ h_2 h_1 \overline{h}_2 \, , \ h_2^2 \overline{h}_2 \ .$$

Note, that 6 of them are (functionally) linearly independent and hence one has 6 additional vectors $a \otimes |h\rangle$. Altogether, there are $4(m-2) + 6 = 4m - 2$ linearly independent vectors.

($3 \Rightarrow 1$) Follows from Theorem 3.2. □

A similar problem was analyzed in [27] in the context of optimal decomposable maps. Recall that Φ is decomposable if $\Phi = \Phi_1 + \Phi_2 \circ {\rm t}$, where Φ_1 and Φ_2 are completely positive. Equivalently, the corresponding entanglement witness W is decomposable if $W = Q_1 + (\mathbb{1}_\mathcal{H} \otimes {\rm t}) Q_2$, where $Q_1, Q_2 \in \mathcal{B}_+(\mathcal{H} \otimes \mathcal{K})$. Let us recall that $S \subset \mathcal{H} \otimes \mathcal{K}$ is a *completely entangled subspace* (CES) iff there is no nonzero product vectors in S. The authors of [27] proved the following

Theorem 4.2 *Let $\Phi : \mathcal{B}(\mathbb{C}^2) \longrightarrow \mathcal{B}(\mathbb{C}^m)$ be a positive decomposable map. The following conditions are equivalent*

1. *Φ is optimal,*

2. $\Phi(a) = \mathrm{Tr}_{\mathbb{C}^2}(W\, a^{\mathrm{t}} \otimes \mathbb{I}_m)$ *, where* $W = (\mathbb{1}_2 \otimes \mathrm{t})Q$ *and* $Q \geq 0$ *is supported on a CES,*

3. P_Φ *spans* $\mathbb{C}^2 \otimes \mathbb{C}^m$*.*

Note, that we replaced optimality by exposedness, an arbitrary CES by a 1-dimensional CES supporting a positive operator

$$Q = \sum_{i,j=1}^{2} |i\rangle\langle j| \otimes V^{\mathrm{t}}|i\rangle\langle j|\overline{V} \ ,$$

with $\mathrm{rank}(V) = 2$ (clearly, if $\mathrm{rank}(V) = 1$, then Q is no longer supported on a CES). Finally, we replaced *weak* spanning property

$$\dim \mathrm{span}_{\mathbb{C}} \{\, |\overline{x}\rangle \otimes |h\rangle \ : \ \Phi(|x\rangle\langle x|)|h\rangle = 0 \,\} = 2m \ ,$$

by much stronger property (*strong spanning*)

$$\mathrm{rank}\, \Phi(\mathbb{I}_2) + \dim \mathrm{span}_{\mathbb{C}} \{\, |\overline{x}\rangle \otimes |x\rangle \otimes |h\rangle \ : \ \Phi(|x\rangle\langle x|)|h\rangle = 0 \,\} = 4m \ .$$

5 A class of exposed decomposable maps $\mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^m)$

It was already shown by Marciniak [21] that all extremal decomposable maps $\mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^m)$ are exposed, i.e. maps of the form $\Phi(a) = V^\dagger a V$ and $\Phi(a) = V^\dagger a^{\mathrm{t}} V$ are exposed. Now we show that being exposed these maps in general do not satisfy the *strong spanning property* (18).

Proposition 5.1 *Consider a positive decomposable map* $\Phi : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^m)$ *defined by* $\Phi(a) = V^\dagger a^{\mathrm{t}} V$*. One has*

$$\dim N_\Phi = \begin{cases} m(n^2-1) & , \ \mathrm{rank}(V) > 1 \\ mn^2 - (2m-1) & , \ \mathrm{rank}(V) = 1 \end{cases} \tag{28}$$

Proof: it is clear that it is enough to consider $a \in \mathcal{B}(\mathbb{C}^n)_+$ being rank-1 projector, i.e. $a = |x\rangle\langle x|$. Note, that using a linear transformation one can transform V to the following form $V = \sum_{i=1}^r |e_i\rangle\langle f_i|$, where $\{e_i\}_{i=1}^n, \{f_j\}_{j=1}^m$ are orthonormal bases in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively.

Let $|\tilde{x}\rangle$ and $|\tilde{h}\rangle$ be vectors in $\mathbb{C}^r$ built from the first r coordinates of $|x\rangle$ and $|h\rangle$, respectively. For a given vector $|x\rangle$, the orthogonal complement of $|\tilde{x}\rangle$ is spanned by $r-1$ vectors

$$v_2 = |-x_2, x_1, 0, \ldots, 0\rangle \ , \quad v_3 = |-x_2, 0, x_1, 0, \ldots, 0\rangle \ , \ \ldots, \ v_r = |-x_r, \ldots, x_1\rangle \ .$$

The general vector $|h\rangle$ orthogonal to $|x\rangle$ is then of the form $\sum_{i=2}^r \alpha_i |v_i\rangle \oplus |\hat{h}_i\rangle$ (where $\sum_{i=2}^r \alpha_i |\hat{h}_i\rangle = |h_{r+1}, \ldots, h_m\rangle$). Observe, that $|h_{r+1}, \ldots, h_m\rangle$ can be arbitrary. Now, a general vector $|h\rangle$ which is orthogonal to $|x\rangle$ is a linear combination of vectors from $r-1$ subspaces:

$$\begin{aligned} H_2(x) &= \mathrm{span}_{\mathbb{C}}\{\, |-x_2, x_1, 0, \ldots, 0\rangle \,\} \oplus \mathbb{C}^{m-r} \ , \\ H_3(x) &= \mathrm{span}_{\mathbb{C}}\{\, |-x_3, 0, x_1, 0, \ldots, 0\rangle \,\} \oplus \mathbb{C}^{m-r} \ , \\ &\vdots \\ H_r(x) &= \mathrm{span}_{\mathbb{C}}\{\, |-x_r, 0, \ldots, 0, x_1\rangle \,\} \oplus \mathbb{C}^{m-r} \ . \end{aligned}$$

Consider the subspace $W_2 \subset \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^m \approxeq \mathcal{B}(\mathbb{C}^n) \otimes \mathbb{C}^m$ spanned by the vectors $|\overline{x}\rangle \otimes |x\rangle \otimes |h\rangle$, where $|h\rangle \in H_2(x)$, that is, $|\overline{x}\rangle \otimes |y\rangle$, where

$$|y\rangle = |x_1, \ldots, x_r, x_{r+1}, \ldots, x_n\rangle \otimes |-x_2, x_1, 0, \ldots, 0, h_{r+1}, \ldots, h_m\rangle = \sum_{i,j} y_{ij}\, e_i \otimes f_j \ . \tag{29}$$

The coordinates of $|y\rangle$ are monomials of degree 2 in variables $\{x_1, \ldots, x_n, h_{r+1}, \ldots, h_m\}$. Note that $|y\rangle$ has in general $(2 + m - r) \times n$ non-zero coordinates, which satisfy one linear condition $y_{11} + y_{22} = 0$. Hence $\dim W_2 = n(n[m - r + 2] - 1)$. Using the same argument one shows that $\dim W_2 = \ldots = \dim W_r$. It is easy to show that

$$W_i \cap W_j = W_2 \cap \cdots \cap W_r \ , \tag{30}$$

for each pair $i \neq j$. Moreover, the constructions of $H_i(x)$ imply

$$W_2 \cap \cdots \cap W_r = \mathbb{C}^n \otimes (\mathrm{span}_{\mathbb{C}}\{e_2, \ldots, e_n\} \otimes f_1 \oplus \mathbb{C}^n \otimes \mathrm{span}_{\mathbb{C}}\{f_{r+1}, \ldots, f_m\}),$$

and hence its dimension equals $n(n - 1 + [m - r]n)$. Let $W = \mathrm{span}_{\mathbb{C}}(W_2 \cup \ldots \cup W_r)$. One finds

$$\begin{aligned} \dim W &= \sum_{i=2}^{r} \dim W_i - (r - 2) \cdot \dim(W_2 \cap \cdots \cap W_r) \\ &= (r - 1)n((m - r + 2)n - 1) - (r - 2)n(n - 1 + (m - r)n) = n^2 m - n \ . \end{aligned}$$

Note that if $r = 1$, one consider vectors $|\overline{x}\rangle \otimes |x\rangle \otimes |h\rangle$ such that $x_1 h_1 = 0$. Vectors with $x_1 = 0$ form a $(n - 1)^2 m$ dimensional subspace. Vectors with $h_1 = 0$ form a $n^2(m - 1)$ dimensional subspace. The intersection of these subspaces is $(n - 1)^2(m - 1)$ dimensional. Finally, one gets $(n - 1)^2 m + n^2(m - 1) - (n - 1)^2(m - 1) = n^2 m - (2n - 1)$ linearly independent vectors. □

It is therefore clear that the *strong spanning property*

$$\mathrm{rank}\, \Phi(\mathbb{I}_n) + \dim \mathrm{span}_{\mathbb{C}} \{\, |\overline{x}\rangle \otimes |x\rangle \otimes |h\rangle \ : \ \Phi(|x\rangle\langle x|)|h\rangle = 0 \,\} = mn^2 \ ,$$

supplemented by irreducibility provides only a sufficient condition for exposedness in the same way as *weak spanning property*

$$\dim \mathrm{span}_{\mathbb{C}} \{\, |\overline{x}\rangle \otimes |h\rangle \ : \ \Phi(|x\rangle\langle x|)|h\rangle = 0 \,\} = nm \ ,$$

provides only a sufficient condition for optimality. Note, that $\Phi(a) = V^\dagger a^{\mathrm{t}} V$ has a *strong spanning property* iff $\mathrm{rank}(V) = n$. However, Φ is exposed for any V [21].

6 Conclusions

We have provided a sufficient condition for exposedness – *strong spanning property* (18). It was shown that in the class of decomposable maps $\mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^m)$ this condition is also necessary if $n = 2$. This result provides an analog of the result of [27] in the context of optimal maps/witnesses. One calls a block-positive operator $W \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^m)$ irreducible iff W cannot be written as $W_1 \oplus W_2$, where W_1 and W_2 are block-positive. One has the following

Proposition 6.1 *Let $W \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^m)$ be a block-positive irreducible operator. If*

$$\dim(\mathrm{Im}\mathrm{Tr}_{\mathbb{C}^n} W) + \dim\{a \otimes h : \mathrm{Tr}_{\mathbb{C}^n}[W\, a^{\mathrm{t}} \otimes \mathbb{I}_m]|h\rangle = 0\} = n^2 m \ , \tag{31}$$

then W is exposed.

If $n = 2$, then one proves the following

Proposition 6.2 *Let $W \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^m)$ be a block-positive but not positive decomposable operator (i.e. decomposable entanglement witness). The following conditions are equivalent*

1. *W is exposed,*
2. *$W = (\mathbb{1}_2 \otimes \mathrm{t})Q$, and Q is Schmidt rank 2 projector,*
3. *There are $3m$ linearly independent vectors $|\overline{x}\rangle\langle\overline{x}| \otimes |h\rangle \in \mathcal{B}_+(\mathbb{C}^2 \otimes \mathbb{C}^n)$ such that*

$$\langle x \otimes h|W|x \otimes h\rangle = 0 \ .$$

In the forthcoming paper we use the *strong spanning property* to analyze exposed positive indecomposable maps.

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