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Trends in Logic XIII

Gentzen's and Jaśkowski's heritage
80 years of natural deduction
and sequent calculi

Editors

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AXIOMATISATIONS OF MINIMAL MODAL LOGICS DEFINING JAŚKOWSKI-LIKE DISCUSSIVE LOGICS

Abstract Jaśkowski's discussive logic $\mathbf{D_2}$ was formulated with the help of the modal logic $\mathbf{S5}$ as follows: $A \in \mathbf{D_2}$ iff $\ulcorner \Diamond A^{\bullet} \urcorner \in \mathbf{S5}$, where $(-)^{\bullet}$ is a translation of discussive formulae into the modal language. Thus, the key role in the definition of the logic $\mathbf{D_2}$ is played by the logic $\mathbf{S5}$. In the literature there are considered other modal logics that are also defining the same logic $\mathbf{D_2}$.

There are also investigated translations that are determining other Jaśkowski-like logics. In [3, 5], instead of the original translation with "right"-discussive conjunction, another translation is considered, where "left"-discussive conjunction is treated as Jaśkowski's one. In [2], it has been shown that this new transformation yields a logic different from $\mathbf{D_2}$. Ciuciura denotes the obtained logic by '$\mathbf{D_2^*}$'. There are two other possibilities as regards the internal translation of discussive conjunctions.

The question arises (which has been stated by João Marcos), what does it change if we consider the weakest in a given class modal logics that determine these "new" discussive logics. In [11] the smallest modal logics defining respective Jaśkowski-like discussive logics are considered. In the present paper we give more elegant axiomatisations of these logics.

Keywords: Jaśkowski's discussive logic, Jaśkowski-like discussive logics, axiomatisations of Jaśkowski-like discussive logics, minimal modal logics defining Jaśkowski logic, minimal modal logics defining Jaśkowski-like discussive logics

1 BASIC NOTIONS AND FACTS

1.1 *Some facts of modal logic*

MODAL LANGUAGE. Modal formulae are formed in the standard way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', ...; truth-value operators: '$\neg$', '$\vee$', '$\wedge$', '$\rightarrow$', and '$\leftrightarrow$' (connectives of negation, disjunction, conjunction, material implication and material equivalence, respectively); modal operators: the necessity sign '$\Box$' and the possibility sign '$\Diamond$'; and brackets. By $\mathrm{For_m}$ we denote the set of modal formulae. Of course, the set $\mathrm{For_m}$ includes the set of all classical formulae (without '$\Box$' and '$\Diamond$'); let **Taut** be the set of all classical tautologies and **PL** — the set of all modal formulae being instances of elements of **Taut**. Besides, for any $\varphi, \psi, \chi \in \mathrm{For_m}$, let $\chi[^{\varphi}/_{\psi}]$ be any formula that results from χ by replacing none, one, or more occurrences of φ, in χ, by ψ.

For any $\psi \in \mathrm{For_m}$ let $\mathrm{Sub}(\psi)$ be the set of all modal formulae being substitution instances of ψ. For any $\Phi \subseteq \mathrm{For_m}$ let $\mathrm{Sub}(\Phi) := \bigcup_{\varphi\in\Phi} \mathrm{Sub}(\varphi)$. We have $\psi \in \mathrm{Sub}(\psi)$ and $\Phi \subseteq \mathrm{Sub}(\Phi)$. Moreover, we put $\Diamond\Phi := \{\psi : \exists_{\varphi\in\Phi}\ \psi = \ulcorner\Diamond\varphi\urcorner\} = \{\ulcorner\Diamond\varphi\urcorner : \varphi \in \Phi\}$ and $\Box\Phi := \{\ulcorner\Box\varphi\urcorner : \varphi \in \Phi\}$.

MODAL LOGICS. A **modal logic** is any set L of modal formulae satisfying following conditions:

- **Taut** $\subseteq L$,
- L includes the following set of formulae

$$\left\{ \ulcorner\chi[^{\neg\Box\neg\varphi}/_{\Diamond\varphi}] \leftrightarrow \chi\urcorner \ : \ \varphi, \chi \in \mathrm{For_m} \right\}, \qquad (rep^{\Box})$$

- L is closed under the following two rules: **modus ponens** for '$\rightarrow$':

$$\varphi,\ \varphi \rightarrow \psi \ \ / \ \ \psi \qquad \text{(mp)}$$

and **uniform substitution**:

$$\varphi \ \ / \ \ \mathrm{s}\,\varphi \qquad \text{(sb)}$$

where $s\varphi$ is the result of uniform substitution of formulae for propositional letters in φ.

CHOSEN CLASSES OF LOGICS. We say that a modal logic L is an **rte-logic** iff L is closed under replacement of tautological equivalents, i.e., for any $\varphi, \psi, \chi \in \mathrm{For_m}$:

$$\text{if } \ulcorner\varphi \leftrightarrow \psi\urcorner \in \mathrm{PL} \text{ and } \chi \in L, \text{ then } \chi[^{\varphi}/_{\psi}] \in L. \tag{rte}$$

A modal logic is rte-logic iff it includes the following set

$$\left\{ \ulcorner\chi[^{\varphi}/_{\psi}] \leftrightarrow \chi\urcorner : \varphi, \psi, \chi \in \mathrm{For_m} \text{ and } \ulcorner\varphi \leftrightarrow \psi\urcorner \in \mathrm{PL} \right\}. \tag{rep_{PL}}$$

LEMMA 1.1. A modal logic contains the formula:

$$\Box p \to p \tag{T}$$

iff it contains its dual version:

$$p \to \Diamond p \tag{T°}$$

LEMMA 1.2. An rte-logic contains the following formulae:

$$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q) \tag{R}$$
$$\Diamond\Box p \to p \tag{B}$$
$$\Diamond\Box p \to \Box p \tag{5}$$

iff it contains, respectively, theirs dual versions:

$$\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q) \tag{R°}$$
$$p \to \Box\Diamond p \tag{B°}$$
$$\Diamond p \to \Box\Diamond p \tag{5°}$$

In [1] a modal logic is called **classical modal** (**cm-logic** for short) iff it is an rte-logic which contains

$$\Box(p \to q) \to (\Box p \to \Box q) \tag{K}$$
$$\Box(p \to p) \tag{N}$$

Thus, all cm-logics include the set $\Box\mathsf{PL} := \{\Box\tau : \tau \in \mathsf{PL}\}$.

We say that a logic is **congruential** iff it is closed under the congruence rule

$$\varphi \leftrightarrow \psi \;\; / \;\; \Box\varphi \leftrightarrow \Box\psi \tag{cgr}$$

A logic is congruential iff it is closed under replacement

$$\varphi \leftrightarrow \psi \;\; / \;\; \chi[^{\varphi}/_{\psi}] \leftrightarrow \chi \tag{rep}$$

Every congruential logic is an rte-logic.

We say that a logic L is **monotonic** iff L is closed under the monotonicity rule:

$$\varphi \rightarrow \psi \;\; / \;\; \Box\varphi \rightarrow \Box\psi \tag{mon}$$

Every monotonic logic is closed under (rep), i.e. is congruential.

We say that a logic is **regular** iff it contains (K) and is closed under (mon).

A logic is **normal** iff it contains (K) and is closed under the necessitation rule

$$\varphi \;\; / \;\; \Box\varphi \tag{nec}$$

All normal logics are regular and cm-logics.

For all sets X and $\mathcal{A}$ of modal formulae and any set of rules $\mathcal{R}$ in $\mathsf{For}_{\mathsf{m}}$ we say that the pair $\langle\mathcal{A},\mathcal{R}\rangle$ is an **axiomatization** of X iff X is the smallest set including $\mathcal{A}$ and closed under all rules from $\mathcal{R}$.

1.2 *The discussive logic* D_2 *and other Jaśkowski-like logics*

DISCUSSIVE LANGUAGE. The logic D_2 is defined as a set of discussive formulae of a certain kind. These formulae are formed in the standard way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', ...; truth-value operators: '$\neg$' and '$\vee$' (negation and disjunction); discussive connectives: '$\wedge^{\mathsf{d}}$', '$\rightarrow^{\mathsf{d}}$', '$\leftrightarrow^{\mathsf{d}}$' (conjunction, implication and equivalence); and brackets. Let $\mathsf{For}^{\mathsf{d}}$ be the set of all these formulae.

DEFINITION OF DISCUSSIVE LOGIC $\mathbf{D_2}$. The logic $\mathbf{D_2}$ was formulated with the help of the modal logic $\mathbf{S5}$ as follows (see [7, 8]):

$$\mathbf{D_2} := \{ A \in \mathrm{For^d} : \ulcorner \Diamond A^{\bullet} \urcorner \in \mathbf{S5} \},$$

where $(-)^{\bullet}$ is a translation of discussive formulae into modal language, i.e., it is a function $(-)^{\bullet}$ from $\mathrm{For^d}$ into $\mathrm{For_m}$ such that:

1. $(a)^{\bullet} = a$, for any propositional letter a,
2. for any $A, B \in \mathrm{For^d}$:
   (a) $(\neg A)^{\bullet} = \ulcorner \neg A^{\bullet} \urcorner$,
   (b) $(A \vee B)^{\bullet} = \ulcorner A^{\bullet} \vee B^{\bullet} \urcorner$,
   (c) $(A \wedge^{\mathrm{d}} B)^{\bullet} = \ulcorner A^{\bullet} \wedge \Diamond B^{\bullet} \urcorner$,
   (d) $(A \rightarrow^{\mathrm{d}} B)^{\bullet} = \ulcorner \Diamond A^{\bullet} \rightarrow B^{\bullet} \urcorner$,
   (e) $(A \leftrightarrow^{\mathrm{d}} B)^{\bullet} = \ulcorner (\Diamond A^{\bullet} \rightarrow B^{\bullet}) \wedge \Diamond(\Diamond B^{\bullet} \rightarrow A^{\bullet}) \urcorner$.

Of course, $\mathbf{D_2}$ is closed under (sb) with respect to $\mathrm{For^d}$. Moreover, $\mathbf{D_2}$ is closed under modus ponens for '$\rightarrow^{\mathrm{d}}$':

$$A,\ A \rightarrow^{\mathrm{d}} B \ /\ B \qquad (\mathrm{mp_d})$$

because $\mathbf{S5}$ is closed under the following rule:

$$\Diamond\varphi,\ \Diamond(\Diamond\varphi \rightarrow \psi) \ /\ \Diamond\psi \qquad (RC)$$

DEFINITIONS OF JAŚKOWSKI-LIKE LOGICS. In [3, 5] a logic $\mathbf{D_2^*}$ was formulated with the help of the modal logic $\mathbf{S5}$ as follows:

$$\mathbf{D_2^*} := \{ A \in \mathrm{For^d} : \ulcorner \Diamond A^{*} \urcorner \in \mathbf{S5} \},$$

where $(-)^*$ is a function from $\mathrm{For^d}$ into $\mathrm{For_m}$ such that for any $A, B \in \mathrm{For^d}$:

(c)* $(A \wedge^{\mathrm{d}} B)^* = \ulcorner \Diamond A^* \wedge B^* \urcorner$,

(e)* $(A \leftrightarrow^{\mathrm{d}} B)^* = \ulcorner \Diamond(\Diamond A^* \rightarrow B^*) \wedge (\Diamond B^* \rightarrow A^*) \urcorner$,

and other cases stay as in the definition of the function $(-)^{\bullet}$.

Additionally a logic $\mathsf{D}_2^{\frown}$ was defined as follows:

$$\mathsf{D}_2^{\frown} := \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^{\frown} \urcorner \in \mathsf{S5} \},$$

where $(-)^{\frown}$ is a function from $\mathrm{For}^{\mathrm{d}}$ into $\mathrm{For}_{\mathrm{m}}$ such that for any $A, B \in \mathrm{For}^{\mathrm{d}}$:

$(\mathrm{c})^{\frown}$ $(A \wedge^{\mathrm{d}} B)^{\frown} = \ulcorner A^{\frown} \wedge B^{\frown} \urcorner$,

$(\mathrm{e})^{\frown}$ $(A \leftrightarrow^{\mathrm{d}} B)^{\frown} = \ulcorner (\Diamond A^{\frown} \rightarrow B^{\frown}) \wedge (\Diamond B^{\frown} \rightarrow A^{\frown}) \urcorner$.

and, as previously, other cases stay the same. (Notice that in the translation for conjunction '$\Diamond$' is not used.)

And finally, a logic D_2^{**} was formulated also with the help of the modal logic $\mathsf{S5}$ as follows:

$$\mathsf{D}_2^{**} := \{ A \in \mathrm{For}^{\mathrm{d}} : \ulcorner \Diamond A^{\times} \urcorner \in \mathsf{S5} \},$$

where $(-)^{\times}$ is a function from $\mathrm{For}^{\mathrm{d}}$ into $\mathrm{For}_{\mathrm{m}}$ such that for any $A, B \in \mathrm{For}^{\mathrm{d}}$:

$(\mathrm{c})^{\times}$ $(A \wedge^{\mathrm{d}} B)^{\times} = \ulcorner \Diamond A^{\times} \wedge \Diamond B^{\times} \urcorner$,

$(\mathrm{e})^{\times}$ $(A \leftrightarrow^{\mathrm{d}} B)^{\times} = \ulcorner \Diamond(\Diamond A^{\times} \rightarrow B^{\times}) \wedge \Diamond(\Diamond B^{\times} \rightarrow A^{\times}) \urcorner$,

and again, other cases stay unchanged.

Thus, all these logics have different conditions for conjunction. Notice that for each translation — call it 'any', for all $A, B \in \mathrm{For}^{\mathrm{d}}$: $(A \leftrightarrow^{\mathrm{d}} B)^{\mathrm{any}} = ((A \rightarrow^{\mathrm{d}} B) \wedge^{\mathrm{d}} (B \rightarrow^{\mathrm{d}} A))^{\mathrm{any}}$. Of course, these logics are also closed under (sb) and $(\mathrm{mp}_{\mathrm{d}})$.

In [2] Ciuciura observed that $\mathsf{D}_2^{*} \nsubseteq \mathsf{D}_2$. It was shown that one of the axioms of the logic D_2^{*} is not a thesis of the logic D_2. We also have:

FACT 1.3 ([11]). Every two logics among D_2, D_2^{*}, $\mathsf{D}_2^{\frown}$, and D_2^{**} cross each other.

2 MODAL LOGICS DEFINING D_2, D_2^{*}, $\mathsf{D}_2^{\frown}$ AND D_2^{**}

There is a procedure (see [9]) that for a given class of logics fulfilling some natural conditions, returns, in the considered class, the minimal logic which has the same theses beginning with '◇' as S5. The same can be repeated for D_2^*, D_2^-, and D_2^{**}.

We say that a modal logic L **defines** D_2 (resp. D_2^*, D_2^-, D_2^{**}) iff

- $D_2 = \{A \in For^d : \ulcorner \Diamond A^\bullet \urcorner \in L\}$ (resp.
- $D_2^* = \{A \in For^d : \ulcorner \Diamond A^* \urcorner \in L\}$,
- $D_2^- = \{A \in For^d : \ulcorner \Diamond A^\frown \urcorner \in L\}$
- $D_2^{**} = \{A \in For^d : \ulcorner \Diamond A^\times \urcorner \in L\}$).

There are known other modal logics defining D_2. The same holds for the other three discussive logics.

We see that while expressing the logic D_2 we refer to modal logics which

$$\text{have the same theses beginning with '}\Diamond\text{' as S5.} \qquad (\dagger)$$

Let $S5_\diamond$ be the set of all these logics, that is,

$$L \in S5_\diamond \quad \text{iff} \quad \forall_{\varphi \in For_m}(\ulcorner \Diamond\varphi \urcorner \in L \iff \ulcorner \Diamond\varphi \urcorner \in S5).$$

By the definition we see:

FACT 2.1. For any $L \in S5_\diamond$:

1. $\{\ulcorner \Diamond\varphi \urcorner : \ulcorner \Diamond\varphi \urcorner \in S5\} \subseteq L$,
2. If $L \in S5_\diamond$, then L defines D_2, D_2^*, D_2^- and D_2^{**}.

Recall that $rteS5^M$, $cmS5^M$, $eS5^M$, $mS5^M$, $rS5^M$ and $S5^M$ are respectively, the smallest rte-, cm-, congruential, monotonic, regular and normal logic in $S5_\diamond$. Thus, by Fact 2.1 each of them defines logics D_2^*, D_2^- and D_2^{**}.

Let $(-)^{any}$ be any translation of discussive formulae into modal language, i.e., $(-)^{any}$ is a function from For^d into For_m, and let

$$D_2^{any} := \{A \in For^d : \ulcorner \Diamond A^{any} \urcorner \in S5\},$$

COROLLARY 2.2 ([11]). The logics $rteS5^M$, $cmS5^M$, $eS5^M$, $mS5^M$, $rS5^M$, and $S5^M$ are the smallest rte-, cm-, congruential, monotonic, regular, and normal logic in $S5_\diamond$ defining D_2^{any}, respectively.

FACT 2.3 ([9]). For any rte-logic L: L defines $\mathbf{D_2}$ iff $L \in \mathbf{S5}_\diamond$.

In the proof of the next fact a function $(-)^{\circ_1}$ from For_m into For^d which <<un-modalizes>> every modal formula was used:

1. $(a)^{\circ_1} = a$, for any propositional letter a,
2. for any $\varphi, \psi \in \mathrm{For}_\mathrm{m}$:
   (a) $(\neg\varphi)^{\circ_1} = \ulcorner\neg\varphi^{\circ_1}\urcorner$,
   (b) $(\varphi \vee \psi)^{\circ_1} = \ulcorner\varphi^{\circ_1} \vee \psi^{\circ_1}\urcorner$,
   (c) $(\varphi \wedge \psi)^{\circ_1} = \ulcorner\neg(\neg\varphi^{\circ_1} \vee \neg\psi^{\circ_1})\urcorner$,
   (d) $(\varphi \rightarrow \psi)^{\circ_1} = \ulcorner\neg\varphi^{\circ_1} \vee \psi^{\circ_1}\urcorner$,
   (e) $(\varphi \leftrightarrow \psi)^{\circ_1} = \ulcorner\neg(\neg(\neg\varphi^{\circ_1} \vee \psi^{\circ_1}) \vee \neg(\neg\psi^{\circ_1} \vee \varphi^{\circ_1}))\urcorner$,
   (f) $(\Diamond\varphi)^{\circ_1} = \ulcorner\varphi^{\circ_1} \wedge^\mathrm{d} (p \vee \neg p)\urcorner$,
   (g) $(\Box\varphi)^{\circ_1} = \ulcorner\neg\varphi^{\circ_1} \rightarrow^\mathrm{d} \neg(p \vee \neg p)\urcorner$.

Next we observe that for any $A, B \in \mathrm{For}_\mathrm{m}$, $\S \in \{\neg, \Diamond\}$ and $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ the following formulae belong to $\mathbf{PL}$:

$$\begin{aligned} (\S A)^{\circ *} &\leftrightarrow \S A^{\circ *} \\ (A * B)^{\circ *} &\leftrightarrow (A^{\circ *} * B^{\circ *}) \\ (\Box A)^{\circ *} &\leftrightarrow \neg\Diamond\neg A^{\circ *} \end{aligned} \tag{$\star$}$$

And finally we see that for any formulae $A_1, \ldots, A_n$, C we obtain:

$$C^{\circ *} \in L \quad \text{iff} \quad C[{}^{\Box A_1}/_{\neg\Diamond\neg A_1}, \ldots, {}^{\Box A_n}/_{\neg\Diamond\neg A_n}] \in L\,.$$

FACT 2.4 ([11]). For any rte-logic L: L defines $\mathbf{D}_2^*$ iff $L \in \mathbf{S5}_\diamond$.

On the other hand in the proof of the below fact another function $(-)^{\circ_2}$ from For_m into For^d is used where for any $\varphi \in \mathrm{For}_\mathrm{m}$:

(f) $(\Diamond\varphi)^{\circ_2} = \ulcorner\neg(\varphi^{\circ_2} \rightarrow^\mathrm{d} \neg(p \vee \neg p))$,

The other cases are as in the formulation of the function $(-)^{\circ_1}$.

FACT 2.5 ([11]). For any rte-logic L: L defines $\mathbf{D}_2^-$ iff $L \in \mathbf{S5}_\diamond$.

And finally, in the proof of Fact 2.6 a function $(-)^{\circ_3}$ from For_m into For^d is needed such that for any $\varphi \in \mathrm{For}_\mathrm{m}$:

(f) $(\Diamond\varphi)^{\circ_3} = \ulcorner\varphi^{\circ_3} \wedge^{d} \varphi^{\circ_3}\urcorner$.

Again, the other cases stay unchanged.

FACT 2.6 ([11]). For any rte-logic L: L defines D_2^{**} iff $L \in S5_\diamond$.

COROLLARY 2.7 ([11]). The logic rte$S5^M$ (resp. cm$S5^M$, e$S5^M$, m$S5^M$, r$S5^M$, $S5^M$) is the smallest rte- (resp. cm-, congruential, monotonic, regular, normal) modal logic defining the logics D_2, D_2^*, D_2^-, and D_2^{**}.

Taking into account the above Corollary, we see that to find differences between logics defining respective discussive logics one has to search for modal logics that are weaker than rte$S5^M$. There are considered ([11]) the weakest modal logics defining respectively D_2^*, D_2^-, and D_2^{**}. In the case of these modal logics, we do not have all theses of S5 that begin with '$\Diamond$'.

3 THE SMALLEST MODAL LOGICS DEFINING D_2^*, D_2^-, D_2^{**}

3.1 *Logics* A, A*, A⁻, *and* A×

Let A, A^*, A^-, and $A^\times$ be the smallest logics defining D_2, D_2^*, D_2^-, and D_2^{**}, respectively. We define the following set of modal formulae:

$$\begin{aligned}
\text{Gen}&:=\{\varphi \in \text{For}_m : \exists_{A\in D_2}\ \varphi = \ulcorner\Diamond A^{\bullet}\urcorner\}\\
&=\{\ulcorner\Diamond A^{\bullet}\urcorner \in \text{For}_m : A \in D_2\},\\
\text{Gen}^*&:=\{\varphi \in \text{For}_m : \exists_{A\in D_2^*}\ \varphi = \ulcorner\Diamond A^{*}\urcorner\}\\
&=\{\ulcorner\Diamond A^{*}\urcorner \in \text{For}_m : A \in D_2^*\},\\
\text{Gen}^\frown&:=\{\varphi \in \text{For}_m : \exists_{A\in D_2^-}\ \varphi = \ulcorner\Diamond A^{\frown}\urcorner\}\\
&=\{\ulcorner\Diamond A^{\frown}\urcorner \in \text{For}_m : A \in D_2^-\},\\
\text{Gen}^\times&:=\{\varphi \in \text{For}_m : \exists_{A\in D_2^{**}}\ \varphi = \ulcorner\Diamond A^{\times}\urcorner\}\\
&=\{\ulcorner\Diamond A^{\times}\urcorner \in \text{For}_m : A \in D_2^{**}\}.
\end{aligned}$$

LEMMA 3.1 ([11]). Every modal logic defining D_2 (resp. D_2^*, D_2^- and D_2^{**}) includes the set Sub(Gen) (resp. Sub(Gen^*), Sub($\text{Gen}^\frown$), Sub($\text{Gen}^\times$)).

Let Ax_{PL} be the set of modal formulae such that the pair $\langle \text{Ax}_{PL}, \{(\text{mp})\}\rangle$ is an axiomatization of PL.

FACT 3.2 ([11]). A (resp. A^*, A^-, $A^\times$) is the smallest modal logic including the set Gen (resp. Gen^*, $Gen^\frown$, $Gen^\times$). Consequently, A (resp. A^*, A^-, $A^\times$) is axiomatized by the sum of sets Ax_{PL}, $(rep^\Box)$, and Sub(Gen) (resp. Sub(Gen^*), Sub($Gen^\frown$), Sub($Gen^\times$)) and (mp) as the only rule.

COROLLARY 3.3 ([11]). Every two logics among A, A^*, A^-, and $A^\times$ cross each other.

From facts 2.4–2.6 we obtain:

FACT 3.4 ([10, 11]). The logic A is not an rte-logic, so $A \subsetneq rteS5^M$. Moreover, none of the logics A^*, A^-, and $A^\times$ is an rte-logic.

3.2 *Simplified axiomatisations of the considered JaÂlJkowski-like discussive logics*

Although Fact 3.2 gives an axiomatisations of logics A, A^*, A^-, and $A^\times$, it is not elegant since the sets Gen, $Gen^\frown$, Gen^* and $Gen^\times$ are infinite (other constituents of sums constituting axiomatisations of the considered modal logics can be easily replaced by respective finite sets). We recall Kotas's method of axiomatisation of D_2, since it can also be adopted to finally give axiomatisations of the considered modal logics.

For any rule R on For_m we define the following rules $R^\Diamond$ and $R^\Box$ on For_m:

$$R^\Diamond := \{\langle \Diamond\varphi_1, \ldots, \Diamond\varphi_n, \Diamond\psi\rangle : \langle \varphi_1, \ldots, \varphi_n, \psi\rangle \in R\},$$
$$R^\Box := \{\langle \Box\varphi_1, \ldots, \Box\varphi_n, \Box\psi\rangle : \langle \varphi_1, \ldots, \varphi_n, \psi\rangle \in R\}.$$

Moreover, for any set of rules $\mathcal{R}$ on For_m we put $\mathcal{R}^\Diamond := \{ R^\Diamond : R \in \mathcal{R} \}$ and $\mathcal{R}^\Box := \{ R^\Box : R \in \mathcal{R} \}$.

Now, let Ax^{fin}_{Taut} be any finite axiomatization of **Taut** with (mp) and (sb). Next we consider the following rules:

$$\Box\varphi \;/\; \varphi \qquad (nec^{-1})$$
$$\Diamond\varphi \;/\; \varphi \qquad (pos^{-1})$$

In [12] a set $\mathbf{M\text{-}S5} := \{\varphi \in \mathrm{For_m} : \Diamond\varphi \in \mathbf{S5}\}$ was considered. Adopting axiomatisation given in [4] we see that for the case where '$\Diamond$' is a primitive symbol of the language it has the following form:

FACT 3.5 ([4]). 1. The set $\mathbf{M\text{-}S5}$ is axiomatized by the sum of sets $\Box\mathrm{Ax}^{\mathrm{fin}}_{\mathrm{Taut}}$, $(rep^{\Box})$, $\{\Box K, \Box T, \Box 5\}$, and the rules (sb), (nec^{-1}), (pos^{-1}), (nec)$^{\Box}$, (mp)$^{\Box}$.

2. The set $\Box\mathbf{S5}$ is axiomatized by the sum of the sets $\Box\mathrm{Ax}^{\mathrm{fin}}_{\mathrm{Taut}}$, $(rep^{\Box})$, $\{\Box K, \Box T, \Box 5\}$, and the rules (sb), (nec)$^{\Box}$, (mp)$^{\Box}$.

It appears that unmodalizing functions used in proofs of facts 2.4–2.6 are variants of the function used in [4]. Let $(-)^{\circ}\colon \mathrm{For_m} \longrightarrow \mathrm{For^d}$ be a function such that for any $\varphi \in \mathrm{For_m}$:

(f) $(\Diamond\varphi)^{\circ} = (p \vee \neg p) \wedge^{\mathrm{d}} \varphi^{\circ}$,

(g) $(\Box\varphi)^{\circ} = \neg((p \vee \neg p) \wedge^{\mathrm{d}} \neg \varphi^{\circ})$,

and other conditions stay as in the definition of the function o_1.

Now we have

LEMMA 3.6 ([4]). 1. For any $A \in \mathrm{For^d}$, if $A \in \mathbf{D_2}$, then $A^{\bullet} \in \mathbf{M\text{-}S5}$.

2. For any $\phi \in \mathrm{For_m}$, if $\varphi \in \mathbf{M\text{-}S5}$, then $\varphi^{\circ} \in \mathbf{D_2}$.

Let us recall the following notation (see [10]). For any $\Gamma \subseteq \mathrm{For^d}$ and any translation \$ from $\mathrm{For^d}$ into $\mathrm{For_m}$ we put

$$\Gamma^{\Diamond\$} := \{\ulcorner\Diamond A^{\$}\urcorner \in \mathrm{For_m} : A \in \Gamma\}.$$

Of course, for $\$ = \bullet$ we have Gen $= \mathbf{D}_2^{\Diamond\bullet}$.

Moreover, for any rule R on $\mathrm{For^d}$ we define the following rule $R^{\Diamond\$}$ on $\mathrm{For_m}$:

$$R^{\Diamond\$} := \{\langle \varphi_1, \dots, \varphi_n, \psi\rangle : \exists_{A_1,\dots,A_n,B\in\mathrm{For^d}}\ \varphi_1 = \ulcorner\Diamond A_1^{\$}\urcorner, \dots, \varphi_n = \ulcorner\Diamond A_n^{\$}\urcorner, \psi = \ulcorner\Diamond B^{\$}\urcorner \text{ and } \langle A_1, \dots, A_n, B\rangle \in R\}.$$

Thus, for any $A_1, \dots, A_n, B \in \mathrm{For^d}$:

$$\langle A_1, \dots, A_n, B\rangle \in R \text{ iff } \langle \Diamond A_1^{\$}, \dots, \Diamond A_n^{\$}, \Diamond B^{\$}\rangle \in R^{\Diamond\$}.$$

For $\mathcal{R}$ being a set of rules on $\mathrm{For_m}$ let $\mathcal{R}^{\diamond s} := \{R^{\diamond s} : R \in \mathcal{R}\}$.

Similarly as in the case of modal logics, for all sets X and $\mathcal{A}$ of discussive formulae and any set of rules $\mathcal{R}$ in $\mathrm{For^d}$ we say that the pair $\langle \mathcal{A}, \mathcal{R}\rangle$ is an **axiomatization** of X iff X is the smallest set including $\mathcal{A}$ and closed under all rules from $\mathcal{R}$.

FACT 3.7 ([10]). Let $\langle \mathcal{A}, \{(\mathrm{mp_d})\}\rangle$ be an axiomatization of $\mathsf{D_2}$. Then $\langle \mathrm{Ax_{PL}} \cup (rep^{\Box}) \cup \mathcal{A}^{\diamond\bullet}, \{(\mathrm{mp_d})^{\diamond\bullet}, (\mathrm{mp})\}\rangle$ and $\langle \mathrm{Ax_{PL}} \cup (rep^{\Box}) \cup \mathcal{A}^{\diamond\bullet}, \{(RC), (\mathrm{mp})\}\rangle$ are axiomatizations of A. Consequently, A is the smallest modal logic which includes the set $\mathcal{A}^{\diamond\bullet}$ and is closed under the rule $(\mathrm{mp_d})^{\diamond\bullet}$ (resp. (RC)).

One can extend the above lemma to a theorem (see [10, Fact 4.2]) that can be used to obtain an axiomatisation of the logic A. We can use Kotas's axiomatisation [4, 6] of $\mathsf{D_2}$. To be able to express Kotas's result, we recall his abbreviation:

$$p \to^1_s q := \neg((r \vee \neg r) \wedge^{\mathrm{d}} \neg(\neg p \vee q))$$

THEOREM 3.8 ([4]). *The logic* $\mathsf{D_2}$ *is axiomatised by the sum of the sets* $(\Box\mathrm{Ax^{fin}_{Taut}})^{\circ}$, $(\Box(rep^{\Box}))^{\circ}$, $\{(\Box K)^{\circ}, (\Box T)^{\circ}, (\Box 5)^{\circ}\}$, *and the formulae* $\ulcorner(p \mathbin{\S} q)^{\bullet\circ} \to^1_s (p \mathbin{\S} q)\urcorner$ *and* $\ulcorner(p \mathbin{\S} q) \to^1_s (p \mathbin{\S} q)^{\bullet\circ}\urcorner$, *for* $\S \in \{\wedge^{\mathrm{d}}, \vee, \to^{\mathrm{d}}, \leftrightarrow^{\mathrm{d}}\}$, *and the rules* $(\mathrm{sb})^{\circ}$, $(\mathrm{nec}^{-1})^{\circ}$, $(\mathrm{pos}^{-1})^{\circ}$, $(\mathrm{nec})^{\Box\circ}$, $(\mathrm{mp}_{\to^1_s})$, $(\mathrm{mp})^{\Box\circ}$.

Using translations $(-)^*$ and $(-)^{\circ_1}$ (resp. $(-)^{\frown}$ and $(-)^{\circ_2}$; $(-)^{\circ_3}$ and $(-)^{\times}$) we extend Kotas' Lemma 3.6 to the case of $\mathsf{D_2^*}$, $\mathsf{D_2^-}$, and $\mathsf{D_2^{**}}$.

LEMMA 3.9. 1. (a) For any $A \in \mathrm{For^d}$, if $A \in \mathsf{D_2^*}$, then $A^* \in$ M-S5.

(b) For any $\phi \in \mathrm{For_m}$, if $\varphi \in$ M-S5, then $\varphi^{\circ_1} \in \mathsf{D_2^*}$.

2. (a) For any $A \in \mathrm{For^d}$, if $A \in \mathsf{D_2^-}$, then $A^{\frown} \in$ M-S5.

(b) For any $\phi \in \mathrm{For_m}$, if $\varphi \in$ M-S5, then $\varphi^{\circ_2} \in \mathsf{D_2^-}$.

3. (a) For any $A \in \mathrm{For^d}$, if $A \in \mathsf{D_2^{**}}$, then $A^{\times} \in$ M-S5.

(b) For any $\phi \in \mathrm{For_m}$, if $\varphi \in$ M-S5, then $\varphi^{\circ_3} \in \mathsf{D_2^{**}}$.

We can easily obtain axiomatisations of D_2^*, D_2^- and D_2^{**}. Now we will use respective abbreviations for those logics:

$$p \to_s^2 q := \neg(\neg(\neg p \vee q) \wedge^{\mathrm{d}} (r \vee \neg r))$$
$$p \to_s^3 q := (\neg(\neg p \vee q) \to^{\mathrm{d}} \neg(r \vee \neg r))$$

We see that in the next theorem, in the case of D_2^{**} one can use either $\to_s^1$ or $\to_s^2$. Besides, the implication $\to_s^3$ can be used in each case.

THEOREM 3.10. 1. *The logic* D_2^* *is axiomatised by the sum of the sets* $(\Box\mathrm{Ax}^{\mathrm{fin}}_{\mathrm{Taut}})^{\circ_1}$, $\Box((rep^{\Box}))^{\circ_1}$, $\{(\Box K)^{\circ_1}, (\Box T)^{\circ_1}, (\Box 5)^{\circ_1}\}$, *and* $\ulcorner(p \,\S\, q)^{*\circ_1} \to_s^2 (p \,\S\, q)\urcorner$ *and* $\ulcorner(p \,\S\, q) \to_s^2 (p \,\S\, q)^{*\circ_1}\urcorner$, *for* $\S \in \{\wedge^{\mathrm{d}}, \vee, \to^{\mathrm{d}}, \leftrightarrow^{\mathrm{d}}\}$ *as axioms, and the rules* $(\mathrm{sb})^{\circ_1}$, $(\mathrm{nec}^{-1})^{\circ_1}$, $(\mathrm{pos}^{-1})^{\circ_1}$, $(\mathrm{nec})^{\Box\circ_1}$, $(\mathrm{mp}_{\to_s^2})$, $(\mathrm{mp})^{\Box\circ_1}$.

2. *The logic* D_2^- *is axiomatised by the sum of three sets* $(\Box\mathrm{Ax}^{\mathrm{fin}}_{\mathrm{Taut}})^{\circ_2}$, $(\Box(rep^{\Box}))^{\circ_2}$, $\{(\Box K)^{\circ_2}, (\Box T)^{\circ_2}, (\Box 5)^{\circ_2}\}$ *and the formulae* $\ulcorner(p \,\S\, q)^{\frown\circ_2} \to_s^3 (p \,\S\, q)\urcorner$ *and* $\ulcorner(p \,\S\, q) \to_s^3 (p \,\S\, q)^{\frown\circ_2}\urcorner$, *for* $\S \in \{\wedge^{\mathrm{d}}, \vee, \to^{\mathrm{d}}, \leftrightarrow^{\mathrm{d}}\}$, *as axioms, and the rules* $(\mathrm{sb})^{\circ_2}$, $(\mathrm{nec}^{-1})^{\circ_2}$, $(\mathrm{pos}^{-1})^{\circ_2}$, $(\mathrm{nec})^{\Box\circ_2}$, $(\mathrm{mp}_{\to_s^3})$, $(\mathrm{mp})^{\Box\circ_2}$.

3. *The logic* D_2^{**} *is axiomatised by the sum of three sets* $(\Box\mathrm{Ax}^{\mathrm{fin}}_{\mathrm{Taut}})^{\circ_3}$, $(\Box(rep^{\Box}))^{\circ_3}$, $\{(\Box K)^{\circ_3}, (\Box T)^{\circ_3}, (\Box 5)^{\circ_3}\}$, *and* $\ulcorner(p \,\S\, q)^{\times\circ_3} \to_s^2 (p \,\S\, q)\urcorner$ *and* $\ulcorner(p \,\S\, q) \to_s^2 (p \,\S\, q)^{\times\circ_3}\urcorner$, *for* $\S \in \{\wedge^{\mathrm{d}}, \vee, \to^{\mathrm{d}}, \leftrightarrow^{\mathrm{d}}\}$ *as axioms, and the rules* $(\mathrm{sb})^{\times\circ_3}$, $(\mathrm{nec}^{-1})^{\circ_3}$, $(\mathrm{pos}^{-1})^{\circ_3}$, $(\mathrm{nec})^{\Box\circ_3}$, $(\mathrm{mp}_{\to_s^2})$, $(\mathrm{mp})^{\Box\circ_3}$.

The obtained axiomatisations of the logics D_2^*, D_2^- and D_2^{**} can be used to give axiomatisations of logics A^*, A^-, and $\mathsf{A}^\times$. Fact 3.7 can be extended to any axiomatization of D_2 and also of D_2^-, D_2^*, and D_2^{**}. In such a way we obtain an extension of the mentioned Fact 4.2 from [10] to the case of D_2^-, D_2^* and D_2^{**}.

THEOREM 3.11. *Let* $\langle \mathcal{A}, \mathcal{R} \rangle$ *be an axiomatization of* D_2 *(resp.* D_2^-, D_2^*, D_2^{**}*).*

1. *The pairs*

- $\langle \mathrm{Ax}_{\mathsf{PL}} \cup (rep^{\Box}) \cup \mathcal{A}^{\Diamond\bullet}, \mathcal{R}^{\Diamond\bullet} \cup \{(\mathrm{mp})\} \rangle$,
- $\langle \mathrm{Ax}_{\mathsf{PL}} \cup (rep^{\Box}) \cup \mathcal{A}^{\Diamond *}, \mathcal{R}^{\Diamond *} \cup \{(\mathrm{mp})\} \rangle$,
- $\langle \mathrm{Ax}_{\mathsf{PL}} \cup (rep^{\Box}) \cup \mathcal{A}^{\Diamond\frown}, \mathcal{R}^{\Diamond\frown} \cup \{(\mathrm{mp})\} \rangle$,
- $\langle \mathrm{Ax}_{\mathsf{PL}} \cup (rep^{\Box}) \cup \mathcal{A}^{\Diamond\times}, \mathcal{R}^{\Diamond\times} \cup \{(\mathrm{mp})\} \rangle$

are axiomatizations of the logics A, A*, A⁻, *and* A×, *respectively.*

2. *The logic* A *(resp.* A*, A⁻, A×*) is the smallest modal logic which includes the set* $\mathcal{A}^{\diamond\bullet}$ *(resp.* $\mathcal{A}^{\diamond\cdot}$, $\mathcal{A}^{\diamond-}$, $\mathcal{A}^{\diamond\times}$*) and is closed under all rules from the set* $\mathcal{R}^{\diamond\bullet}$ *(resp.* $\mathcal{R}^{\diamond\cdot}$, $\mathcal{R}^{\diamond-}$ *and* $\mathcal{R}^{\diamond\times}$*).*

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