Spectral properties of the squeeze operator

Dariusz Chruściński
Institute of Physics, Nicolaus Copernicus University
ul. Grudziądzka 5/7, 87-100 Toruń, Poland

Abstract

We show that a single-mode squeeze operator $S(z)$ being an unitary operator with a purely continuous spectrum gives rise to a family of discrete real generalized eigenvalues. These eigenvalues are closely related to the spectral properties of $S(z)$ and the corresponding generalized eigenvectors may be interpreted as resonant states well known in the scattering theory. It turns out that these states entirely characterize the action of $S(z)$. This result is then generalized to N-mode squeezing.

1 Introduction

Squeezed states play a prominent role in the modern quantum optics, see e.g. [1, 2, 3]. They are quantum states for which no classical analog exists. Recently, they have drown a great deal of interest in connection with quantum teleportation. Squeezed states of light were successfully teleported in the experiment reported by the group of Furusawa [4]. Mathematical properties of these states were investigated in a series of papers [5, 6, 7, 8, 9]. In the present paper we analyze the spectral properties of single-mode $S(z)$ and N-mode $S_N(\widehat{Z})$ squeeze operators. Clearly, squeeze operators are unitary and hence their spectra define a subset of complex numbers with modulus one. It is easy to show that the spectrum of the squeeze operator operator is purely continuous and cover the entire unit circle on the complex plane. However, it is not the whole story. Actually, it is easy to note that $S(z)$ displays two families of discrete real eigenvalues. Clearly, these eigenvalues are not proper, that is, the corresponding eigenvectors do not belong to the corresponding Hilbert space of square integrable functions. One of these families was reported in a series of papers by Jannussis et al. [10, 11]. However, the interpretation of this result was not clear (it was criticized by Ma et al. [12] who stressed that proper eigenvalues of the squeeze operator do not exist). In the present paper we are going to clarify this problem. In particular we show that $S(z)$ being an unitary operator with a purely continuous spectrum does indeed possesses discrete eigenvalues which are closely related to its spectral properties. The corresponding eigenvectors may be interpreted as resonant states well known in the scattering theory [13, 14]. We show that restricting to a suitable class of states (e.g. coherent states do belong to this class) the action of $S(z)$ may be entirely characterized in terms of these discrete eigenvalues and the corresponding eigenvectors. This observation is then generalized to two-mode squeezing and finally to N-mode squeezing.

2 Single-mode squeezing

A single-mode squeeze operator is defined by

$$S(z) = \exp\left(\frac{1}{2}\left[za^{\dagger 2} - z^* a^2\right]\right) \ , \tag{2.1}$$

where z is a complex number and a ($a^\dagger$) is the photon annihilation (creation) operator which obey the standard commutation relation $[a, a^\dagger] = 1$. Clearly, $S(z)$ may be represented as $S(z) = \exp(iH(z))$, with

$$H(z) = \frac{1}{2i}\left(za^{\dagger 2} - z^* a^2\right) \ . \tag{2.2}$$

Now, to investigate spectral properties of $H(z)$ let us consider a unitarily equivalent operator $R^\dagger(\varphi)H(z)R(\varphi)$, where $R(\varphi)$ is a single-mode rotation [15, 16, 17]

$$R(\varphi) = \exp(i\varphi\, a^\dagger a) \ . \tag{2.3}$$

One has

$$R^\dagger(\varphi)H(z)R(\varphi) = H(ze^{-2i\varphi}) \ , \tag{2.4}$$

and hence, for $\varphi = \theta/2$, where $z = re^{i\theta}$, it shows that $H(z)$ is unitarily equivalent to $H(r)$. Introducing two quadratures x and p *via*

$$a = \frac{x + ip}{\sqrt{2}} \ , \qquad a^\dagger = \frac{x - ip}{\sqrt{2}} \ , \tag{2.5}$$

one finds the following formula for $H(r)$:

$$H(r) = -\frac{r}{2}\,(xp + px) \ . \tag{2.6}$$

Spectral properties of $H(r)$ were recently investigated in [18] (see also [19]) in connection with quantum dissipation. Note, that the classical Hamilton equations implied by the Hamiltonian $H = -rxp$:

$$\dot{x} = -rx \ , \qquad \dot{p} = rp \ , \tag{2.7}$$

describe the damping of x and pumping of p. This is a classical picture of the squeezing process. It turns out that $H(r)$ has purely continuous spectrum covering the whole real line. Hence, it is clear that it does not have any proper eigenvalue. Using standard Schrödinger representation for x and $p = -id/dx$, the corresponding eigen-problem $H(r)\psi = E\psi$ may be rewritten as follows

$$x\frac{d}{dx}\psi(x) = -\left(i\frac{E}{r} + \frac{1}{2}\right)\psi(x) \ . \tag{2.8}$$

Note, that $H(r)$ is parity invariant and hence each generalized eigenvalue $E \in \mathbb{R}$ is doubly degenerated. Therefore, two independent solutions of (2.8) are given by

$$\psi^E_\pm(x) = \frac{1}{\sqrt{2\pi r}}\, x_\pm^{-(iE/r+1/2)} \ , \tag{2.9}$$

where $x_\pm^\lambda$ are distributions defined as follows [20] (see also [21]):

$$x_+^\lambda := \begin{cases} x^\lambda & x \geq 0 \\ 0 & x < 0 \end{cases} , \qquad x_-^\lambda := \begin{cases} 0 & x \geq 0 \\ |x|^\lambda & x < 0 \end{cases} , \tag{2.10}$$

with $\lambda \in \mathbb{C}$ (basic properties of $x_\pm^\lambda$ are collected in [18]). These generalized eigenvectors $\psi_\pm^E$ are complete

$$\int \overline{\psi_\pm^E(x)} \psi_\pm^E(x')\, dE = \delta(x - x') , \tag{2.11}$$

and δ-normalized

$$\int \overline{\psi_\pm^{E_1}(x)} \psi_\pm^{E_2}(x)\, dx = \delta(E_1 - E_2) . \tag{2.12}$$

Hence they give rise to the following spectral resolution of $H(r)$:

$$H(r) = \sum_\pm \int E\, |\psi_\pm^E\rangle\langle\, \psi_\pm^E|\, dE , \tag{2.13}$$

and the corresponding spectral resolutions of squeeze operator $S(r)$ immediately follows

$$S(r) = \sum_\pm \int e^{iE}\, |\psi_\pm^E\rangle\langle\, \psi_\pm^E|\, dE . \tag{2.14}$$

Now, let us observe that $FH = -HF$, where F denotes the Fourier transformation. Hence, if $H(r)\psi^E = E\psi^E$, then $H(r)F[\psi^{-E}] = EF[\psi^{-E}]$. Therefore, the family $F[\psi_\pm^{-E}]$ defines another system of complete and δ-normalized generalized eigenvectors of $H(r)$. Note that the action of $S(r)$ is defined by

$$S(r)\psi(x) = e^{-r/2}\psi(e^{-r}x) , \tag{2.15}$$

and its Fourier transform

$$F[S(r)\psi](p) = e^{r/2}F[\psi](e^r p) , \tag{2.16}$$

due to $FS(r) = S(-r)F$, that is, if the fluctuations of p are reduced then the fluctuation of x are amplified and vice versa.

3 Discrete real eigenvalues of $S(r)$

Surprisingly, apart from the continuous spectrum $H(r)$ gives rise to the following families of complex discrete eigenvalues [18]

$$H(r)f_n^\pm = \pm E_n\, f_n^\pm , \tag{3.1}$$

where

$$E_n = ir\left(n + \frac{1}{2}\right) , \tag{3.2}$$

and

$$f_n^-(x) = \frac{(-1)^n}{\sqrt{n!}}\,\delta^{(n)}(x)\ , \qquad f_n^+(x) = \frac{x^n}{\sqrt{n!}}\ . \tag{3.3}$$

Interestingly they satisfy the following properties:

$$\int_{-\infty}^{\infty} f_n^+(x)\, f_m^-(x)\, dx = \delta_{nm}\ , \tag{3.4}$$

and

$$\sum_{n=0}^{\infty} f_n^+(x)\, f_n^-(x') = \delta(x-x')\ . \tag{3.5}$$

It implies that

$$S(r) f_n^{\pm} = e^{\pm i E_n}\, f_n^{\pm}\ , \tag{3.6}$$

which shows that $S(r)$ displays two families of purely real generalized eigenvalues

$$s_n^{\pm} = \exp\left[\pm r\left(n+\frac{1}{2}\right)\right]\ . \tag{3.7}$$

A family s_n^+ was already derived by Jannussis et al., see e.g. formula (2.3) in [11], but they overlooked the second one s_n^-. How to interpret these eigenvalues? It turns out that one recover E_n and $f_n^{\pm}$ by studying a continuation of generalized eigenvectors $\psi_{\pm}^E$ and $F[\psi_{\pm}^{-E}]$ into the energy complex plane $E \in \mathbb{C}$. Both ψ^E and $F[\psi^{-E}]$ display singular behavior when E is complex: $\psi_{\pm}^E$ has simple poles at $E = -E_n$, whereas $F[\psi_{\pm}^{-E}]$ has simple poles at $E = +E_n$, with E_n defined in (3.2). Moreover, their residues correspond, up to numerical factors, to the eigenvectors $f_n^{\pm}$:

$$\mathrm{Res}(\psi_{\pm}^E(x); -E_n)\ \sim\ f_n^-\ , \tag{3.8}$$

and

$$\mathrm{Res}(F[\psi_{\pm}^{-E}(x)]; +E_n)\ \sim\ f_n^+\ . \tag{3.9}$$

Such eigenvectors are well known in scattering theory as resonant states, see e.g. [13] and references therein. In the so called rigged Hilbert space to quantum mechanics these states are also called Gamov vectors [14]. To see the connection with the scattering theory let us observe that under the following canonical transformation:

$$x = \frac{rQ-P}{\sqrt{2r}}\ , \qquad p = \frac{rQ+P}{\sqrt{2r}}\ , \tag{3.10}$$

$H(r)$ transforms into the unitarily equivalent operator [22]

$$H(r)\ \longrightarrow\ H_{\mathrm{io}} = \frac{1}{2}(P^2 - r^2Q^2)\ , \tag{3.11}$$

which represents the Hamiltonian of the called an inverted or reversed oscillator (or equivalently a potential barrier '$-r^2Q^2/2$') and it was studied by several authors in various contexts [23, 24, 25, 26, 27, 28]. An inverted oscillator H_{io} corresponds to the harmonic oscillator with a purely imaginary frequency $\omega = \pm ir$ and hence the harmonic oscillator spectrum '$\omega(n+1/2)$' implies '$\pm ir(n+1/2)$' as generalized eigenvalues of H_{io}.

4 A new representation of $S(r)$

Interestingly, the action of $S(r)$ may be entirely characterized in terms of $f_n^\pm$ and $s_n^\pm$. Indeed, consider a space $\mathcal{D}$ of smooth functions $\psi = \psi(x)$ with compact supports, i.e. $\psi(x) = 0$ for $|x| > a$ for some positive a (depending upon chosen ψ), see e.g. [29]. Clearly, $\mathcal{D}$ defines a subspace of square integrable functions $L^2(\mathbb{R})$. Moreover, let $\mathcal{Z} = F[\mathcal{D}]$, that is, $\psi \in \mathcal{Z}$ if $\psi = F[\phi]$ for some $\phi \in \mathcal{D}$. It turns out [29] that $\mathcal{D}$ and $\mathcal{Z}$ are isomorphic and $\mathcal{D} \cap \mathcal{Z} = \emptyset$. Now, any function ϕ from $\mathcal{Z}$ may be expanded into Taylor series and hence

$$\phi(x) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} f_n^+(x) \langle\, f_n^- | \phi \,\rangle \ . \tag{4.1}$$

On the other hand, for any $\phi \in \mathcal{D}$, its Fourier transform $F[\phi] \in \mathcal{Z}$, and

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2\pi}} \int e^{ikx} F[\phi](k) dk = \frac{1}{\sqrt{2\pi}} \int e^{ikx} \sum_{n=0}^{\infty} \frac{F[\phi]^{(n)}(0)}{n!} k^n \, dk \\ &= \sum_{n=0}^{\infty} F[f_n^+](x) \langle\, f_n^- | F[\phi] \,\rangle = \sum_{n=0}^{\infty} f_n^-(x) \langle\, f_n^+ | \phi \,\rangle \ . \end{aligned} \tag{4.2}$$

Hence, we have two decompositions of the identity operator

$$\mathbb{1} = \sum_{n=0}^{\infty} | f_n^+ \,\rangle\langle\, f_n^- | \qquad \text{on} \quad \mathcal{Z} \ , \tag{4.3}$$

and

$$\mathbb{1} = \sum_{n=0}^{\infty} | f_n^- \,\rangle\langle\, f_n^+ | \qquad \text{on} \quad \mathcal{D} \ . \tag{4.4}$$

It implies the following representations of the squeeze operator $S(r)$:

$$S(r) = \sum_{n=0}^{\infty} s_n^- \, | f_n^+ \,\rangle\langle\, f_n^- | \qquad \text{on} \quad \mathcal{Z} \ , \tag{4.5}$$

and

$$S(r) = \sum_{n=0}^{\infty} s_n^+ \, | f_n^- \,\rangle\langle\, f_n^+ | \qquad \text{on} \quad \mathcal{D} \ . \tag{4.6}$$

It should be stressed that the above formulae for $S(r)$ are not spectral decompositions and they valid only on $\mathcal{Z}$ and $\mathcal{D}$, respectively (its spectral decomposition is given in formula (2.14)). Note, that $S(r)$ maps $\mathcal{Z}$ into $\mathcal{D}$ and using (4.5) one has

$$S^\dagger(r) = \sum_{n=0}^{\infty} s_n^- \, | f_n^- \,\rangle\langle\, f_n^+ | = S(-r) \qquad \text{on} \quad \mathcal{D} \ . \tag{4.7}$$

Conversely, $S(r)$ represented by (4.6) maps $\mathcal{D}$ into $\mathcal{Z}$ and $S^\dagger(r) = S(-r)$ on $\mathcal{Z}$. It shows that squeezing of x (p) corresponds to amplifying of p (x). Clearly, a general quantum state $\psi \in L^2(\mathbb{R})$ belongs neither to $\mathcal{D}$ nor to $\mathcal{Z}$. An example of quantum states belonging to $\mathcal{Z}$ is a family of Glauber coherent states $|\alpha\rangle$. Consider e.g. a squeezed vacuum $S(r)\psi_0$, where $\psi_0(x) = \pi^{-1/4}e^{-x^2/2}$. One has

$$\psi_0(x) = \frac{1}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{(2n)!}}\, f_{2n}^{+}(x) \ , \tag{4.8}$$

and

$$S(r)\psi_0(x) = \frac{e^{-r/2}}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{(-e^{-2r})^n}{\sqrt{(2n)!}}\, f_{2n}^{+}(x) = \frac{e^{-r/2}}{\pi^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{(2n)!}}\, f_{2n}^{+}(e^{-r}x) \ . \tag{4.9}$$

The similar formulae hold for $S(r)|\alpha\,\rangle$.

5 Two-mode squeezing

Consider now a two-mode squeeze operator [15, 16, 17]

$$S_2(z) = \exp\left(z a_1^\dagger a_2^\dagger - z^* a_1 a_2\right) \ , \tag{5.1}$$

where $a_k^\dagger$ and a_k are creation and annihilation operators for two modes $k = 1, 2$. Introducing 2-dimensional vectors

$$\underline{a}^T = (a_1, a_2) \ , \qquad (\underline{a}^\dagger)^T = (a_1^\dagger, a_2^\dagger) \ , \tag{5.2}$$

one finds

$$S_2(z) = \exp\left(z(\underline{a}^\dagger)^T \sigma_1 \underline{a}^\dagger - z^* \underline{a}^T \sigma_1 \underline{a}\right) \ , \tag{5.3}$$

where σ_1 stands for the corresponding Pauli matrix. Using well-known relation

$$e^{i\pi/4\,\sigma_2}\, \sigma_1\, e^{-i\pi/4\,\sigma_2} = \sigma_3 \ , \tag{5.4}$$

one obtains

$$\begin{aligned} e^{i\pi/4\,\sigma_2}\, S_2(z)\, e^{-i\pi/4\,\sigma_2} &= \exp\left(z(\underline{a}^\dagger)^T \sigma_3 \underline{a}^\dagger - z^* \underline{a}^T \sigma_3 \underline{a}\right) \\ &= S^{(1)}(z) S^{(2)}(-z) \ , \end{aligned} \tag{5.5}$$

where $S^{(k)}$ denotes a single-mode (k) squeeze operator. Now, since $S^{(k)}(z)$ is unitarily equivalent to $S^{(k)}(r)$ a two-mode squeeze operator $S_2(z)$ is unitarily equivalent to $S^{(1)}(r)S^{(2)}(-r)$. Hence, the spectral properties of $S_2(z)$ easily follows. In particular defining the space $\mathcal{D}$ of smooth functions $\psi = \psi(x_1, x_2)$ with compact supports and $\mathcal{Z} = F[\mathcal{D}]$ one obtains the following representations:

$$S^{(1)}(r)S^{(2)}(-r) = \sum_{nm=0}^{\infty} s_{nm}^{-}\, |f_{nm}^{+}\,\rangle\langle\, f_{nm}^{-}| \qquad \text{on} \quad \mathcal{Z} \ , \tag{5.6}$$

and

$$S^{(1)}(r)S^{(2)}(-r) = \sum_{nm=0}^{\infty} s^{+}_{nm}\, |f^{-}_{nm}\,\rangle\langle\, f^{+}_{nm}| \qquad \text{on} \quad \mathcal{D}\ , \tag{5.7}$$

where

$$f^{\pm}_{nm}(x_1,x_2) = f^{\pm}_{n}(x_1)f^{\pm}_{m}(x_2)\ , \tag{5.8}$$

and

$$s^{\pm}_{nm} = e^{\pm r(n-m)}\ . \tag{5.9}$$

Jannussis et al. [11] claimed that the eigenvalues of $S_2(z)$ are given by $e^{2(m-n)}$, see e.g. formula (5.7) in [11]. Their result has the similar form as s^{-}_{nm} but of course it is incorrect. Note that eigenvalues of [11] do not depend upon the squeezing parameter z as was already observed in [12].

6 N-mode squeezing

Following [30] one defines an N-mode squeeze operator

$$S_N(\widehat{Z}) = \exp\left(\frac{1}{2}\,(\underline{a}^{\dagger})^T\,\widehat{Z}\,\underline{a}^{\dagger} - \frac{1}{2}\,\underline{a}^{\dagger}\,\widehat{Z}^{\dagger}\,\underline{a}\right) \tag{6.1}$$

where $\widehat{Z}$ is an $N\times N$ symmetric (complex) matrix and

$$\underline{a}^T = (a_1,a_2,\ldots,a_N)\ .$$

Defining an N-mode rotation operator

$$R_N(\widehat{\Phi}) = \exp\left(i(\underline{a}^{\dagger})^T\,\widehat{\Phi}\,\underline{a}\right)\ , \tag{6.2}$$

with $\widehat{\Phi}$ being an $N\times N$ hermitian matrix, one shows [17, 30]

$$R^{\dagger}_N(\widehat{\Phi})\,S_N(\widehat{Z})\,R_N(\widehat{\Phi}) = S_N\left(e^{-i\widehat{\Phi}}\,\widehat{Z}\,e^{-i\widehat{\Phi}^T}\right)\ . \tag{6.3}$$

Now, by a suitable choice of $\widehat{\Phi}$ one obtains

$$e^{-i\widehat{\Phi}}\,\widehat{Z}\,e^{-i\widehat{\Phi}^T} = \widehat{Z}_{\rm D}\ , \tag{6.4}$$

where $\widehat{Z}_{\rm D}$ is a diagonal matrix, i.e. $(\widehat{Z}_{\rm D})_{kl} = z_k\delta_{kl}$. Hence, an N-mode squeeze operator $S_N(\widehat{Z})$ is unitarily equivalent to

$$R^{\dagger}_N(\widehat{\Phi})\,S_N(\widehat{Z})\,R_N(\widehat{\Phi}) = S^{(1)}(z_1)\,S^{(2)}(z_2)\,\ldots\,S^{(N)}(z_N)\ , \tag{6.5}$$

and therefore its properties are entirely governed by the properties of the single-mode squeeze operator $S(z)$. In particular $S_N(\widehat{Z})$ gives rise to a discrete family of generalized eigenvalues being combinations of generalized eigenvalues of $S^{(k)}(z_k)$. Defining the corresponding subspaces $\mathcal{D}$ and $\mathcal{Z}$ in the Hilbert space $L^2(\mathbb{R}^N)$ one easily finds

$$R_N^\dagger(\widehat{\Phi})\, S_N(\widehat{Z})\, R_N(\widehat{\Phi}) = \sum_{n_1,\ldots,n_N=0}^{\infty} s^-_{n_1\ldots n_N}\, |f^+_{n_1\ldots n_N}\rangle\langle f^-_{n_1\ldots n_N}| \qquad \text{on} \quad \mathcal{Z}\ , \tag{6.6}$$

and

$$R_N^\dagger(\widehat{\Phi})\, S_N(\widehat{Z})\, R_N(\widehat{\Phi}) = \sum_{n_1,\ldots,n_N=0}^{\infty} s^+_{n_1\ldots n_N}\, |f^-_{n_1\ldots n_N}\rangle\langle f^+_{n_1\ldots n_N}| \qquad \text{on} \quad \mathcal{D}\ , \tag{6.7}$$

where

$$f^\pm_{n_1\ldots n_N}(x_1,\ldots,x_N) = f^\pm_{n_1}(x_1)\,\ldots f^\pm_{n_N}(x_N)\ , \tag{6.8}$$

and

$$s^\pm_{n_1\ldots n_N} = \exp\left\{\pm\left[r_1\left(n_1+\frac{1}{2}\right)+\ldots+r_N\left(n_N+\frac{1}{2}\right)\right]\right\}\ , \tag{6.9}$$

with $r_k = |z_k|$.

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