Quantum damped oscillator II: Bateman's Hamiltonian vs. 2D Parabolic Potential Barrier.

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Abstract

We show that quantum Bateman's system which arises in the quantization of a damped harmonic oscillator is equivalent to a quantum problem with 2D parabolic potential barrier known also as 2D inverted isotropic oscillator. It turns out that this system displays the family of complex eigenvalues corresponding to the poles of analytical continuation of the resolvent operator to the complex energy plane. It is shown that this representation is more suitable than the hyperbolic one used recently by Blasone and Jizba.

1 Introduction

In the previous paper [1] we have investigated a quantization of a 1D damped harmonic oscillator defined by the following equation of motion

$$\ddot{x} + 2\gamma\dot{x} + \kappa x \;=\; 0 \;, \tag{1.1}$$

where $\gamma > 0$ denotes the damping constant. To quantize this system we follow an old observation of Bateman [2] and double the number of degrees of freedom, that is together with (1.1) we consider

$$\ddot{y} - 2\gamma\dot{y} + \kappa y \;=\; 0 \;, \tag{1.2}$$

i.e. an amplified oscillator. The detailed historical review of the Bateman idea may be found in [3]. For more recent papers see e.g. [4] and [5]. The enlarged system is a Hamiltonian one and it is governed by the following classical Bateman Hamiltonian:

$$H(x,y,p_x,p_y) = p_xp_y - \gamma(xp_x - yp_y) + \omega^2xy \;, \tag{1.3}$$

where $\omega = \sqrt{\kappa - \gamma^2}$.[1] Now, performing a linear canonical transformation $(x,y,p_x,p_y) \longrightarrow (x_1,x_2,p_1,p_2)$:

$$x_1 \;=\; \frac{p_y}{\sqrt{\omega}} \;, \qquad p_1 \;=\; -\sqrt{\omega}\, y \tag{1.4}$$

$$x_2 \;=\; -\sqrt{\omega}\, x \;, \qquad p_2 \;=\; -\frac{p_x}{\sqrt{\omega}} \;, \tag{1.5}$$

[1]Throughout the paper we shall consider the underdamped case, i.e. $\kappa > \gamma^2$.

and applying a standard symmetric Weyl ordering one obtains the following quantum Hamiltonian

$$\hat{H} = \omega\, \hat{\mathbf{p}} \wedge \hat{\mathbf{x}} - \gamma\, \hat{\mathbf{p}} \odot \hat{\mathbf{x}}\ , \tag{1.6}$$

where $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$, $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2)$ and we define two natural operations:

$$\hat{\mathbf{p}} \wedge \hat{\mathbf{x}} = \hat{p}_1\hat{x}_2 - \hat{p}_2\hat{x}_1\ , \qquad \hat{\mathbf{p}} \odot \hat{\mathbf{x}} = \hat{\mathbf{x}} \odot \hat{\mathbf{p}} = \frac{1}{2}\sum_{k=1}^{2}(\hat{x}_k\hat{p}_k + \hat{p}_k\hat{x}_k)\ .$$

Note, that $[\hat{\mathbf{p}} \wedge \hat{\mathbf{x}}, \hat{\mathbf{p}} \odot \hat{\mathbf{x}}] = 0$. This operator was carefully analyzed in [1]. In particular it was shown that the family of complex eigenvalues

$$\hat{H}|\mathfrak{f}^{\pm}_{nl}\rangle = E^{\pm}_{nl}|\mathfrak{f}^{\pm}_{nl}\rangle\ , \tag{1.7}$$

with

$$E^{\pm}_{nl} = \hbar\omega l \pm i\hbar\gamma(|l| + 2n + 1)\ , \tag{1.8}$$

found already by Feshbach and Tikochinsky [6], corresponds to the poles of the resolvent operator $\hat{\mathrm{R}}(\hat{H}, z) = (\hat{H} - z)^{-1}$. Therefore, the corresponding generalized eigenvectors $|\mathfrak{f}^{\pm}_{nl}\rangle$ may be interpreted as resonant states of the Bateman system. It shows that dissipation of energy is directly related to the presence of resonances.

In the present paper we continue to study this system but in a different representation. Let us observe that performing the linear canonical transformation $(\mathbf{x}, \mathbf{p}) \longrightarrow (\mathbf{u}, \mathbf{v})$:

$$\mathbf{x} = \frac{\gamma\mathbf{u} - \mathbf{v}}{\sqrt{2\gamma}}\ , \qquad \mathbf{p} = \frac{\gamma\mathbf{u} + \mathbf{v}}{\sqrt{2\gamma}}\ , \tag{1.9}$$

one obtains for the Hamiltonian

$$\hat{H} = \omega\, \hat{\mathbf{v}} \wedge \hat{\mathbf{u}} + \hat{H}_{\rm iho}\ , \tag{1.10}$$

where

$$\hat{H}_{\rm iho} = \frac{1}{2}\,(\hat{\mathbf{v}}^2 - \gamma^2\hat{\mathbf{u}}^2)\ , \tag{1.11}$$

represents a Hamiltonian of a 2D isotropic inverted harmonic oscillator (iho) or, equivalently, a 2D potential barrier $-\gamma^2\hat{\mathbf{u}}^2$. Now, $\omega\, \hat{\mathbf{v}} \wedge \hat{\mathbf{u}}$ generates an SO(2) rotation on (u_1, u_2)–plane. Therefore, in the rotating frame the problem is described by the following Schrödinger equation

$$i\hbar\dot{\psi}_{\rm rf} = \hat{H}_{\rm iho}\psi_{\rm rf}\ , \tag{1.12}$$

where the rotating frame wave function $\psi_{\rm rf} = \exp(i\omega\, \hat{\mathbf{v}} \wedge \hat{\mathbf{u}}t/\hbar)\, \psi$.

A 1D inverted (or reversed) oscillator was studied by several authors in various contexts [7, 8, 9, 10, 11, 12, 13]. Recently, this system was studied in the context of dissipation in quantum mechanics and a detailed analysis of its resonant states was performed in [14]. The present paper is mostly devoted to analysis of a 2D iho. We find its energy eigenvectors and show that they are singular when one continues energy into complex plane. The complex poles correspond to resonant states of the 2D potential barrier [15, 16].

Finally, we analyze the Bateman system in the hyperbolic representation used recently in [5] by Blasone and Jizba. It turns out that this representation in not appropriate to describe resonant states and hence the family of generalized complex eigenvalues found in [5] is not directly related to the spectral properties of the Bateman Hamiltonian. We stress that it does not prove that these representation are physically inequivalent. Clearly they are. Different representation lead to different mathematical realization which is connected with different functional spaces and different boundary conditions. These may lead to different analytical properties and hence some representation may display resonant states while others not.

From the mathematical point of view the natural language to analyze the spectral properties of Bateman's system is the so called rigged Hilbert space approach to quantum mechanics [17, 18, 19, 20]. We show (cf. Section 4) that there are two dense subspaces $\Phi_\pm \in L^2(\mathbb{R}^2_{\mathbf{u}})$ such that restriction of the unitary group $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ to $\Phi_\pm$ does no longer define a group but gives rise to two semigroups: $\hat{U}_-(t) = \hat{U}(t)|_{\Phi_-}$ defined for $t \geq 0$ and $\hat{U}_+(t) = \hat{U}(t)|_{\Phi_+}$ defined for $t \leq 0$. It means that the quantum damped oscillator corresponds to the following Gel'fand triplets:

$$\Phi_\pm \subset L^2(\mathbb{R}^2_{\mathbf{u}}) \subset \Phi'_\pm \ , \tag{1.13}$$

and hence it serves as a simple example of Arno Bohm theory of resonances [20].

2 2D inverted oscillator and complex eigenvalues

2.1 2D harmonic oscillator

Let us briefly recall the spectral properties of the 2D harmonic oscillator (see e.g. [21, 22]):

$$\hat{H}_{\rm ho} = -\frac{\hbar^2}{2}\,\triangle_2 + \frac{\Omega^2}{2}\,\rho^2 \ , \tag{2.1}$$

where the 2D Laplacian reads

$$\triangle_2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} \ , \tag{2.2}$$

and (ρ,φ) are standard polar coordinates on (u_1,u_2)–plane. The corresponding eigenvalue problem

$$\hat{H}_{\rm ho}\psi^{\rm ho}_{nl} = \varepsilon^{\rm ho}_{nl}\psi^{\rm ho}_{nl} \ , \tag{2.3}$$

is solved by

$$\psi^{\rm ho}_{nl}(\rho,\varphi) = R_{nl}(\rho)\Phi_l(\varphi) \ , \tag{2.4}$$

where

$$\Phi_l(\varphi) = \frac{e^{il\varphi}}{\sqrt{2\pi}} \ , \qquad l = 0, \pm 1, \pm 2, \dots \ , \tag{2.5}$$

and the radial functions

$$R_{nl}(\rho) = C_{nl}\,(\sqrt{\Omega/\hbar}\,\rho)^{|l|}\,\exp(-\Omega\rho^2/2\hbar)\,{}_1F_1(-n, |l|+1, \Omega\,\rho^2/\hbar) \ , \tag{2.6}$$

where the normalization constant reads as follows

$$C_{nl} = \frac{\sqrt{2\Omega/\hbar}}{|l|!}\sqrt{\frac{(n+|l|)!}{n!}} \ , \qquad n = 0, 1, 2, \dots \ . \tag{2.7}$$

Finally, the corresponding eigenvalues $\varepsilon_{nl}^{\rm ho}$ are given by the following formula

$$\varepsilon_{nl}^{\rm ho} = \hbar\Omega(|l| + 2n + 1) \ . \tag{2.8}$$

Note, that using well known relation between confluent hypergeometric function ${}_1F_1$ and generalized Laguerre polynomials [24, 26]

$$L_n^\mu(z) = \frac{\Gamma(n+\mu+1)}{\Gamma(n+1)\Gamma(\mu+1)}\, {}_1F_1(-n, \mu+1, z) \ , \tag{2.9}$$

one may rewrite R_{nl} alternatively as follows

$$R_{nl}(\rho) = \sqrt{2\Omega/\hbar}\sqrt{\frac{n!}{(n+|l|)!}}\, (\sqrt{\Omega/\hbar}\,\rho)^{|l|}\, \exp(-\Omega\rho^2/2\hbar)\, L_n^{|l|}(\Omega\,\rho^2/\hbar) \ . \tag{2.10}$$

It is evident that the family $\psi_{ln}^{\rm ho}$ is orthonormal

$$\langle\, \psi_{nl}^{\rm ho} | \psi_{n'l'}^{\rm ho}\, \rangle = \delta_{nn'}\delta_{ll'} \ , \tag{2.11}$$

and complete

$$\sum_{n=0}^\infty \sum_{l=-\infty}^\infty \overline{\psi_{nl}^{\rm ho}(\rho,\varphi)}\, \psi_{nl}^{\rm ho}(\rho',\varphi') = \frac{1}{\rho}\, \delta(\rho-\rho')\delta(\varphi-\varphi') \ , \tag{2.12}$$

where $\langle\ |\ \rangle$ denotes the standard scalar product in the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}_+, \rho d\rho) \otimes L^2([0,2\pi), d\varphi) \ . \tag{2.13}$$

2.2 Scaling and complex eigenvalues

Let us note that $\hat{H}_{\rm iho}$ defined in (1.11) corresponds to the Hamiltonian of the harmonic oscillator with purely imaginary frequency $\Omega = \pm i\gamma$. The connection with a harmonic oscillator may be established by the following scaling operator

$$\hat{V}_\lambda := \exp\left(\frac{\lambda}{\hbar}\, \hat{\mathbf{v}} \odot \hat{\mathbf{u}}\right) \ , \tag{2.14}$$

with $\lambda \in \mathbb{R}$. Using commutation relation $[\hat{u}_k, \hat{v}_l] = i\hbar\delta_{kl}$, this operator may be rewritten as follows

$$\hat{V}_\lambda = e^{-i\lambda}\, \exp\left(-i\lambda\, \rho\frac{\partial}{\partial\rho}\right) \ , \tag{2.15}$$

and therefore it defines a complex dilation, i.e. the action of $\hat{V}_\lambda$ on a function $\psi = \psi(\rho,\varphi)$ is given by

$$\hat{V}_\lambda\, \psi(\rho,\varphi) = e^{-i\lambda}\, \psi(e^{-i\lambda}\, \rho, \varphi) \ . \tag{2.16}$$

In particular one easily finds:

$$\hat{V}_\lambda\, \hat{H}_{\rm iho}\, \hat{V}_\lambda^{-1} = \frac{1}{2}\, e^{2i\lambda}\, \left(-\hbar^2\triangle_2 - e^{-4i\lambda}\gamma^2\rho^2\right) \ . \tag{2.17}$$

Therefore, for $e^{4i\lambda} = -1$, i.e. $\lambda = \pm\pi/4$, one has

$$\hat{V}_{\pm\pi/4}\,\hat{H}_{\rm iho}\,\hat{V}^{-1}_{\pm\pi/4} = \pm i\,\left(-\frac{\hbar^2}{2}\,\triangle_2 + \frac{\gamma^2}{2}\,\rho^2 \right)\ . \tag{2.18}$$

Now, let us introduce

$$\mathfrak{u}^{\pm}_{nl} = \hat{V}_{\mp\pi/4}\,\psi^{\rm ho}_{nl}\ , \tag{2.19}$$

that is

$$\mathfrak{u}^{\pm}_{nl}(\rho,\varphi) = \sqrt{\pm i}\,\psi^{\rm ho}_{nl}(\sqrt{\pm i}\rho,\varphi)\ . \tag{2.20}$$

It is evident that

$$\hat{H}_{\rm iho}\,\mathfrak{u}^{\pm}_{nl} = \varepsilon^{\pm}_{nl}\,\mathfrak{u}^{\pm}_{nl}\ , \tag{2.21}$$

where

$$\varepsilon^{\pm}_{nl} = \pm i\,\varepsilon^{\rm ho}_{nl} = \pm i\hbar\gamma(|l| + 2n + 1)\ . \tag{2.22}$$

We stress that $\hat{V}_\lambda$ is not unitary (for $\lambda \in \mathbb{R}$) and hence in general $\hat{V}_\lambda\psi$ does not belong to $\mathcal{H}$ even for $\psi \in \mathcal{H}$. In particular the generalized eigenvectors $\mathfrak{u}^{\pm}_{nl}$ do not belong to $\mathcal{H}$ (the radial part $R_{nl}(\sqrt{\pm i}\rho)$ is not an element from $L^2(\mathbb{R}_+, \rho d\rho)$).

Proposition 1 *Two families of generalized eigenvectors $\mathfrak{u}^{\pm}_{nl}$ satisfy the following properties:*

1. *they are bi-orthonormal*

$$\int_0^{2\pi}\!\!\int_0^{\infty} \overline{\mathfrak{u}^{\pm}_{nl}(\rho,\varphi)}\,\mathfrak{u}^{\mp}_{n'l'}(\rho,\varphi)\,\rho d\rho\,d\varphi = \delta_{nn'}\delta_{ll'}\ , \tag{2.23}$$

2. *they are bi-complete*

$$\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty} \overline{\mathfrak{u}^{\pm}_{nl}(\rho,\varphi)}\,\mathfrak{u}^{\mp}_{nl}(\rho',\varphi') = \frac{1}{\rho}\,\delta(\rho-\rho')\delta(\varphi-\varphi')\ . \tag{2.24}$$

The proof follows immediately from orthonormality and completness of oscillator eigenfunctions $\psi^{\rm ho}_{nl}$.

3 Spectral properties of the Bateman Hamiltonian

Now, we solve the corresponding spectral problem for the Bateman Hamiltonian (1.10). Note that $\hat{H}$ is bounded neither from below nor from above and hence its spectrum $\sigma(\hat{H}) = (-\infty,\infty)$. The corresponding generalized eigenvectors satisfy

$$\hat{H}\psi_{\varepsilon,l} = E_{\varepsilon,l}\psi_{\varepsilon,l}\ , \tag{3.1}$$

where $l \in \mathbb{Z}$ and $\varepsilon \in \mathbb{R}$. Assuming the following factorized form of $\psi_{\varepsilon,l}$

$$\psi_{\varepsilon,l}(\rho,\varphi) = R_{\varepsilon,l}(\rho)\Phi_l(\varphi)\ , \tag{3.2}$$

one has

$$E_{\varepsilon,l} = \omega\hbar l + \varepsilon\ , \tag{3.3}$$

with

$$\hat{H}_{\rm iho} R_{\varepsilon,l} = \varepsilon R_{\varepsilon,l} \ . \tag{3.4}$$

The above equation rewritten in terms of (ρ,φ)-variables takes the following form

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{|l|^2}{\rho^2} + \frac{\gamma^2}{\hbar^2}\,\rho^2 + \frac{2\varepsilon}{\hbar^2} \right) R_{\varepsilon,l} = 0 \ , \tag{3.5}$$

and its solution reads as follows

$$R_{\varepsilon,l}(\rho) = N_{\varepsilon,l}\, (\sqrt{i\gamma/\hbar}\,\rho)^{|l|}\, \exp(-i\gamma\rho^2/2\hbar)\, {}_1F_1(a,|l|+1,i\gamma\rho^2/\hbar) \ , \tag{3.6}$$

with

$$a = \frac{1}{2}\left(|l| + 1 - \frac{\varepsilon}{i\gamma\hbar} \right) \ . \tag{3.7}$$

The normalization factor $N_{\varepsilon,l}$ is chosen such that

$$\int_0^\infty \overline{R_{\varepsilon,l}(\rho)}\, R_{\varepsilon',l}(\rho)\, \rho d\rho = \delta(\varepsilon-\varepsilon') \ . \tag{3.8}$$

It turns out (see Appendix A) that

$$N_{\varepsilon,l} = \sqrt{\frac{\gamma}{\pi|l|!}}\, (-i)^a\, \Gamma(a) \ , \tag{3.9}$$

with a defined in (3.7).

Proposition 2 *The family of generalized eigenvectors $\psi_{\varepsilon,l}$ satisfy the following properties:*

1. *orthonormality*

$$\int_0^{2\pi}\!\!\int_0^\infty \overline{\psi_{\varepsilon,l}(\rho,\varphi)}\, \psi_{\varepsilon',l'}(\rho,\varphi)\, \rho d\rho\, d\varphi = \delta(\varepsilon-\varepsilon')\delta_{ll'} \ , \tag{3.10}$$

2. *completeness*

$$\sum_{l=-\infty}^{\infty} \int_{-\infty}^\infty d\varepsilon\ \overline{\psi_{\varepsilon,l}(\rho,\varphi)}\, \psi_{\varepsilon,l}(\rho',\varphi') = \frac{1}{\rho}\,\delta(\rho-\rho')\delta(\varphi-\varphi') \ . \tag{3.11}$$

Let us define another family of generalized energy eigenvectors

$$\chi_{\varepsilon,l} = \mathcal{T}\psi_{\varepsilon,l} \ , \tag{3.12}$$

where the anti-unitary operator $\mathcal{T}$ is defined as follows

$$\mathcal{T}\psi_{\varepsilon,l}(\rho,\varphi) = \overline{R_{\varepsilon,l}(\rho)}\Phi_l(\varphi) \ . \tag{3.13}$$

It easy to show that Bateman's Hamiltonian $\hat{H}$ is $\mathcal{T}$–invariant

$$\mathcal{T}\hat{H}\mathcal{T}^\dagger = \hat{H} \ . \tag{3.14}$$

Moreover, if $\psi(t) = \hat{U}(t)\psi_0$, then $\mathcal{T}\psi(t) = \hat{U}(-t)(\mathcal{T}\psi_0)$, which shows that $\mathcal{T}$ is a time reversal operator. Finally, Proposition 2 gives rise to the following spectral representation of the Bateman Hamiltonian

$$\hat{H} = \sum_{l=-\infty}^{+\infty} \int_{-\infty}^\infty d\varepsilon\, E_{\varepsilon,l} |\psi_{\varepsilon,l}\,\rangle\langle\,\psi_{\varepsilon,l}| = \sum_{l=-\infty}^{+\infty} \int_{-\infty}^\infty d\varepsilon\, E_{\varepsilon,l} |\chi_{\varepsilon,l}\,\rangle\langle\,\chi_{\varepsilon,l}| \ , \tag{3.15}$$

with $E_{\varepsilon,l}$ defined in (3.3).

4 Analyticity, resolvent and resonances

Now, we continue energy eigenfunctions $\psi_{\varepsilon,l}$ and $\chi_{\varepsilon,l}$ into complex ε-plane. Note, that ε-dependence enters $R_{\varepsilon,l}$ via the normalization factor $N_{\varepsilon,l}$ and the function ${}_1F_1(a,|l|+1,i\gamma\rho^2/\hbar)$ (a is ε-dependent, see (3.7)). It is well known (see e.g. [26]) that confluent hypergeometric function ${}_1F_1(a,b,z)$ defines a convergent series for all values of complex parameters a, b and z provided $a \neq -n$ and $b \neq -m$, with m and n positive integers. Moreover, if $a = -n$ and $b \neq -m$, then ${}_1F_1(a,b,z)$ is a polynomial of degree n in z. In our case $b = |l|+1$ which is never negative and hence ${}_1F_1(a,|l|+1,i\gamma\rho^2/\hbar)$ is analytic in ε. However, it is no longer true for the normalization constant $N_{\varepsilon,l}$ given by (3.9). The Γ-function has simple poles at $a = -n$, with $n = 0,1,2,\ldots$, which correspond to

$$\varepsilon = \varepsilon_{nl} = i\gamma\hbar(|l| + 2n + 1) \ , \tag{4.1}$$

on the complex ε-plane. On the other hand the time-reversed function $\overline{R_{\varepsilon,l}}$ has simple poles at $\varepsilon = \overline{\varepsilon_{nl}} = -\varepsilon_{nl}$.

It is, therefore, natural to introduce two classes of functions that respect these analytical properties of $\psi_{\varepsilon,l}$ and $\chi_{\varepsilon,l}$. Recall [28] that a smooth function $f = f(\varepsilon)$ is in the Hardy class from above $\mathcal{H}^2_+$ (from below $\mathcal{H}^2_-$) if $f(\varepsilon)$ is a boundary value of an analytic function in the upper, i.e. $\mathrm{Im}\,\varepsilon \geq 0$ (lower, i.e. $\mathrm{Im}\,\varepsilon \leq 0$) half complex ε-plane vanishing faster than any power of ε at the upper (lower) semi-circle $|\varepsilon| \to \infty$. Define

$$\Phi_- := \left\{ \phi \in \mathcal{S}(\mathbb{R}^2_{\mathbf{u}}) \,\middle|\, f(\varepsilon) := \langle\, \chi_{\varepsilon,l} | \phi \,\rangle \in \mathcal{H}^2_- \right\} \ , \tag{4.2}$$

and

$$\Phi_+ := \left\{ \phi \in \mathcal{S}(\mathbb{R}^2_{\mathbf{u}}) \,\middle|\, f(\varepsilon) := \langle\, \psi_{\varepsilon,l} | \phi \,\rangle \in \mathcal{H}^2_+ \right\} \ , \tag{4.3}$$

where $\mathcal{S}(\mathbb{R}^2_{\mathbf{u}})$ denotes the Schwartz space [29], i.e. the space of $C^\infty(\mathbb{R}^2_{\mathbf{u}})$ functions $f = f(u_1,u_2)$ vanishing at infinity ($|\mathbf{u}| \longrightarrow \infty$) faster than any polynomial.

It is evident from (3.13) that

$$\Phi_+ = \mathcal{T}(\Phi_-) \ . \tag{4.4}$$

The main result of this section consists in the following

Theorem 1 *For any function $\phi^\pm \in \Phi_\pm$ one has*

$$\phi^+ = \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} \mathfrak{u}^+_{nl} \langle\, \mathfrak{u}^-_{nl} | \phi^+ \,\rangle \ , \tag{4.5}$$

and

$$\phi^- = \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} \mathfrak{u}^-_{nl} \langle\, \mathfrak{u}^+_{nl} | \phi^- \,\rangle \ . \tag{4.6}$$

For the proof see Appendix B. The above theorem implies the following spectral resolutions of the Hamiltonian:

$$\hat{H}_- \equiv \hat{H}\Big|_{\Phi_-} = \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} E^-_{nl}\, |\mathfrak{u}^-_{nl}\rangle\langle\mathfrak{u}^+_{nl}| \ , \tag{4.7}$$

and

$$\hat{H}_+ \equiv \hat{H}\Big|_{\Phi_+} = \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty} E^+_{nl}\,|\mathfrak{u}^+_{nl}\rangle\langle\mathfrak{u}^-_{nl}|\ . \tag{4.8}$$

In the above formulae $E^{\pm}_{nl}$ is given by (1.8).

The same techniques may be applied for the resolvent operator

$$\hat{R}(z,\hat{H}) = \frac{1}{\hat{H}-z}\ . \tag{4.9}$$

One obtains

$$\begin{aligned}\hat{R}_+(z,\hat{H}) &= \sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\varepsilon}{E_{\varepsilon,l}-z}\,|\psi_{\varepsilon,l}\rangle\langle\psi_{\varepsilon,l}|\ \Big|_{\Phi_+}\\ &= \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}\frac{1}{E^+_{n,l}-z}\,|\mathfrak{u}^-_{nl}\rangle\langle\,\mathfrak{u}^+_{nl}|\ ,\end{aligned} \tag{4.10}$$

on Φ_+, and

$$\begin{aligned}\hat{R}_-(z,\hat{H}) &= \sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\varepsilon}{E_{\varepsilon,l}-z}\,|\chi_{\varepsilon,l}\rangle\langle\chi_{\varepsilon,l}|\ \Big|_{\Phi_-}\\ &= \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}\frac{1}{E^-_{n,l}-z}\,|\mathfrak{u}^+_{nl}\rangle\langle\,\mathfrak{u}^-_{nl}|\ ,\end{aligned} \tag{4.11}$$

on Φ_-. Hence, $\hat{R}_+(z,\hat{H})$ has poles at $z=E^+_{nl}$, and $\hat{R}_-(z,\hat{H})$ has poles at $z=E^-_{nl}$. As usual eigenvectors $\mathfrak{u}^+_{nl}$ and $\mathfrak{u}^-_{nl}$ corresponding to poles of the resolvent are interpreted as resonant states. Note, that the Cauchy integral formula implies

$$\hat{P}^+_{nl}\ :=\ |\mathfrak{u}^-_{nl}\rangle\langle\,\mathfrak{u}^+_{nl}|\ =\ \frac{1}{2\pi i}\oint_{\Gamma^+_{nl}}\hat{R}_+(z,\hat{H})dz\ , \tag{4.12}$$

where Γ^+_{nl} is a clockwise closed curve that encircles the singularity $z=E^+_{nl}$. Similarly,

$$\hat{P}^-_{nl}\ :=\ |\mathfrak{u}^+_{nl}\rangle\langle\,\mathfrak{u}^-_{nl}|\ =\ \frac{1}{2\pi i}\oint_{\Gamma^-_{nl}}\hat{R}_-(z,\hat{H})dz\ , \tag{4.13}$$

where Γ^-_{nl} is an anti-clockwise closed curve that encircles the singularity $z=E^-_{nl}$. One easily shows that

$$\hat{P}^{\pm}_{nl}\cdot\hat{P}^{\pm}_{n'l'} = \delta_{nn'}\delta_{ll'}\,\hat{P}^{\pm}_{nl}\ , \tag{4.14}$$

and hence the spectral decompositions of (4.7) and (4.7) may be written as follows:

$$\hat{H}_{\pm} = \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty} E^{\pm}_{nl}\,\hat{P}^{\pm}_{nl}\ . \tag{4.15}$$

Finally, let us note, that restriction of the unitary group $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ to $\Phi_\pm$ no longer defines a group. It gives rise to two semigroups:

$$\hat{U}_-(t) := e^{-i\hat{H}_- t/\hbar} \; : \; \Phi_- \longrightarrow \Phi_- \; , \qquad \text{for} \quad t \geq 0 \; , \tag{4.16}$$

and

$$\hat{U}_+(t) := e^{-i\hat{H}_+ t/\hbar} \; : \; \Phi_+ \longrightarrow \Phi_+ \; , \qquad \text{for} \quad t \leq 0 \; . \tag{4.17}$$

Using (4.15) and the formula for E_{nl} one finds:

$$\phi^-(t) = \hat{U}_-(t)\phi^- = \sum_{l=-\infty}^{\infty} e^{-i\omega l t} \sum_{n=0}^{\infty} e^{-\gamma(2n+|l|+1)t}\, \hat{P}^-_{nl} \; , \tag{4.18}$$

for $t \geq 0$, and

$$\phi^+(t) = \hat{U}_+(t)\phi^+ = \sum_{l=-\infty}^{\infty} e^{-i\omega l t} \sum_{n=0}^{\infty} e^{\gamma(2n+|l|+1)t}\, \hat{P}^+_{nl} \; , \tag{4.19}$$

for $t \leq 0$. We stress that ϕ^-_t (ϕ^+_t) does belong to $L^2(\mathbb{R}^2_{\mathbf{u}})$ also for $t < 0$ ($t > 0$). However, $\phi^-_t \in \Phi_-$ ($\phi^+_t \in \Phi_+$) only for $t \geq 0$ ($t \leq 0$). This way the irreversibility enters the dynamics of the reversed oscillator by restricting it to the dense subspace $\Phi_\pm$ of $L^2(\mathbb{R}^2_{\mathbf{u}})$.

From the mathematical point of view the above construction gives rise to so called rigged Hilbert spaces (or Gel'fand triplets) [17, 18, 19, 20]:

$$\Phi_- \subset \mathcal{H} \subset \Phi'_- \; , \tag{4.20}$$

and

$$\Phi_+ \subset \mathcal{H} \subset \Phi'_+ \; , \tag{4.21}$$

where $\Phi'_\pm$ denote dual spaces, i.e. linear functionals on $\Phi_\pm$. Note, that generalized eigenvectors $\mathbf{u}^\pm_{nl}$ are not elements from $\mathcal{H}$. However, they do belong to $\Phi'_\pm$. The first triplet $(\Phi_-, \mathcal{H}, \Phi'_-)$ is corresponds to the evolution for $t \geq 0$, whereas the second one $(\Phi_+, \mathcal{H}, \Phi'_+)$ corresponds to the evolution for $t \leq 0$.

5 Bateman's system in hyperbolic representation

In a recent paper [5] Blasone and Jizba used another representation. They transform Bateman's Hamiltonian (1.3) into the following form

$$H(y_1, y_2, w_1, w_2) = \frac{1}{2}(w_1^2 - w_2^2) - \gamma(y_1 w_2 + y_2 w_1) + \frac{1}{2}\omega^2(y_1^2 - y_2^2) \; , \tag{5.1}$$

with the new positions

$$y_1 = \frac{x+y}{\sqrt{2}} \; , \qquad y_2 = \frac{x-y}{\sqrt{2}} \; , \tag{5.2}$$

and new canonical momenta

$$w_1 = \frac{p_x + p_y}{\sqrt{2}} \; , \qquad w_2 = \frac{p_x - p_y}{\sqrt{2}} \; . \tag{5.3}$$

Now, introducing hyperbolic coordinates (ϱ, u):

$$y_1 = \varrho \cosh u \ , \qquad y_2 = \varrho \sinh u \ , \tag{5.4}$$

the canonical quantization leads to the following Hamiltonian defined on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, \varrho d\varrho) \otimes L^2(\mathbb{R}, du)$:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm iho} \ , \tag{5.5}$$

with

$$\hat{H}_0 = -\frac{\hbar^2}{2} \Box_2 + \frac{\omega^2}{2} \varrho^2 \ , \tag{5.6}$$

and the iho part $\hat{H}_{\rm iho}$

$$\hat{H}_{\rm iho} = i\gamma\hbar \frac{\partial}{\partial u} \ . \tag{5.7}$$

In the above formulae $\Box_2$ denotes the 2D wave operator, that is

$$\Box_2 = \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} = \frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial \varrho} - \frac{1}{\varrho^2}\frac{\partial^2}{\partial u^2} \ . \tag{5.8}$$

Clearly, in the (ϱ, u) variables the formula for $\hat{H}_{\rm iho}$ considerably simplifies and $\hat{H}_{\rm iho}$ represents the generator of SO(1,1) hyperbolic rotation on the (y_1, y_2)–plane. In this particular representation $\hat{H}_{\rm iho}$ defines a self-adjoint operator on $L^2(\mathbb{R}, du)$. The corresponding eigen-problem is immediately solved

$$\hat{H}_{\rm iho}\Phi_\nu = \gamma\hbar\nu\, \Phi_\nu \ , \tag{5.9}$$

with $\Phi_\nu(u) = e^{-i\nu u}/\sqrt{2\pi}$, and hence it reproduces the continuous spectrum of 2D iho $\sigma(\hat{H}_{\rm iho}) = (-\infty, \infty)$. However, there is a crucial difference between elliptic (ρ, φ) and (ϱ, u) representations. The generalized eigenvectors Φ_ν may be analytically continued on the entire complex ν–plane. Therefore, the hyperbolic representation does not display the family of resonances corresponding to complex eigenvalues ε_{nl} defined in (4.1). Of course one may by hand fix the values of ν to $\nu = i(2n + |l| + 1)$ but then the corresponding discrete Φ_{nl} family is neither bi-orthogonal nor bi-complete (cf. Proposition 2).

To show how the complex eigenvalues of Blasone and Jizba [5] appear let us consider $\hat{H}_0$ defined in (5.6). Note, that $\hat{H}_0$ resembles 2D harmonic oscillator given by (2.1). There is, however, crucial difference between $\hat{H}_0$ and $\hat{H}_{\rm ho}$. The hyperbolic operator '$-\Box_2$', contrary to the elliptic one '$-\triangle_2$', is not positively defined and hence it allows for negative eigenvalues. It is clear, since in the elliptic (ρ, φ)–representation $\hat{H}_0 = i\omega\hbar\, \partial_\varphi$ defines a self-adjoint operator on $L^2([0, 2\pi), d\varphi)$ with purely discrete spectrum $\omega\hbar l$ $(l \in \mathbb{Z})$. Now, the spectral analysis of the Bateman Hamiltonian represented by (5.5) is straightforward:

$$\hat{H}\psi_{\epsilon\nu} = \mathcal{E}_{\epsilon\nu}\psi_{\epsilon\nu} \ , \tag{5.10}$$

with

$$\mathcal{E}_{\epsilon\nu} = \epsilon + \gamma\hbar\nu \ , \tag{5.11}$$

and the following factorized form of $\psi_{\epsilon\nu}$:

$$\psi_{\epsilon\nu}(\varrho, u) = \mathcal{R}_{\epsilon\nu}(\varrho)\, \Phi_\nu(u) \ . \tag{5.12}$$

The radial function $\mathcal{R}_{\epsilon\nu}$ solves

$$\hat{H}_0 \mathcal{R}_{\epsilon\nu} = \epsilon\, \mathcal{R}_{\epsilon\nu} \ , \tag{5.13}$$

and in analogy to (3.6) it is given by

$$\mathcal{R}_{\epsilon\nu}(\varrho) = N_{\epsilon\nu}\, (\sqrt{\omega/\hbar}\varrho)^{i\nu} \exp(-\omega\varrho^2/2\hbar)\, U(b, i\nu+1, \omega\varrho^2/\hbar) \ , \tag{5.14}$$

with

$$b = \frac{1}{2}\left(i\nu + 1 - \frac{\epsilon}{\hbar\omega}\right) \ . \tag{5.15}$$

In (5.14) we have used instead of the standard confluent hypergeometric function ${}_1F_1$ so called Tricomi function U (see e.g. [27]).[2] It is defined by

$$U(a,c,z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, {}_1F_1(a,c,z) + \frac{\Gamma(c-1)}{\Gamma(a)}\, z^{1-c}\, {}_1F_1(a-c+1, 2-c, z) \ . \tag{5.16}$$

A Tricomi function $U(a,c,z)$ is an analytical function of its arguments and for $a = -n$ ($n = 0,1,2,\ldots$) it defines a polynomial of order n in z:

$$U(-n, \alpha+1, z) = (-1)^n n!\, L_n^\alpha(z) \ . \tag{5.17}$$

Moreover, using the following property of U (an analog of (B.5) for ${}_1F_1$)

$$U(a,c,z) = z^{1-c}\, U(1+a-c, 2-c, z) \ , \tag{5.18}$$

one obtains

$$\int_0^\infty \overline{\mathcal{R}_{\epsilon\nu}(\varrho)}\mathcal{R}_{\epsilon\nu}(\varrho)\, \varrho d\varrho = \frac{\hbar}{2\omega}\, |N_{\epsilon\nu}|^2 \int_0^\infty z^{i\nu}\, e^{-z} U^2(b, i\nu+1, z)\, dz \ , \tag{5.19}$$

with $z = \omega\varrho^2/\hbar$. Now for $b = -n$, $\mathcal{R}_{\epsilon\nu}$ belongs to the Hilbert space $L^2(\mathbb{R}_+, \varrho d\varrho)$. It implies

$$\epsilon = \hbar\omega(2n + 1 + i\nu) \ . \tag{5.20}$$

and hence it reproduces discrete spectrum '$\hbar\omega \times$ integer' iff $i\nu = l = 0, \pm1, \pm2, \ldots$. Now, using (5.17), (5.19) and

$$\int_0^\infty e^{-z} z^\alpha L_n^\alpha(z) L_m^\alpha(z)\, dz = \frac{1}{n!}\, \Gamma(n+\alpha+1)\, \delta_{nm} \ , \tag{5.21}$$

with $\alpha > -1$, one obtains the following family $\mathcal{R}_{nl} \in L^2(\mathbb{R}_+, \varrho d\varrho)$:[3]

$$\mathcal{R}_{nl}(\varrho) = \sqrt{\frac{2\omega/\hbar}{n!\Gamma(n+l+1)}}\ (\sqrt{\omega/\hbar}\varrho)^l \exp(-\omega\varrho^2/2\hbar)\, L_n^l(\omega\varrho^2/\hbar) \ , \tag{5.22}$$

[2] Actually, in [27] (and also in [23]) this function is denoted by G. We follow the notation of Abramowitz and Stegun [26].

[3] There is a difference in normalization factor in formulae (37) in [5]. It follows from slightly different definition of L_n^α.

with $n = 0, 1, 2, \dots$, and $l = 0, 1, 2, \dots$, satisfying

$$\int_0^\infty \overline{\mathcal{R}_{nl}(\varrho)} \mathcal{R}_{n'l}(\varrho) \, \varrho d\varrho = \delta_{nn'} \ . \tag{5.23}$$

We stress that the family $\mathcal{R}_{nl}$ is defined for $l \geq 0$ only (otherwise it can not be normalized!). Finally, defining

$$\phi_{nl}(\varrho, u) = \frac{1}{\sqrt{2\pi}} \mathcal{R}_{nl}(\varrho) \, e^{-ul} \ , \tag{5.24}$$

one has

$$\hat{H} \phi_{nl} = \mathcal{E}_{nl} \, \phi_{nl} \ , \tag{5.25}$$

with

$$\mathcal{E}_{nl} = \hbar\omega(2n + l + 1) - i\hbar\gamma \, l \ . \tag{5.26}$$

There is, however, crucial difference between families $\mathfrak{u}^{\pm}_{nl}(\rho, \varphi)$ and $\phi_{nl}(\varrho, u)$. The family $|\mathfrak{u}^{\pm}_{nl}\rangle$ corresponds to the poles of $\psi_{\varepsilon,l}$ from (3.2). No such correspondence holds for $|\phi_{nl}\rangle$ and $\psi_{\epsilon\nu}$ from (5.12). In particular there is no analog of Theorem 1 for $|\phi_{nl}\rangle$. Moreover, $\mathcal{E}_{nl}$ contrary to E_{nl} from (1.8) does not fit the formula for complex eigenvalues of Feshbach and Tikochinsky [6] (see detailed discussion in [1]). It defines simply another family which is however not directly related to the spectral properties of the Bateman Hamiltonian.

Appendix A

To compute $N_{\varepsilon,l}$ in (3.6) let us analyze the quantity $I_\varepsilon = \int_0^\infty \overline{R_{\varepsilon,l}(\rho)} \, R_{\varepsilon,l}(\rho) \, \rho d\rho$. Clearly, this integral diverges ($I_\varepsilon = \delta(0)$), however, its structure enables one the calculation of $N_{\varepsilon,l}$. One has

$$I_\varepsilon = \frac{1}{2\gamma} \, |N_{\varepsilon,l}|^2 \int_0^\infty z^{|l|} \, {}_1F_1(a, |l| + 1, iz) \, {}_1F_1(\overline{a}, |l| + 1, -iz) \, dz \ , \tag{A.1}$$

where we defined $z = \gamma\rho^2$. Now, the integral in (A.1) belongs to the general class

$$J = \int_0^\infty e^{-\lambda z} \, z^{\mu - 1} \, {}_1F_1(\alpha, \mu, kz) \, {}_1F_1(\alpha', \mu, k'z) \, dz \ , \tag{A.2}$$

given by the following formula (see Appendix f in [23]):

$$J = \Gamma(\mu) \, \lambda^{\alpha - \alpha' - \mu} (\lambda - k)^{-\alpha} (\lambda - k')^{-\alpha'} \, {}_2F_1 \left(\alpha, \alpha', \mu; \frac{kk'}{(\lambda - k)(\lambda - k')} \right) \ . \tag{A.3}$$

Using the above formula with $\lambda = 0$, $\mu = |l| + 1$, $\alpha = a$, $\alpha' = \overline{a}$ and $k = -k' = i$ one finds

$$I_\varepsilon = \frac{1}{2\gamma} \, |N_{\varepsilon,l}|^2 \, (-i)^{-a} \, \overline{(-i)^{-a}} \, {}_2F_1(a, \overline{a}, |l| + 1; 1) \ . \tag{A.4}$$

Finally, noting that

$${}_2F_1(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \ ,$$

one has

$$I_\varepsilon = \frac{|l|!}{2\gamma}\, |N_{\varepsilon,l}|^2\, (-i)^{-a}\, \overline{(-i)^{-a}}\, \frac{\Gamma(0)}{\Gamma(a)\Gamma(\overline{a})} \ . \tag{A.5}$$

Therefore, comparing (A.5) with $I_\varepsilon = \delta(0)$ one finds

$$N_{\varepsilon,l} = \sqrt{\frac{\gamma}{\pi |l|!}}\, (-i)^a\, \Gamma(a) \ , \tag{A.6}$$

which proves (3.9).

Appendix B

Due to the Gel'fand-Maurin spectral theorem [17, 18] an arbitrary function $\phi^+ \in \Phi_+$ may be decomposed with respect to the basis $\psi_{\varepsilon,l}$

$$\phi^+ = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} d\varepsilon\, \psi_{\varepsilon,l} \langle \psi_{\varepsilon,l} | \phi^+ \rangle \ . \tag{B.1}$$

Now, since $\langle\, \psi_{\varepsilon,l} | \phi^+ \,\rangle \in \mathcal{H}^2_+$, we may close the integration contour along the upper semi-circle $|\varepsilon| \to \infty$. Applying the Residue Theorem one obtains

$$\phi^+(\rho,\varphi) = 2\pi i \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \mathrm{Res}\, \psi_{\varepsilon,l}(\rho,\varphi)\Big|_{\varepsilon=\varepsilon_{nl}} \langle\, \psi_{\varepsilon,l} | \phi^+ \,\rangle\Big|_{\varepsilon=\varepsilon_{nl}} . \tag{B.2}$$

Using the well known formula for the residuum

$$\mathrm{Res}\, \Gamma(a)\Big|_{a=-n} = \frac{(-1)^n}{n!} \ , \tag{B.3}$$

one obtains

$$\mathrm{Res}\, \psi_{\varepsilon,l}\Big|_{\varepsilon=\varepsilon_{nl}} = \frac{-i}{\sqrt{i^{2n+|l|+1}}} \sqrt{\frac{1}{2\pi\hbar}} \sqrt{\frac{(n+|l|)!}{n!|l|!}}\, \mathfrak{u}^+_{nl} \ . \tag{B.4}$$

Moreover, the analytical function $\overline{\psi_{\varepsilon,l}}$ computed at $\varepsilon = \varepsilon_{nl}$ reads:

$$\overline{\psi_{\varepsilon,l}}\Big|_{\varepsilon=\varepsilon_{nl}} = i^{n+|l|+1} \sqrt{\frac{\gamma}{\pi|l|!}}\, (\sqrt{-i\gamma/\hbar}\, \rho)^{|l|}\, \exp(i\gamma\rho^2/2\hbar)\, {}_1F_1(n+|l|+1, |l|+1, -i\gamma\rho^2/\hbar)\, \overline{\Phi_l(\varphi)} \ .$$

Due to the well known relation [24, 25, 26]

$${}_1F_1(a,b,z) = e^z\, {}_1F_1(b-a, b, -z) \ , \tag{B.5}$$

one finds

$$\overline{\psi_{\varepsilon,l}}\Big|_{\varepsilon=\varepsilon_{nl}} = \sqrt{i^{2n+|l|+1}} \sqrt{\frac{\hbar}{2\pi}} \sqrt{\frac{n!|l|!}{(n+|l|)!}}\, \overline{\mathfrak{u}^-_{nl}} \ . \tag{B.6}$$

and hence the formula (4.5) follows. In a similar way one shows (4.6).

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