Wigner function for damped systems

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Abstract

Both classical and quantum damped systems give rise to complex spectra and corresponding resonant states. We investigate how resonant states, which do not belong to the Hilbert space, fit the phase space formulation of quantum mechanics. It turns out that one may construct out of a pair of resonant states an analog of the stationary Wigner function.

1. Introduction

Recently, it was observed by Kossakowski [1], [2] that quantized simple damped systems (e.g. damped harmonic oscillator) give rise to discrete complex spectra. The corresponding eigenvectors may be interpreted as resonant states. Such states play important role in quantum mechanics and it is widely believed that they are responsible for the irreversible dynamics of physical systems (see e.g. [3]). In a recent paper [4] it was shown that the damping behaviour in a classical system may be also interpreted as appearance of resonant states for the corresponding Koopman operator. Consider a classical Hamiltonian system defined by the Hamilton function $H : P \to \mathbb{R}$. Assuming that a Hamiltonian flow is complete one introduces a self-adjoint operator L_H in the Hilbert space $L^2(P, d\mu)$, where $d\mu$ denotes the standard Liouville measure on P. This so called Koopman operator is defined by

$$L_H\, f := i\{f, H\} \ , \tag{1.1}$$

where $\{\ ,\ \}$ denotes the Poisson bracket in the algebra of classical observables $C^\infty(P)$. Studying simple examples of classical damped systems we showed [4] that corresponding Koopman operators have discrete complex spectra. Clearly, the generalized eigenvectors do not belong to the Hilbert space $L^2(P, d\mu)$ and the appropriate mathematical language to deal with this situation is the rigged Hilbert space or the Gelfand triplet (cf. e.g. [5],[6]), that is, any generalized eigenvector belongs to a dual space D^* where D is an appropriate space of test functions in $L^2(P, d\mu)$.

In the present paper we shall study the phase space formulation of quantum damped systems. Clearly, this formulation, called also the deformation quantization, is perfectly equivalent to the standard Hilbert space approach. However, as we already mentioned, resonant states lie outside the Hilbert space, and hence, it would be interesting to find how they fit phase space approach. Any vector $\psi \in \mathcal{H}$ gives rise to a Wigner function W_ψ on a classical phase space P. As was shown already by Wigner [7] this function is real and produces marginal probability densities $\int W_\psi(x, p)dx$ and $\int W_\psi(x, p)dp$. The classical limit of W_ψ reproduces a classical probability distribution on P. Moreover, if ψ is an eigenvalue of the Hamilton operator $\widehat{H}$, i.e. $\widehat{H}\psi = E\psi$, then the corresponding Wigner function W_ψ satisfies the following eigenvalue problem:

$$H \star W_\psi = W_\psi \star H = EW_\psi \ , \tag{1.2}$$

where H is a classical Hamiltonian on P and $f \star g$ denotes "quantum deformation" of a usual commutative product of functions $f \cdot g$ (see the next section).

It turns out that resonant states appear always in pairs:

$$\widehat{H}\psi^{\pm} = E^{\pm}\psi^{\pm} \ , \tag{1.3}$$

such that

$$E^{-} = \overline{E^{+}} \ . \tag{1.4}$$

We show that each pair gives rise to a pair of function $F^{\pm}$ on P satisfying

$$H \star F^{\pm} = F^{\pm} \star H = E^{\pm}F^{\pm} \ , \tag{1.5}$$

with

$$F^{-} = \overline{F^{+}} \quad \text{and} \quad E^{-} = \overline{F^{+}} \ . \tag{1.6}$$

Moreover, if $\psi^{+} = \psi^{-} = \psi \in \mathcal{H}$, then $E^{+} = E^{-} \in \mathbb{R}$ and $F^{+} = F^{-} = W_{\psi}$, that is, $F^{\pm}$ may be considered as a generalization of Wigner function for resonant states. Functions $F^{\pm}$ do indeed satisfy basic properties of Wigner function: they may be normalized so that $\int_P F^{\pm}\, d\mu = 1$, and they give rise to marginal probability distributions: if $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ are canonical coordinates on P, then

$$\pi^{\pm}_{\mathbf{x}}(x_1, \ldots, x_n) := \int F^{\pm}(x_1, \ldots, x_n, p_1, \ldots, p_n)\, dp_1 \ldots dp_n \ , \tag{1.7}$$

$$\pi^{\pm}_{\mathbf{p}}(p_1, \ldots, p_n) := \int F^{\pm}(x_1, \ldots, x_n, p_1, \ldots, p_n)\, dx_1 \ldots dx_n \ , \tag{1.8}$$

are classical probability distributions on P. Actually, in the examples considered in this paper one finds

$$\pi^{\pm}_{\mathbf{x}}(x_1, \ldots, x_n) = \delta(x_1) \ldots \delta(x_n) \ , \tag{1.9}$$

$$\pi^{\pm}_{\mathbf{p}}(p_1, \ldots, p_n) = \delta(p_1) \ldots \delta(p_n) \ . \tag{1.10}$$

It seems that the above result does contradict Heisenberg uncertainty relations. Note, however, that resonant states do not belong to the Hilbert space and hence the probabilistic interpretation is not clear.

In the next section we recall phase space approach to quantum mechanics. Section 3 shows how this approach works for the harmonic oscillator. Following sections discuss damped systems: a toy model of a damped motion $\dot{x} = -\gamma x$, and damped harmonic oscillator. We end up with some conclusions.

2. Phase space formulation of quantum mechanics

Deformation quantization consists in replacing a commutative algebra of functions $C^{\infty}(P)$ over a classical phase space P by a noncommutative algebra $(C^{\infty}(P), \star)$. For simplicity let us assume that $P = \mathbb{R}^{2N}$. The $\star$-product operation

$$\star \ : \ C^{\infty}(P) \ \times \ C^{\infty}(P) \ \longrightarrow \ C^{\infty}(P)$$

is defined by:

$$f \star g := f \, \exp\left[\frac{i\hbar}{2}\overleftrightarrow{\Lambda}\right] g \ , \tag{2.1}$$

where Λ denotes a bidifferential operator

$$f \overleftrightarrow{\Lambda} g := \{f, g\} \ . \tag{2.2}$$

The above structure was introduced long ago by Groenewold [8] and later on it was used by Moyal [9] to construct the so called phase-space formulation of quantum mechanics (see e.g. recent review by Zachos [10]). The equivalence of the above approach to the standard Hilbert space one is based on the well known Weyl correspondence: if $\widehat{A}$ is self-adjoint operator on $\mathcal{H} = L^2(\mathbb{R}^N, d\mathbf{x})$, then one defines the symbol $A(\mathbf{u},\mathbf{v})$ of the operator $\widehat{A}$ by:

$$A(\mathbf{u},\mathbf{v}) := \mathrm{Tr}\left(\widehat{A}\, e^{(-i/\hbar)(\mathbf{u}\widehat{\mathbf{p}}+\mathbf{v}\widehat{\mathbf{x}})}\right) , \tag{2.3}$$

where $\widehat{\mathbf{x}} = (\widehat{x}_1,\ldots,\widehat{x}_N)$ and $\widehat{\mathbf{p}} = (\widehat{p}_1,\ldots,\widehat{p}_N)$ are standard position and momentum operators in $L^2(\mathbb{R}^N, d\mathbf{x})$. Conversely, given a symbol $A(\mathbf{u},\mathbf{v})$ one construct a corresponding operator:

$$\widehat{A} = \int d\mathbf{u} \int d\mathbf{v}\, A(\mathbf{u},\mathbf{v})\, e^{(-i/\hbar)(\mathbf{u}\widehat{\mathbf{p}}+\mathbf{v}\widehat{\mathbf{x}})} . \tag{2.4}$$

Now, if $\widehat{C} = \widehat{A}\widehat{B}$, then

$$C = A \star B , \tag{2.5}$$

where A, B, C are symbols of $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$, respectively. In this approach the von Neumann equation for the density operator $\widehat{\rho}$

$$i\hbar\, \partial_t \widehat{\rho} = [\widehat{H}, \widehat{\rho}] , \tag{2.6}$$

is replaced by the Moyal equation for the corresponding Wigner function W, i.e. symbol of $\widehat{\rho}$:

$$i\hbar\, \partial_t W = \{H, W\}_{\mathrm{M}} , \tag{2.7}$$

where the Moyal brackets is given by:

$$\{H, W\}_{\mathrm{M}} := H \star W - W \star H . \tag{2.8}$$

Using (2.3) it is easy to show that Wigner function corresponding to $\widehat{\rho}$ is given by [7]

$$W(\mathbf{x},\mathbf{p}) := \frac{1}{(2\pi)^N} \int d\mathbf{y}\, e^{-i\mathbf{p}\mathbf{y}} \left\langle \mathbf{x} - \frac{\hbar}{2}\mathbf{y} \right| \widehat{\rho} \left| \mathbf{x} + \frac{\hbar}{2}\mathbf{y} \right\rangle , \tag{2.9}$$

where $|\mathbf{x}\,\rangle$ is a generalized eigenvector of $\widehat{\mathbf{x}}$, i.e. $\widehat{\mathbf{x}}|\mathbf{x}\,\rangle = \mathbf{x}|\mathbf{x}\,\rangle$. This way quantum mechanics may be formulated entirely in terms of objects living on a classical phase space P. The very definition of the $\star$-product implies

$$f \star g - g \star f = i\hbar\{f, g\} + O(\hbar^2) , \tag{2.10}$$

and hence in the classical limit $\hbar \to 0$ the Moyal equation (2.7) reproduces the Liouville equation

$$\partial_t f_{\mathrm{cl}} = \{H, f_{\mathrm{cl}}\} , \tag{2.11}$$

for the classical density probability f_{cl} on P. Now, the unitary evolution of $\widehat{\rho}$:

$$\widehat{\rho}(t) = U(t)\,\widehat{\rho}\,U^{-1}(t) , \tag{2.12}$$

with $U(t) = \exp(-(i/\hbar)t\widehat{H})$, is replaced by the following formula for $W(t)$:

$$W(t) = U_\star(t) \star W \star U_\star^{-1}(t) , \tag{2.13}$$

where the so called $\star$–exponential $U_\star$ is defined by [12]:

$$\begin{aligned} U_\star(t) &= \mathrm{Exp}\,(-(it/\hbar)H) \\ &:= 1 + (-it/\hbar)H + \frac{1}{2!}(-it/\hbar)^2 H \star H + \frac{1}{3!}(-it/\hbar)^3 H \star H \star H + \ldots \end{aligned} \tag{2.14}$$

In the classical limit

$$W(t) \;\longrightarrow\; f_{\rm cl}(t) = e^{-itL_H}\, f_{\rm cl}\ , \tag{2.15}$$

that is,

$$f_{\rm cl}(\mathbf{x},\mathbf{p},t) = f_{\rm cl}(\mathbf{x}(-t),\mathbf{p}(-t),0)\ , \tag{2.16}$$

where $\mathbf{x}(t)$ and $\mathbf{p}(t)$ stand for the classical evolution of $\mathbf{x}$ and $\mathbf{p}$, respectively. Actually, using (2.1), one finds that the quantum evolution of $\mathbf{x}$ and $\mathbf{p}$ (in the Heisenberg picture)

$$\dot{\mathbf{x}} \;=\; \frac{\mathbf{x}\star H - H\star\mathbf{x}}{i\hbar} = \{\mathbf{x},H\}\ , \tag{2.17}$$

$$\dot{\mathbf{p}} \;=\; \frac{\mathbf{p}\star H - H\star\mathbf{p}}{i\hbar} = \{\mathbf{p},H\}\ , \tag{2.18}$$

is the same as the classical one.

Finally, let us turn to the energy spectrum. In the standard approach one solves for the standard eigenvalue problem for the quantum Hamiltonian $\widehat{H}$:

$$\widehat{H}\psi = E\psi\ . \tag{2.19}$$

It is easy to see that the corresponding Wigner function W satisfies:

$$W_\psi \star H = H\star W_\psi = EW_\psi\ . \tag{2.20}$$

Actually, one may prove (see e.g. [13]) that any real solution W of (2.20) corresponds to a Wigner function for ψ satisfying (2.19). Moreover, if ψ_n define an orthonormal basis in $\mathcal{H}$, then corresponding Wigner functions W_n satisfy:

$$W_n\star W_m = \frac{1}{(2\pi\hbar)^N}\ \delta_{nm}W_n\ , \tag{2.21}$$

and hence one obtains the following resolution of identity on P:

$$\sum_n W_n = \frac{1}{(2\pi\hbar)^N}\ , \tag{2.22}$$

which is phase space analog of the Hilbert space formula $\sum_n P_n = \mathbb{1}$, where P_n is a 1-dimensional projector onto the eigenspace generated by ψ_n. For more properties of Wigner function see e.g. the review article [11].

3. Harmonic oscillator

To get the feeling how this approach works in practice let us consider 1-dimensional harmonic oscillator described by the following Hamiltonian:

$$H_{\rm ho}(x,p) = \frac{\omega}{2}(p^2+x^2)\ . \tag{3.1}$$

Now, let us study the corresponding eigenvalue problem (2.20). Equation $H\star W = EW$ gives:

$$\left[x^2+p^2-\frac{\hbar^2}{4}(\partial_x^2+\partial_p^2)+\frac{i\hbar}{2}(x\partial_p - p\partial_x)-\frac{2E}{\omega}\right]W = 0\ , \tag{3.2}$$

whereas $W\star H = EW$:

$$\left[x^2+p^2-\frac{\hbar^2}{4}(\partial_x^2+\partial_p^2)-\frac{i\hbar}{2}(x\partial_p - p\partial_x)-\frac{2E}{\omega}\right]W = 0\ . \tag{3.3}$$

Therefore,

$$(x\partial_p - p\partial_x)\, W = 0 \ , \tag{3.4}$$

which means that W is a zero-mode of the Koopman operator $L_{H_{\rm ho}}$ [4]. Taking into account (3.2) and (3.3) we obtain:

$$\left[x^2 + p^2 - \frac{\hbar^2}{4}(\partial_x^2 + \partial_p^2) - \frac{2E}{\omega}\right] W = 0 \ . \tag{3.5}$$

Introducing a new variable ξ:

$$\xi := \frac{2}{\hbar}\,(x^2 + p^2) \tag{3.6}$$

equation (3.5) may be rewritten as follows:

$$\left[\frac{\xi}{4} - \xi\partial_\xi^2 - \partial_\xi - \frac{E}{\omega}\right] W(\xi) = 0 \ . \tag{3.7}$$

Finally, defining $L = L(\xi)$ by

$$W(\xi) =: e^{-\xi/2}\, L(\xi) \ , \tag{3.8}$$

equation (3.7) implies:

$$\left[\xi\partial_\xi^2 + (1-\xi)\partial_\xi + \frac{E}{\hbar\omega} - \frac{1}{2}\right] L(\xi) = 0 \ , \tag{3.9}$$

which is the defining equation of Laguerre's polynomials:

$$L_n(\xi) = \frac{1}{n!}\, e^\xi \partial_\xi (e^{-\xi}\, \xi) \ , \tag{3.10}$$

for $n = E/\hbar\omega - 1/2 = 0, 1, \ldots$. This way one recovers well known oscillator spectrum. The corresponding Wigner functions W_n read

$$W_n = \frac{(-1)^n}{\pi\hbar}\, e^{-\xi/2}\, L_n(\xi) \ . \tag{3.11}$$

The reader will easily check that W_n defined in (3.11) do indeed satisfy formula (2.21).

It is well known that only W_0 which is given by the Gaussian distribution

$$W_0 = \frac{1}{\pi\hbar}\, e^{-\xi/2} = \frac{1}{\pi\hbar}\, e^{-(x^2+p^2)/\hbar} \ , \tag{3.12}$$

defines a probability distribution on P. However, in the classical limit $\hbar \longrightarrow 0$ all Wigner functions W_n tend to well defined classical probability distributions. For example

$$W_0(x,p) \ \longrightarrow \ \delta(x)\delta(p) \ . \tag{3.13}$$

There is an alternative way to find the eigen-Wigner functions W_n. One introduces phase-space analogs of creation and annihilation operators:

$$a = \frac{x + ip}{\sqrt{2\hbar}} \ , \qquad a^* = \frac{x - ip}{\sqrt{2\hbar}} \ , \tag{3.14}$$

satisfying standard commutation relation:

$$\{a, a^*\}_{\rm M} = a \star a^* - a^* \star a = 1 \ . \tag{3.15}$$

It is easy to rewrite the formula for the $\star$-product (2.1) in terms of a and a^*:

$$f \star g = f\, e^{\frac{1}{2}\left(\overleftarrow{\partial}\overrightarrow{\partial}^* - \overleftarrow{\partial}^*\overrightarrow{\partial}\right)}\, g\ , \tag{3.16}$$

where $\partial = \partial/\partial a$ and $\partial^* = \partial/\partial a^*$. Oscillator Hamiltonian (3.1) takes in the new variables the following form:

$$H_{\rm ho} = \hbar\omega \left(a^* \star a + \frac{1}{2} \right)\ . \tag{3.17}$$

Now, let us define W_0 as a $\star$-Fock vacuum, that is,

$$a \star W_0 = W_0 \star a^* = 0\ , \tag{3.18}$$

and the corresponding excited states:

$$W_n \ \propto\ a^{*n} \star W_0 \star a^n\ , \tag{3.19}$$

Noting that $\xi = 4|a|^2$ it is easy to check that W_n defined in (3.19) agrees with the formula (3.11).

Finally, let us turn to the time evolution defined in (2.13). The corresponding $\star$-exponential (2.14) was found in [12] and is given by:

$$\mathrm{Exp}(-(i/\hbar)tH_{\rm ho}) = \frac{1}{\cos(t/2)}\ \exp\left[-2(i/\hbar)\tan(t/2)H_{\rm ho}\right]\ . \tag{3.20}$$

Actually, for the harmonic oscillator the quantum evolution has the same from as the classical one. It is evident from (3.2) and (3.3) that

$$\{H_{\rm ho}, W\}_{\rm M} = i\hbar\{H_{\rm ho}, W\}\ , \tag{3.21}$$

and hence, the Moyal equation (2.7) is the same as the Liouville equation (2.11). Therefore, due to (2.16)

$$W(x,p,t) = W(x(-t), p(-t), 0)\ . \tag{3.22}$$

4. Toy model of damped system

Now, we apply this scheme to the simple damped system described by the following equation:

$$\dot{x} = -\gamma x\ , \tag{4.1}$$

where $\gamma > 0$ is a damping constant. Clearly, this system is not Hamiltonian. However, following [14] we may lift an arbitrary dynamics on a configuration space Q

$$\dot{x} = \mathbf{X}(x)\ , \tag{4.2}$$

where $\mathbf{X}$ is a vector field on Q, to the Hamiltonian dynamics on the corresponding phase space $P = T^*Q$. We define the corresponding Hamiltonian

$$H\ :\ P\ \longrightarrow\ \mathbb{R}\ , \tag{4.3}$$

by

$$H(\alpha_x) := \alpha_x(\mathbf{X}(x))\ , \tag{4.4}$$

for $\alpha_x \in T_x^*Q$. Using canonical coordinates $(x_1,\ldots,x_N,p_1,\ldots,p_N)$ on P we may rewrite a formula for H in a more familiar way:

$$H(x,p) = \sum_{k=1}^{N} p_k\, X_k(x)\ . \tag{4.5}$$

The corresponding Hamilton equations read as follows:

$$\dot{x}_k \;=\; \{x_k, H\} = X_k(x)\ , \tag{4.6}$$

$$\dot{p}_k \;=\; \{p_k, H\} = -\sum_{l=1}^{N} p_l \frac{\partial X_l(x)}{\partial x_k}\ , \tag{4.7}$$

for $k = 1,\ldots,N$. In the above formulae $\{\ ,\ \}$ denotes the canonical Poisson bracket on T^*Q:

$$\{F,G\} = \sum_{k=1}^{N}\left(\frac{\partial F}{\partial x_k}\frac{\partial G}{\partial p_k} - \frac{\partial G}{\partial x_k}\frac{\partial F}{\partial p_k}\right)\ . \tag{4.8}$$

Clearly, the formulae (4.6) reproduce our initial dynamical system (4.2) on Q.

Now, applying the above procedure to (4.1) one obtains the Hamiltonian system on $\mathbb{R}^2$ with the Hamiltonian given by:

$$H_{\rm d}(x,p) = -\gamma x p\ . \tag{4.9}$$

This system was analyzed in [4] where both *classical spectrum* of the corresponding Koopman operator $L_{H_{\rm d}}$ and quantum spectrum of

$$\widehat{H}_{\rm d} = -\frac{\gamma}{2}\,(\widehat{x}\widehat{p} + \widehat{p}\widehat{x})\ , \tag{4.10}$$

were found:

$${\rm Spec}(L_{H_{\rm d}}) = \{\, i\gamma n \mid n \in \mathbb{Z}\,\}\ , \tag{4.11}$$

and

$${\rm Spec}(\widehat{H}_{\rm d}) = \left\{\, i\hbar\gamma\left(n + \frac{1}{2}\right)\ \Big|\ n \in \mathbb{Z}\,\right\}\ . \tag{4.12}$$

Both spectra are discrete and purely imaginary. It should be stressed that both $L_{H_{\rm d}}$ and $\widehat{H}_{\rm d}$ are self-adjoint operators on the corresponding Hilbert spaces $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R})$, respectively. The corresponding eigenvectors, which obviously do not belong to the Hilbert space, are usually called resonant states (see e.g. [3]). It was found in [4], [2] that for

$$\psi_n^+(x) := x^n \quad\text{ and }\quad \psi_n^-(x) := (-i\hbar)^n \delta^{(n)}(x)\ ,\quad n = 0,1,2,\ldots\ , \tag{4.13}$$

one has:

$$\widehat{H}\psi_n^{\pm} = \pm i\hbar\gamma\left(n + \frac{1}{2}\right)\psi_n^{\pm}\ . \tag{4.14}$$

Evidently, these states living outside the Hilbert space $L^2(\mathbb{R})$ can not be used to construct stationary Wigner functions. Indeed, defining

$$W_n^{\pm}(x,p)\ \propto\ \int dy e^{-ipy}\overline{\psi_n^{\pm}}(x - \frac{\hbar}{2}y)\,\psi_n^{\pm}(x + \frac{\hbar}{2}y)\ , \tag{4.15}$$

one obtains

$$W_n^{\pm}(x,p,t) = e^{\pm(2n+1)\gamma t}\, W_n^{\pm}(x,p,0)\ , \tag{4.16}$$

which shows that $W_n^{\pm}$ are non-stationary. To find the analogs of stationary Wigner functions consider the eigenvalue problem:

$$H_{\rm d} \star F = EF \qquad \text{and} \qquad F \star H_{\rm d} = EF \ . \tag{4.17}$$

Equation $H_{\rm d} \star F = EF$ gives:

$$\left[xp + \frac{\hbar^2}{4}\partial^2_{xp} + \frac{i\hbar}{2}(p\partial_p - x\partial_x) - \frac{E}{\gamma} \right] F = 0 \ , \tag{4.18}$$

whereas $F \star H_{\rm d} = EF$:

$$\left[xp + \frac{\hbar^2}{4}\partial^2_{px} - \frac{i\hbar}{2}(p\partial_p - x\partial_x) - \frac{E}{\gamma} \right] F = 0 \ . \tag{4.19}$$

Therefore, in analogy to (3.4) one finds

$$(p\partial_p - x\partial_x)\, F = 0 \ , \tag{4.20}$$

which means that F is a zero-mode of the corresponding Koopman operator $L_{H_{\rm d}}$. Taking into account (4.18) and (4.19) we obtain:

$$\left[xp + \frac{\hbar^2}{4}\partial^2_{xp} + \frac{E}{\gamma} \right] F = 0 \ . \tag{4.21}$$

Introducing a new variable:

$$\eta := \frac{4xp}{i\hbar} \ , \tag{4.22}$$

one may rewrite (4.21) as follows:

$$\left(\frac{\eta}{4} - \eta\partial^2_\eta - \partial_\eta - \frac{iE}{\hbar\gamma} \right) F(\eta) = 0 \ . \tag{4.23}$$

Finally, defining $L(\eta)$:

$$F(\eta) = e^{-\eta/2} L(\eta) \ , \tag{4.24}$$

one finds:

$$\left[\eta\partial^2_\eta + (1-\eta)\partial_\eta - \left(\frac{iE}{\hbar\gamma} - \frac{1}{2} \right) \right] L(\eta) = 0 \ , \tag{4.25}$$

which is defining equation for Laguerre polynomials (cf. (3.9)). Hence we may define

$$F_n^+ = C_n\, e^{-\eta/2} L_n(\eta) \ , \qquad n = 0, 1, 2, \ldots \ , \tag{4.26}$$

where nth polynomial L_n is given by (3.10) and C_n is a normalization constant. The above formula for F_n^+ is an analog of (3.11) for the oscillator Wigner functions. Moreover, it follows from (4.25) that the spectrum is given by:

$$E_n = i\hbar\gamma \left(n + \frac{1}{2} \right) \ , \qquad n = 0, 1, 2, \ldots \ . \tag{4.27}$$

Now, it is easy to check that functions F_n^- defined by:

$$F_n^- := \overline{F_n^+} \ , \tag{4.28}$$

satisfies

$$H_{\rm d} \star F_n^- = F_n^- \star H_{\rm d} = -E_n F_n^- \ . \tag{4.29}$$

It follows immediately from the following property:

$$\overline{f \star g} = \overline{g} \star \overline{f} \ , \tag{4.30}$$

which may be easily proved using the definition of the $\star$-product (2.1).

Now, let us study the basic properties of eigen-functions $F_n^\pm$ and compare these with those of oscillator Wigner functions W_n. Clearly, $F_n^\pm$ contrary to W_n are not real. Observe, that taking a constant C_n in (4.26) according to

$$C_n = \frac{(-1)^n}{\pi\hbar} \ , \tag{4.31}$$

i.e. like in (3.11), one may prove that

$$\int F_n^+(x,p)\, dxdp = \int F_n^-(x,p)\, dxdp = 1 \ , \tag{4.32}$$

in perfect analogy to W_n. Moreover, $F_n^\pm$ give rise to the following marginal probability distributions:

$$\int F_n^\pm(x,p)\, dx \ = \ \delta(p) \ , \tag{4.33}$$

$$\int F_n^\pm(x,p)\, dp \ = \ \delta(x) \ . \tag{4.34}$$

This property seems to violate the Heisenberg uncertainty principle – the particle is localized both in x and p variables. Clearly, we lose the probabilistic interpretation of $F_n^\pm$ since the corresponding eigenvectors $\psi_n^\pm$ (4.13) do not belong to the Hilbert space $L^2(\mathbb{R})$. Interestingly, $F_n^\pm$ satisfy the following condition:

$$F_n^\pm \star F_m^\pm = \frac{1}{2\pi\hbar}\delta_{nm}\ F_n^\pm \ , \tag{4.35}$$

in perfect analogy to (2.21). Therefore, one obtains the corresponding resolution of identity

$$\sum_n F_n^\pm = 2\sum_n \mathrm{Re}\, F_n^+ = \frac{1}{2\pi\hbar} \ . \tag{4.36}$$

Finally, it would be interesting to find relation between the resonant states $\psi_n^\pm$ defined in (4.13) and $F_n^\pm$. Using some simple algebraic manipulations it is easy to show that

$$F_n^+(x,p) = C_n \int dy\, e^{-ipy}\overline{\psi_n^+}(x - \frac{\hbar}{2}y)\, \psi_n^-(x + \frac{\hbar}{2}y) \ , \tag{4.37}$$

and

$$\begin{aligned} F_n^-(x,p) \ &= \ \overline{F_n^+}(x,p) = C_n \int dy\, e^{-ipy}\psi_n^+(x - \frac{\hbar}{2}y)\, \overline{\psi_n^-}(x + \frac{\hbar}{2}y) \\ &= \ C_n \int dy\, e^{-ipy}\overline{\psi_n^-}(x - \frac{\hbar}{2}y)\, \psi_n^+(x + \frac{\hbar}{2}y) \ , \end{aligned} \tag{4.38}$$

with C_n defined in (4.31). Hence, each $F_n^\pm$ is built out of ψ_n^+ and ψ_n^-. Note, that these eigenvectors correspond to E_n and $\overline{E}_n$, respectively. Clearly, if ψ is a proper eigenvectors corresponding to a real eigenvalue E, then using the above prescription for F one recovers the Wigner function corresponding to ψ. Resonant states comes always in pairs and two members of each pair are needed to construct F.

Let us observe, that stationary functions $F_n^\pm$ may be defined in a more transparent way. Rewriting the Hamiltonian (4.9) as follows:

$$H_{\rm d}(x,p) = -\frac{\gamma}{2}(x\star p + p\star x)\ , \tag{4.39}$$

let us define F_0^+ to be a normalized function satisfying the following conditions:

$$p\star F_0^+ = 0 \qquad \text{and} \qquad F_0^+\star x = 0\ , \tag{4.40}$$

which are solved by:

$$F_0^+(x,p) = \frac{1}{\pi\hbar}\, e^{-2ixp/\hbar}\ . \tag{4.41}$$

Having the "+ ground state" F_0^+ one defines "+ excited states" by:

$$F_n^+(x,p)\ \propto\ x^n\star F_0^+(x,p)\star p^n\ . \tag{4.42}$$

Analogously, let us define F_0^- to be a "− ground state" satisfying:

$$x\star F_0^- = 0 \qquad \text{and} \qquad F_0^-\star p = 0\ . \tag{4.43}$$

One finds

$$F_0^-(x,p) = \frac{1}{\pi\hbar}\, e^{2ixp/\hbar} = \overline{F_0^+}(x,p)\ . \tag{4.44}$$

The corresponding "− excited states" read:

$$F_n^-(x,p)\ \propto\ p^n\star F_0^-(x,p)\star x^n\ . \tag{4.45}$$

Using canonical commutation relation

$$x\star p - p\star x = i\hbar\ , \tag{4.46}$$

one easily finds that $F_n^\pm$ do satisfy:

$$H_{\rm d}\star F_n^\pm = F_n^\pm\star H_{\rm d} = \pm i\hbar\gamma\left(n+\frac{1}{2}\right)\ . \tag{4.47}$$

5. Harmonic oscillator vs. damped system

Comparing the spectra of harmonic oscillator and damped system considered in the previous section one finds striking similarity, that is, they are related by the following relation:

$$\omega = \pm i\gamma\ . \tag{5.1}$$

Note, that performing the following canonical transformation:

$$x = \frac{1}{\sqrt{2}}(X+P) \qquad \text{and} \qquad p = \frac{1}{\sqrt{2}}(X-P)\ , \tag{5.2}$$

one obtains

$$H_{\rm d} = -\gamma xp = \frac{\gamma}{2}(P^2 - X^2)\ , \tag{5.3}$$

i.e. in the new variables (X,P), $H_{\rm d}$ corresponds formally to the harmonic oscillator with $\omega = \pm i\gamma$. This correspondence may be easily seen by observing that both Hamiltonians, i.e. (P^2+X^2) and (P^2-X^2) are related by the following $\star$-exponential:

$$V_\lambda := {\rm Exp}(\lambda\, XP/\hbar) = 1 + \frac{\lambda}{\hbar}XP + \frac{\lambda^2}{2\hbar^2}XP\star XP + \dots\ , \tag{5.4}$$

with $\lambda \in \mathbb{R}$. Indeed, one may show that

$$V_\lambda \star X \star V_{-\lambda} = e^{-i\lambda} X \qquad \text{and} \qquad V_\lambda \star P \star V_{-\lambda} = e^{i\lambda} P \ . \tag{5.5}$$

The above formulae imply:

$$V_\lambda \star (P^2 - X^2) \star V_{-\lambda} = e^{2i\lambda}(P^2 - e^{-4i\lambda} X^2) \ , \tag{5.6}$$

and hence, for $\lambda = \pm\pi/4$, one obtains:

$$V_{\pm\pi/4} \star \left[\frac{\gamma}{2}(P^2 - X^2)\right] \star V_{\mp\pi/4} = \pm\frac{i\gamma}{2}(P^2 + X^2) \ , \tag{5.7}$$

i.e. both systems are related by a *complex scaling* $V_{\pm\pi/4}$. Therefore, it should be clear that the corresponding eigen-functions W_n and $F_n^\pm$ are also related by $V_{\pm\pi/4}$. Let us denote

$$H_{\rm ho}^\pm = \pm\frac{i\gamma}{2}(P^2 + X^2) \ . \tag{5.8}$$

Now, if W_n is an oscillator Wigner function satisfying:

$$H_{\rm ho}^\pm \star W_n = W_n \star H_{\rm ho}^\pm = \pm E_n W_n \ , \tag{5.9}$$

with E_n given by (4.27), then $F_n^\pm$ defined by:

$$F_n^\pm := V_{\mp\pi/4} \star W_n \star V_{\pm\pi/4} \ , \tag{5.10}$$

satisfy the corresponding eigen-problem for the damped system:

$$H_{\rm d} \star F_n^\pm = F_n^\pm \star H_{\rm d} = \pm E_n F_n^\pm \ . \tag{5.11}$$

Moreover, it follows from (5.10) that

$$F_n^- = \overline{F_n^+} \ , \tag{5.12}$$

provided W_n is real, and

$$\int F_n^\pm \, dxdp = \int (V_{\pm\pi/4} \star V_{\mp\pi/4}) W_n \, dxdp = \int W_n \, dxdp \ , \tag{5.13}$$

i.e. $F_n^\pm$ are normalized on $\mathbb{R}^2$.

Finally, let us observe that introducing on $\mathbb{R}^2$ polar coordinates (r,φ) the corresponding Koopman operator reads:

$$L_{H_{\rm ho}} = i\omega\partial_\varphi \ , \tag{5.14}$$

and hence, oscillator Wigner functions are $SO(2)$ invariant since $L_{H_{\rm ho}} W_n = 0$. On the other hand using hyperbolic coordinates (s,χ); $P = s\cosh\chi$, $X = s\sinh\chi$, we obtain

$$L_{H_{\rm d}} = i\gamma\partial_\chi \ , \tag{5.15}$$

and hence $F_n^\pm$ are $SO(1,1)$ invariant. Moreover, this observation implies that the corresponding $\star$-exponential $U_*(t)$ has for the damped system following form:

$$\mathrm{Exp}(-(i/\hbar)tH_{\rm d}) = \frac{1}{\cosh(t/2)} \exp\left[-2(i/\hbar)\tanh(t/2)H_{\rm d}\right] \ , \tag{5.16}$$

which follows from (3.20). Note that

$$\{H_{\rm d}, F\}_{\rm M} = i\hbar\,\{H_{\rm d}, F\} \ , \tag{5.17}$$

and hence, like for the harmonic oscillator, quantum and classical evolution are given by the same formulae.

6. Damped harmonic oscillator

Consider now a damped harmonic oscillator described by the following equation of motion:

$$\ddot{x} + 2\gamma\dot{x} + \kappa x = 0 \ . \tag{6.1}$$

As is well known this system plays a prominent role in various branches of physics, especially in quantum optics. The above 2nd order equation may be rewritten as a dynamical system on $\mathbb{R}^2$

$$\begin{aligned} \dot{x}_1 &= -\gamma x_1 + \omega x_2 \ , && (6.2) \\ \dot{x}_2 &= -\gamma x_2 - \omega x_1 \ , && (6.3) \end{aligned}$$

with $\omega = \sqrt{\kappa - \gamma^2}$. Clearly this system is not Hamiltonian if $\gamma \neq 0$. However, applying the procedure of [14] one arrives at the following Hamiltonian system on $\mathbb{R}^4$:

$$\begin{aligned} \dot{x}_1 &= \{x_1, H\} = -\gamma x_1 + \omega x_2 \ , && (6.4) \\ \dot{x}_2 &= \{x_2, H\} = -\omega x_1 - \gamma x_2 \ , && (6.5) \\ \dot{p}_1 &= \{p_1, H\} = +\gamma p_1 + \omega p_2 \ , && (6.6) \\ \dot{p}_2 &= \{p_1, H\} = -\omega p_1 + \gamma p_2 \ , && (6.7) \end{aligned}$$

where the corresponding Hamiltonian function is given by:

$$H_{\rm dho}(x,p) = \omega(p_1x_2 - p_2x_1) - \gamma(p_1x_1 + p_2x_2) \ . \tag{6.8}$$

Let us observe that the above Hamiltonian may be rewritten as follows:

$$H_{\rm dho}(x,p) = \omega(p_1 \star x_2 - p_2 \star x_1) - \frac{\gamma}{2}(p_1 \star x_1 + x_1 \star p_1 + p_2 \star x_2 + x_2 \star p_2) \ . \tag{6.9}$$

Now, let us introduce a new set of variables:

$$\begin{aligned} a_1 &= \frac{x_1 + ix_2}{\sqrt{2\hbar}} \ , \qquad & a_1^* &= \frac{x_1 - ix_2}{\sqrt{2\hbar}} \ , && (6.10) \\ a_2 &= \frac{ip_1 - p_2}{\sqrt{2\hbar}} \ , \qquad & a_2^* &= \frac{-ip_1 - p_2}{\sqrt{2\hbar}} \ , && (6.11) \end{aligned}$$

satisfying the following commutation relations:

$$\{a_1, a_2\}_{\rm M} = \{a_1, a_1^*\}_{\rm M} = \{a_2, a_2^*\}_{\rm M} = 0 \ , \tag{6.12}$$

$$\{a_1, a_2^*\}_{\rm M} = \{a_2, a_1^*\}_{\rm M} = 1 \ . \tag{6.13}$$

Hamiltonian (6.9) takes in new variables the following form:

$$\begin{aligned} H_{\rm dho} &= \hbar\,(\alpha\, a_2^* \star a_1 + \overline{\alpha}\, a_1^* \star a_2 + \omega) \\ &= \hbar\alpha\left(a_2^* \star a_1 + \frac{1}{2}\right) + \hbar\overline{\alpha}\left(a_1^* \star a_2 + \frac{1}{2}\right) \ , && (6.14) \end{aligned}$$

where

$$\alpha = \omega - i\gamma \ . \tag{6.15}$$

Now, we are going to find the spectrum together with the corresponding eigenfunctions:

$$H_{\rm dho} \star F = F \star H_{\rm dho} = EF \ . \tag{6.16}$$

Define $F^{\pm}_{00}$ to be functions corresponding to "$\pm$ ground states", that is,

$$a_1 \star F^+_{00} = 0 \ , \qquad F^+_{00} \star a^*_2 = 0 \ , \tag{6.17}$$

and

$$a_2 \star F^-_{00} = 0 \ , \qquad F^-_{00} \star a^*_1 = 0 \ . \tag{6.18}$$

Unique normalized solutions of (6.17)–(6.18) are given by:

$$F^+_{00} = \frac{1}{(2\pi\hbar)^2}\, e^{\frac{2i}{\hbar}(x_1p_1+x_2p_2)} \ , \tag{6.19}$$

and

$$F^-_{00} = \overline{F^+_{00}} \ . \tag{6.20}$$

Moreover, defining

$$F^+_{nm} \ \propto \ (a^*_2)^n \star F^+_{00} \star a_1^m \ , \tag{6.21}$$

and

$$F^-_{nm} \ \propto \ (a^*_1)^n \star F^-_{00} \star a_2^m \ , \tag{6.22}$$

one shows

$$H_{\rm dho} \star F^+_{nm} = F^+_{nm} \star H_{\rm dho} = E_{nm} F^+_{nm} \ , \tag{6.23}$$

and

$$H_{\rm dho} \star F^-_{nm} = F^-_{nm} \star H_{\rm dho} = \overline{E_{nm}} F^-_{nm} \ , \tag{6.24}$$

with

$$E_{nm} = \hbar\alpha\left(m + \frac{1}{2}\right) - \hbar\overline{\alpha}\left(n + \frac{1}{2}\right) = \hbar\omega(m-n) - i\hbar\gamma(n+m+1) \ . \tag{6.25}$$

Let us compare the above formulation with the standard operator approach (see [1], [4]) based on the following Hamilton operator:

$$\widehat{H}_{\rm dho} = \omega(\widehat{p}_1\widehat{x}_2 - \widehat{p}_2\widehat{x}_1) - \frac{\gamma}{2}(\widehat{p}_1\widehat{x}_1 + \widehat{x}_1\widehat{p}_1 + \widehat{p}_2\widehat{x}_2 + \widehat{x}_2\widehat{p}_2) \ . \tag{6.26}$$

Following (6.10)–(6.11) we introduce $(\widehat{a}_k, \widehat{a}^*_k)$ which satisfy (6.12) and (6.13) with Moyal bracket $\{\ ,\ \}_{\rm M}$ replaced by the commutator. Now, let us introduce "$\pm$ ground states" $\varphi^{\pm}_{00}$ as the states satisfying:

$$\widehat{a}_1\varphi^+_{00} = \widehat{a}^*_1\varphi^+_{00} = 0 \ , \tag{6.27}$$

and

$$\widehat{a}_2\varphi^-_{00} = \widehat{a}^*_2\varphi^-_{00} = 0 \ . \tag{6.28}$$

Moreover, define two families of excited states:

$$\varphi^+_{nm} := \widehat{a}_2^n(\widehat{a}^*_2)^m\, \varphi^+_{00} \ , \tag{6.29}$$

and

$$\varphi^-_{nm} := \widehat{a}_1^n(\widehat{a}^*_1)^m\, \varphi^-_{00} \ . \tag{6.30}$$

It is easy to show that

$$\widehat{H}_{\rm dho}\varphi^+_{nm} = E_{nm}\varphi^+_{nm} \ , \tag{6.31}$$

and

$$\widehat{H}_{\rm dho}\varphi^-_{nm} = \overline{E_{nm}}\varphi^-_{nm} \ , \tag{6.32}$$

with E_{nm} defined in (6.25). Using standard (x_1, x_2)-representation, i.e. $\widehat{x}_k\varphi = x_k\varphi$ and $\widehat{p}_k = -i\hbar\partial_k\varphi$, one easily solves (6.27) and (6.28). Up to non-important constants one obtains:

$$\varphi^+_{00}(x_1, x_2) = \delta(x_1)\delta(x_2) \ , \tag{6.33}$$

and

$$\varphi^-_{00}(x_1, x_2) = 1 \ . \tag{6.34}$$

Clearly, neither φ^+_{00} nor φ^-_{00} belong to $L^2(\mathbb{R}^2)$. Note, that there is a striking similarity between $F^\pm_{nm}$, $\varphi^\pm_{nm}$ and $F^\pm_n$, $\varphi^\pm_n$ from section 4. Now, using a pair of resonant states $\varphi^\pm_{nm}$ one may easily show that

$$F^\pm_{nm} \ \propto \ \int d\mathbf{y}\, e^{-i\mathbf{p}\mathbf{y}}\overline{\varphi^\mp_{nm}}(\mathbf{x} - \frac{\hbar}{2}\mathbf{y})\varphi^\pm_{nm}(\mathbf{x} + \frac{\hbar}{2}\mathbf{y}) \ . \tag{6.35}$$

Moreover, using a straightforward algebra one may prove the following
Proposition. *Normalized functions* $F^\pm_{nm}$ *satisfy:*

$$\pi^\pm_{\mathrm{x}|nm}(\mathbf{x}) \ = \ \int F^\pm(\mathbf{x}, \mathbf{p})\, d\mathbf{p} = \delta(\mathbf{x}) \ , \tag{6.36}$$

$$\pi^\pm_{\mathrm{p}|nm}(\mathbf{p}) \ = \ \int F^\pm(\mathbf{x}, \mathbf{p})\, d\mathbf{x} = \delta(\mathbf{p}) \ , \tag{6.37}$$

and hence

$$\int F^\pm_{nm}(\mathbf{x}, \mathbf{p})\, d\mathbf{x}\, d\mathbf{p} = 1 \ . \tag{6.38}$$

Moreover,

$$F^\pm_{nm} \star F^\pm_{kl} = \frac{1}{(2\pi\hbar)^2}\, \delta_{nk}\delta_{ml}\, F^\pm_{nm} \ , \tag{6.39}$$

in analogy to (2.21). This implies the following resolution of identity:

$$\sum_{n,m} F^\pm_{nm} = \frac{1}{(2\pi\hbar)^2} \ , \tag{6.40}$$

in accordance to (2.22).

7. Another representation

Both oscillator Wigner functions W_n and the corresponding $F^\pm_n$ and $F^\pm_{nm}$ from sections 4. and 6. respectively, are stationary function, i.e. they commute with the corresponding Hamiltonian. In the case of W_n and $F^\pm_n$ this property follows from that fact that

$$W_n = W_n(H_{\mathrm{ho}}) \qquad \text{and} \qquad F^\pm_n = F^\pm(H_{\mathrm{d}}) \ . \tag{7.1}$$

Now, in the case of a damped harmonic oscillator the corresponding Hamiltonian (6.8) may be written as a sum

$$H_{\mathrm{dho}} = H_1 + H_2 \ , \tag{7.2}$$

where

$$H_1 = \omega\, \mathbf{p} \wedge \mathbf{x} \qquad \text{and} \qquad H_2 = -\gamma \mathbf{p} \cdot \mathbf{x} \ . \tag{7.3}$$

Clearly,

$$F^\pm_{nm} = F^\pm_{nm}(H_2) \ , \tag{7.4}$$

and the stationarity of $F^\pm_{nm}$ follows from

$$\{H_1, H_2\}_{\mathrm{M}} = 0 \ . \tag{7.5}$$

Now, we show that it is possible to construct another family G_{nm} such that

$$G_{nm} = G_{nm}(H_1) \ , \tag{7.6}$$

and G_{nm} satisfy the corresponding eigen-problem

$$H_{\rm dho} \star G_{nm} = G_{nm} \star H_{\rm dho} = \mu_{nm} G_{nm} \ . \tag{7.7}$$

Let us define G_{00} as a "ground state" satisfying

$$a_k \star G_{00} = 0 \quad \text{ and } \quad G_{00} \star a^*_k = 0 \ , \qquad k = 1,2 \ . \tag{7.8}$$

Solving (7.8) one finds:

$$G_{00} = e^{\frac{2}{\hbar}(x_1p_2 - x_2p_1)} = e^{\frac{2}{\hbar}\mathbf{x}\wedge\mathbf{p}} \ . \tag{7.9}$$

Clearly, G_{00}, contrary to $F^{\pm}_{00}$, is not integrable over $\mathbb{R}^4$. Using (6.12) and (6.13) it is easy to show that the following set of functions:

$$G_{nm} = (a^*_1)^n \star (a^*_2)^n \star G_{00} \star a_2^n \star a_1^m \ , \tag{7.10}$$

satisfy

$$H_{\rm dho} \star G_{nm} = G_{nm} \star H_{\rm dho} = \mu_{nm} G_{nm} \ , \tag{7.11}$$

with

$$\mu_{nm} = \hbar\alpha\left(n + \frac{1}{2}\right) + \hbar\overline{\alpha}\left(m + \frac{1}{2}\right) = \hbar\omega(n+m+1) - i\hbar\gamma(n-m) \ . \tag{7.12}$$

Note, that

$$G_{nm} = \overline{G_{mn}} \ , \tag{7.13}$$

and

$$\mu_{nm} = \overline{\mu_{mn}} \ . \tag{7.14}$$

Therefore, we have a natural pairing (G_{nm}, G_{mn}) in analogy to (F^+_{nm}, F^-_{nm}). Interestingly, both approaches give completely different spectra of $\widehat{H}_{\rm dho}$: E_{nm} and μ_{nm} defined in (6.25) and (7.12), respectively.

Let us compare the above formulation with the standard operator approach (see [1], [4]) based on the Hamilton operator (6.26). The commutation relations may be easily represented in the space of functions of two variables (x_1, p_2):

$$\widehat{a}_1 = \frac{1}{\sqrt{2\hbar}}\left(x_1 - \hbar\frac{\partial}{\partial p_2}\right) \ , \qquad \widehat{a}^*_1 = \frac{1}{\sqrt{2\hbar}}\left(x_1 + \hbar\frac{\partial}{\partial p_2}\right) \ , \tag{7.15}$$

$$\widehat{a}_2 = \frac{1}{\sqrt{2\hbar}}\left(-p_2 + \hbar\frac{\partial}{\partial x_1}\right) \ , \qquad \widehat{a}^*_2 = \frac{1}{\sqrt{2\hbar}}\left(-p_2 - \hbar\frac{\partial}{\partial x_1}\right) \ . \tag{7.16}$$

Introducing a ground state ψ_{00}:

$$\widehat{a}_1\psi_{00} = \widehat{a}_2\psi_{00} = 0 \ , \tag{7.17}$$

one finds:

$$\psi_{00}(x_1, p_2) = e^{x_1p_2/\hbar} \ . \tag{7.18}$$

Defining

$$\psi_{nm} = (\widehat{a}^*_1)^n(\widehat{a}^*_2)^m\psi_{00} \ , \tag{7.19}$$

one shows

$$\widehat{H}_{\rm dho}\psi_{nm} = \mu_{nm}\psi_{nm} \ , \tag{7.20}$$

with μ_{nm} given by (7.12). One may show that it is possible to construct G_{nm} defined in (7.10) out of resonant states ψ_{nm}. However, contrary to $F^{\pm}_{nm}$, G_{nm} are not normalizable and the striking analogy with Wigner functions is lost.

8. Concluding remarks

In the present paper we analyzed the quantization of simple classical damped systems: a toy model defined by $\dot{x} = -\gamma x$ and the damped harmonic oscillator. Both systems give rise to resonant states and the corresponding energy spectra are discrete and complex. It turns out that resonant states appear always in pairs: if ψ_1 corresponds to E then there exists ψ_2 corresponding to $\overline{E}$. We showed that each pair of such states may be used to construct an analog of the stationary Wigner function. Actually one constructs a pair of stationary functions

$$F \propto \int dy e^{-ipy} \overline{\psi}_1(x - \frac{\hbar}{2}y)\, \psi_2(x + \frac{\hbar}{2}y) \qquad \text{and} \qquad \overline{F}\ .$$

A slightly different approach to quantization of damped oscillator was applied in [15]. In the forthcoming paper we show that both approaches are closely related. In a different context a quantum damped harmonic oscillator was recently analyzed in [16].

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