RUSSELL'S THEORIES OF EVENTS AND INSTANTS FROM THE PERSPECTIVE OF POINT-FREE ONTOLOGIES IN THE TRADITION OF THE LVOV-WARSAW SCHOOL

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Abstract. We classify two of Bertrand Russell's theories of events within the point-free ontology. The first of such approaches was presented informally by Russell in "The World of Physics and the World of Sense" (Lecture IV in *Our Knowledge of the External World* (1914)). Based on this theory, Russell sketched ways to construct instants as collections of events. This paper formalises Russell's approach from (1914). We will also show that in such a reconstructed theory, we obtain all axioms of Russell's second theory from (1936) and all axioms of Thomason's theory of events from (1989). Russell's work certainly influenced the works of Stanisław Leśniewski, his student Alfred Tarski, and Czesław Lejewski—prominent members of the Lvov-Warsaw School (LWS). We see our work in the tradition of the research of Leśniewski and Tarski. Building on the technical tools developed in this environment and in the spirit of the traditional research of the LWS, we engage here, in particular, with two classic works by Russell on fundamental ontology.

Introduction

In (1914) Russell presented the theory of events and instants in a completely informal way. In Section 2, we will formalize its part about events. We will add the missing assumptions that seem necessary to obtain some of the statements given in (1914). We will also clarify some inaccuracies there. Before that, however, in Section 1, we briefly introduce the very idea of point-free ontology. We will refer to its precursors Russell (1914, 1936), Whitehead (1919, 1920) and de Laguna (1922a,b).

In point 2.1, we will cover the basic notions of Russell's theory from (1914). Then, in point 2.2, we will discuss its analytical assumptions, i.e. those that result from the meanings of primitive concepts and do not postulate the existence of events. In point 2.3, we define two new concepts and, following Russell, we introduce a new axiom. This axiom also does not postulate the existence of events, but we do not include it as an analytical proposition. We also show some consequences of the axioms and definitions adopted here. This part of the theory will be "existentially neutral". In Point 2.4, we will introduce the first "existentially involved" axiom and study its consequences. We define further binary relations regarding events in point 2.5. Only some of them are defined by Russell, some are mentioned but not defined, and one is not found in (1914). We do not introduce any new axioms for these new concepts. We want to show what new interesting statements can be obtained using only the axioms from Section 2.

In Section 3 after Russell, we will present the construction of instants (moments). We will show that some of Russell's requirements for moments could not be obtained by merely applying his assumptions. We must use some new axioms regarding events to meet these requirements.

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In Section 4, we will compare Russell's theory from (1914) with his second theory from (1936), which he has already formally presented. Thanks to the additional assumptions we have adopted, we will show that it is possible to reconstruct the latter theory in the first one. We also analyze the problem of the density of the relation *is before* on instants, presented in (Russell, 1936). Let us add that Russell's theory from (1914) is conceptually much more interesting than his theory from (1936).

Section 5 compares Russell's theories with Thomason's theory from (1989). Moreover, in Section 6, we discuss the influence of Russell and Whitehead on the works of Leśniewski, Tarski and Lejewski—prominent members of the Lvov-Warsaw School. We will compare Russell's construction of instants and Tarski's construction of points from (1929; 1956).

In the final section 7, we present the possibility of another way of defining instants, also outlined by Russell (1914). This approach will be similar to the construction of points shown by Grzegorczyk (1950, 1960) and examined in detail in (Gruszczyński and Pietruszczak, 2018).

To keep easy to read this paper, we move the proofs of some facts to Appendix A. In addition, we include the models used in the article in Appendix B.

1. Briefly about point-free ontology

When discussing space and time in an ontology, it is often assumed that they are made of something indivisible, in other words, of "atomic" or "point" objects. In space, these would be geometric points, and in the case of time, instants or, in other words, moments. When combining time and space, space-time is composed of point events. From the perspective of metaphysics, such an assumption is not intuitive because point objects are not components of the real world (i.e. space, time, and space-time, respectively). However, it is very convenient for formal considerations. In "point ontology", time—treated as a distributive set of instants—is structured as the set $\mathbb{R}$ of real numbers. On the other hand, space is given the structure of the set $\mathbb{R}^3$ of ordered triples of real numbers and space-time the set $\mathbb{R}^4$ of ordered fours of real numbers.

However, one can practice ontology without assuming the existence of point entities in the real world. This approach—initiated by Russell (1914, 1936), Whitehead (1919, 1920) and de Laguna (1922a,b)—is called *point-free ontology*. It does not mean that such an ontology does not consider any objects of a 'point nature'. The latter, however, are to be abstractions constructed from the components of the real world. These abstract objects are necessary to obtain a sufficiently rich theory of time, space, or space-time.

Point-free ontology is supposed to concern the real world. We exclude point objects but also those with the character of lines, surfaces, and pieces. Thus, according to the *Elements* of Euclid, we exclude all objects that either do not have length, width, or hight. We also exclude any "mixtures" of such things.

In the sense of a point-free ontology, space is neither a distributive set of dimensionless points nor their mereological sum. Space is the mereological sum of its chunks called *regions*. For Russell (1914) and Whitehead (1919, 1920) points are abstract creations obtained as set-theoretic constructions made of spatial regions (see also Tarski, 1929, 1956; Gruszczyński and Pietruszczak, 2008, 2009):

> [...] one spatial object may be contained within another, and entirely enclosed by the other. This relation of enclosure, by the help of some very natural hypotheses, will enable us to define a "point" as a certain class of spatial objects, namely all those (as it will turn out in the end) which would naturally be said to contain the point. (Russell, 1914, p. 114)

In another edition of (Russell, 1914) we read:

> This relation of enclosure, by the help of some very natural hypotheses, will enable us to define a "point" as a certain set of spatial objects; roughly speaking,

> the set will consist of all volumes which would naturally be said to contain the point.
>
> It should be observed that Dr. Whitehead's abstract logical methods are applicable equally to psychological space, physical space, time, and space-time.
>
> (Russell, 1922, p. 120)

In the sense of point-free ontology, time is associated with events. Russell treated events as "constituents of the real world", not abstracts. Events are not point objects but must be of some finite extent. Russell defined instants-points as abstract creations based on events.

We can say that point-free ontology's main task is the formal construction of abstract objects such as instants and geometric points. Namely, after adding them, we can successfully apply mathematical physics and point geometry methods. Such practices are necessary to discuss the development of research on the nature of time and space. However, we need to find the proper role of point objects in the ontology.

2. A formalization of Russell's first theory about events

2.1. **The primitive notions.** The theory of events presented in (Russell, 1914) is entirely informal. It has three primitive concepts: one non-relational, *being an event*, and two relational: *is earlier than* and *is (at least partially) simultaneous with*. Thus, reconstructing this theory, we will study the relational structures of the form $\langle \mathsf{U}, \mathsf{E}, \mathsf{S}\rangle$, the components of which correspond to the three primitive concepts of Russell's theory: U is a non-empty set (a universe composed of events); E and S are binary relations on U. So, for example, the formulas '$x \in \mathsf{U}$', '$x \mathrel{\mathsf{E}} y$' and '$x \mathrel{\mathsf{S}} y$' are meant to express: x is an event; x is earlier than y (which can be taken as an abbreviation for 'x lasts before y' or 'x is before y'); x is (at least partially) simultaneous with y.

The meaning of the phrase 'is earlier than' is relatively intuitive if we take it as an abbreviation of 'lasts before'. However, we may have difficulties adequately interpreting the phrase 'is (at least partially) simultaneous with'. Note that the words 'is simultaneous with' itself is ambiguous. Firstly, about events, this phrase can be understood as events occurring at the same time. In this case, Russell speaks of *complete simultaneity* (which is reflexive, symmetric and transitive). Secondly, Russell added in parentheses "at least partially". It suggests that the phrase may be understood in a sense "that the times of the two events overlap". More specifically, two events occur in at least one common instant; see further (3.2). Then this phrase determines a reflexive and symmetrical relation but "will not necessarily be transitive" (Russell, 1914, p. 125).

Remark 2.1. In (1936) Russell, in a formal way, introduced a second theory of events based on other primitive concepts than in (1914). They are *being an event* and the relational concept *wholly precedence*. It can therefore be said that the second theory applies to the structures of form $\langle \mathsf{U}, \mathsf{P}\rangle$, where U is a non-empty universe (composed of events) and P is a binary relation on U. In such structures Russell (1936, p. 348)[1] defines the relations $\mathsf{S} := \bar{\mathsf{P}} \setminus \breve{\mathsf{P}}$ and *begins before* $:= \mathsf{S} \mid \mathsf{P}$.[2] These relations are to correspond to S and E in the structure $\langle \mathsf{U}, \mathsf{E}, \mathsf{S}\rangle$ (see further ($\mathrm{df}_{\mathsf{P}}\,\mathsf{S}$) and ($\star_{\mathsf{E}}$)). In Section 4.1, we will show that the 1914 theory we reconstruct is definitionally equivalent to the essential fragment of the 1936 approach. ◇

[1] We will provide pages of the reprint of (Russell, 1936) in *Logic and Knowledge*, Allen & Unwin, 1956.

[2] For any binary relation R, R_1a and R_2 on U we define the following relations on U. The *product* of R_1 and R_2 is the relation $R_1 \cap R_2$ such that for all $x, y \in \mathsf{U}$: $x \ R_1 \cap R_2 \ y$ iff $x \, R_1 \, y \wedge x \, R_1 \, y$. The *difference* between R_1 and R_2 is the relation $R_1 \setminus R_2$ such that for all $x, y \in \mathsf{U}$: $x \ R_1 \setminus R_2 \ y$ iff $x \, R_1 \, y \wedge \neg\, x \, R_1 \, y$. The *complement* of R is the relation $\bar{R}$ such that: $x \, \bar{R} \, y$ iff $\neg\, x \, R \, y$. So $R_1 \setminus R_2 = R_1 \cap \bar{R}_2$. The *converse relation* of R is the relation $\breve{R}$ such that: $x \, \breve{R} \, y$ iff $y \, R \, x$. So $R = \breve{\breve{R}}$. The *relative product* of R_1 and R_2 is the relation $R_1 \mid R_2$ such that: $x \ R_1 \mid R_2 \ y$ iff $\exists_{u \in \mathsf{U}}(x \, R_1 \, u \wedge u \, R_2 \, y)$. So $(R_1 \mid R_2)\breve{} = \breve{R}_2 \mid \breve{R}_1$.

2.2. **Basic analytical assumptions.** There is no doubt that the phrase "is earlier than" is irreflexive, asymmetry and transitive and that these are its analytical properties (in the sense that fluent English speakers think so). So these properties will also have the relation E:[3]

$$\forall_{x\in\mathsf{U}}\ \neg\, x\,\mathsf{E}\,x, \qquad (\mathtt{irr}_{\mathsf{E}})$$

a1 $$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{E}\,y \implies \neg\, y\,\mathsf{E}\,x), \qquad (\mathtt{as}_{\mathsf{E}})$$

a2 $$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{E}\,y \wedge y\,\mathsf{E}\,z \implies x\,\mathsf{E}\,z). \qquad (\mathtt{t}_{\mathsf{E}})$$

Thus, E is a strict partial order. Notice that $(\mathtt{irr}_{\mathsf{E}})$ follows directly from axiom a1 (also $(\mathtt{as}_{\mathsf{E}})$ follows from $(\mathtt{irr}_{\mathsf{E}})$ and $(\mathtt{t}_{\mathsf{E}})$).

The analytical properties of the relation S are its reflexivity and symmetry:

$$\forall_{x\in\mathsf{U}}\ x\,\mathsf{S}\,x, \qquad (\mathtt{r}_{\mathsf{S}})$$

a3 $$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{S}\,y \implies y\,\mathsf{S}\,x). \qquad (\mathtt{s}_{\mathsf{S}})$$

Formula $(\mathtt{r}_{\mathsf{S}})$ will result from a1 and a4 and our definitions (see p. 5).

2.3. **Two auxiliary relations and the fourth axiom.** In addition to the two primitive relations, Russell (1914) uses several auxiliary relations, defined by E and/or S. At this point, we will introduce two such relationships. The first is the L relationship, expressed by the phrase 'is later than', which is the conversion of 'is earlier than'. Therefore, L is defined by $\mathsf{L} := \breve{\mathsf{E}}$, i.e., for all $x, y \in \mathsf{U}$ we put:

$$x\,\mathsf{L}\,y \iff y\,\mathsf{E}\,x\,. \qquad (\mathrm{df}\,\mathsf{L})$$

Directly from $(\mathtt{irr}_{\mathsf{E}})$, $(\mathtt{as}_{\mathsf{E}})$, $(\mathtt{t}_{\mathsf{E}})$ and $(\mathrm{df}\,\mathsf{L})$, L is irreflexive, asymmetric and transitive:

$$\forall_{x\in\mathsf{U}}\ \neg\, x\,\mathsf{L}\,x\,, \qquad (\mathtt{irr}_{\mathsf{L}})$$

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{L}\,y \implies \neg\, y\,\mathsf{L}\,x), \qquad (\mathtt{as}_{\mathsf{L}})$$

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{L}\,y \wedge y\,\mathsf{L}\,z \implies x\,\mathsf{L}\,z). \qquad (\mathtt{t}_{\mathsf{L}})$$

Furthermore, using E and S, Russell defined an auxiliary relation P of *total precedence* ("wholly precedes"):

> When one event is earlier than but not simultaneous with another, we shall say that it "wholly precedes" the other. (Russell, 1914, p. 19)

So P is defined by $\mathsf{P} := \mathsf{E} \setminus \mathsf{S} = \mathsf{E} \cap \bar{\mathsf{S}}$, i.e., for all $x, y \in \mathsf{U}$ we put:

$$x\,\mathsf{P}\,y \iff x\,\mathsf{E}\,y \wedge \neg\, x\,\mathsf{S}\,y. \qquad (\mathrm{df}\,\mathsf{P})$$

Notice that if $y\,\breve{\mathsf{P}}\,x$, i.e. $x\,\mathsf{P}\,y$, then we can say that y *is wholly after* x.

From $(\mathtt{as}_{\mathsf{E}})$ and $(\mathrm{df}\,\mathsf{P})$ we obtain the asymmetry of P; and so also its irreflexivity:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{P}\,y \implies \neg\, y\,\mathsf{P}\,x), \qquad (\mathtt{as}_{\mathsf{P}})$$

$$\forall_{x\in\mathsf{U}}\ \neg\, x\,\mathsf{P}\,x\,. \qquad (\mathtt{irr}_{\mathsf{P}})$$

About the relation P Russell (1914) assumes the following condition:

> [...] we know that of two events which are not simultaneous, there must be one which wholly precedes the other, and in that case the other cannot also wholly precede the one. (Russell, 1914, p. 119)

The second part of the above quote tells about the asymmetry of P, expressed by $(\mathtt{as}_{\mathsf{P}})$, and the first part of it states the connexity of P concerning S, i.e .:

$$\forall_{x,y\in\mathsf{U}}(\neg\, x\,\mathsf{S}\,y \implies x\,\mathsf{P}\,y \vee y\,\mathsf{P}\,x). \qquad (\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$$

[3] We have adopted the following convention for marking some formulas. There may be a label to the right of a given formula to indicate what the formula is saying. Furthermore, if the label appears to the right of a given formula, it means that it is assumed as an axiom of a formalized theory, and the given digit indicates the next number of the axiom. We have adopted as axioms only those formulas that are not derivable from other premises.

In virtue of $(\mathtt{s}_{\mathsf{S}})$, condition $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$ is definitionally equivalent to the following, which states the connexity of E concerning S:

a4 $$\forall_{x,y\in\mathsf{U}}(\neg\, x\,\mathsf{S}\,y \implies x\,\mathsf{E}\,y \vee y\,\mathsf{E}\,x). \qquad (\mathtt{con}^{\mathsf{S}}_{\mathsf{E}})$$

It is obviously that from $(\mathtt{irr}_{\mathsf{E}})$ and $(\mathtt{con}^{\mathsf{S}}_{\mathsf{E}})$ (as well from $(\mathtt{irr}_{\mathsf{P}})$ and $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$) we obtain $(\mathtt{r}_{\mathsf{S}})$, i.e. the reflexivity of S.

Directly from (df P) and $(\mathtt{s}_{\mathsf{S}})$ we obtain the converse implication to $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$; and so:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{S}\,y \iff \neg\, x\,\mathsf{P}\,y \wedge \neg\, y\,\mathsf{P}\,x). \qquad (\mathrm{df}_{\mathsf{P}}\,\mathsf{S})$$

2.4. **The first "existentially involved" axiom.** In analyzing the family of instants, Russell (1914) makes three existentially involved assumptions. At this point, we will explore the first of them.

From Russell's words:

> […] consider all the events which are simultaneous with a given event, and do not begin later, i.e. are not wholly after anything simultaneous with it […]
>
> (Russell, 1914, p. 119)

in the light of the symmetry of S, we can conclude that for Russell the equality $\mathsf{L} = \breve{\mathsf{P}} \mid \mathsf{S}$ holds, i.e.:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{L}\,y \iff \exists_{z\in\mathsf{U}}(z\,\mathsf{P}\,x \wedge z\,\mathsf{S}\,y)). \qquad (\star_{\mathsf{L}})$$

Hence, by (df L) and $(\mathtt{s}_{\mathsf{S}})$, the equality $\mathsf{E} = \mathsf{S} \mid \mathsf{P}$ holds, i.e.:

a5 $$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{E}\,y \iff \exists_{z\in\mathsf{U}}(x\,\mathsf{S}\,z \wedge z\,\mathsf{P}\,y)). \qquad (\star_{\mathsf{E}})$$

For simplicity, we take $(\star_{\mathsf{E}})$ as an axiom of the theory, not $(\star_{\mathsf{L}})$.

Remark 2.2. The "$\Leftarrow$"-part of $(\star_{\mathsf{E}})$ (in short: $(\star^{\Leftarrow}_{\mathsf{E}})$) is "existentially neutral". Only its "$\Rightarrow$"-part (in short: $(\star^{\Rightarrow}_{\mathsf{E}})$) is "existentially involved". In Section 3.3, we will show that $(\star^{\Rightarrow}_{\mathsf{E}})$ will follow from axiom a6, which is also "existentially involved". ⋄

It is obvious that $(\star^{\Leftarrow}_{\mathsf{E}})$ is definitionally equivalent to:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{S}\,z \wedge z\,\mathsf{E}\,y \implies x\,\mathsf{E}\,y \vee z\,\mathsf{S}\,y). \qquad (2.1)$$

Hence, by $(\mathtt{con}^{\mathsf{S}}_{\mathsf{E}})$, we obtain:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{S}\,z \wedge \neg\, x\,\mathsf{E}\,y \wedge \neg\, z\,\mathsf{S}\,y \implies y\,\mathsf{E}\,z).$$

So, by $(\mathtt{s}_{\mathsf{S}})$ and $(\mathtt{as}_{\mathsf{E}})$, we obtain:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{S}\,z \wedge \neg\, x\,\mathsf{E}\,y \wedge x\,\mathsf{S}\,y \wedge \neg\, x\,\mathsf{E}\,z \implies y\,\mathsf{S}\,z). \qquad (2.2)$$

Furthermore, notice that from $(\mathtt{as}_{\mathsf{E}})$ and $(\star^{\Leftarrow}_{\mathsf{E}})$ we have $\mathsf{P} \mid \mathsf{E} \subseteq \bar{\mathsf{S}}$, i.e.:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{P}\,z \wedge z\,\mathsf{E}\,y \implies \neg\, x\,\mathsf{S}\,y).$$

We can prove that having $(\mathtt{t}_{\mathsf{E}})$, $(\mathtt{s}_{\mathsf{S}})$, (df P), $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$ and $(\star_{\mathsf{E}})$ we get that the complement $\bar{\mathsf{E}}$ of E is transitive; and so E is co-transitive:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{E}\,y \implies x\,\mathsf{E}\,z \vee z\,\mathsf{E}\,y). \qquad (\mathtt{ct}_{\mathsf{E}})$$

Russell does not mention this fact in (1914). However, having the transitivity of P, which Russell assumed, is necessary. Also, from $(\mathtt{ct}_{\mathsf{E}})$, we will get other facts needed to reconstruct the family of instants, which cannot be accepted by taking only the transitivity of P.

For any binary relation R on U and any $x \in \mathsf{U}$ we put:

$$\overrightarrow{R}`x := \{u \in \mathsf{U} : u\,R\,x\} \quad \text{and} \quad \overleftarrow{R}`x := \{u \in \mathsf{U} : x\,R\,u\}.$$

From $(\mathtt{ct}_{\mathsf{E}})$ and $(\mathtt{irr}_{\mathsf{E}})$ for all $x, y \in \mathsf{U}$ we obtain:

$$\neg\, x\,\mathsf{E}\,y \iff \overrightarrow{\mathsf{E}}`y \subseteq \overrightarrow{\mathsf{E}}`x \iff \overleftarrow{\mathsf{E}}`x \subseteq \overleftarrow{\mathsf{E}}`y\,. \qquad (2.3)$$

Now we show that from (df P), $(\mathtt{s}_{\mathsf{S}})$, $(\mathtt{as}_{\mathsf{E}})$, $(\mathtt{ct}_{\mathsf{E}})$ and $(\star_{\mathsf{E}}^{\Leftarrow})$ we obtain the transitivity of P. Firstly, notice that from $(\star_{\mathsf{E}}^{\Leftarrow})$ and $(\mathtt{s}_{\mathsf{S}})$ we have:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{P}\,y \wedge \neg\, z\,\mathsf{E}\,y \implies \neg\, x\,\mathsf{S}\,z).$$

Secondly, since from (df P) we have $\mathsf{P} \subseteq \mathsf{E}$, in the light of $(\mathtt{ct}_{\mathsf{E}})$ we obtain:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{P}\,y \wedge \neg\, z\,\mathsf{E}\,y \implies x\,\mathsf{E}\,z).$$

So from both of above conditions and (df P), we obtain:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{P}\,y \wedge \neg\, z\,\mathsf{E}\,y \implies x\,\mathsf{P}\,z). \tag{$\mathtt{t}_{\mathsf{P}}^{\bar{\mathsf{E}}}$}$$

Thirdly, from $(\mathtt{t}_{\mathsf{P}}^{\bar{\mathsf{E}}})$ and $(\mathtt{as}_{\mathsf{E}})$ we get the following condition (which is stronger than the condition expressing the transitivity of P):

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{P}\,y \wedge y\,\mathsf{E}\,z \implies x\,\mathsf{P}\,z). \tag{$\mathtt{t}_{\mathsf{P}}^{\mathsf{E}}$}$$

Hence, since $\mathsf{P} \subseteq \mathsf{E}$, we get the transitivity of P:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{P}\,y \wedge y\,\mathsf{P}\,z \implies x\,\mathsf{P}\,z). \tag{$\mathtt{t}_{\mathsf{P}}$}$$

Furthermore, from $(\mathtt{t}_{\mathsf{P}}^{\mathsf{E}})$ and $(\star_{\mathsf{E}}^{\Leftarrow})$ we have:

$$\forall_{x,y,z,u\in\mathsf{U}}(x\,\mathsf{P}\,y \wedge y\,\mathsf{S}\,z \wedge z\,\mathsf{P}\,u \implies x\,\mathsf{P}\,u). \tag{$\mathtt{t}{+}_{\mathsf{P}}$}$$

Hence, using (df P), $(\mathtt{t}_{\mathsf{P}}^{\bar{\mathsf{E}}})$, $(\mathtt{t}_{\mathsf{P}})$ and $(\mathtt{s}_{\mathsf{S}})$, we obtain:

$$\forall_{x,y,z,u\in\mathsf{U}}(x\,\mathsf{P}\,y \wedge z\,\mathsf{P}\,u \implies x\,\mathsf{P}\,u \vee z\,\mathsf{P}\,y). \tag{Th_{P}}$$

We can prove some facts further used. Firstly, from $(\mathtt{t}_{\mathsf{P}})$ and $(\mathtt{con}_{\mathsf{P}}^{\mathsf{S}})$ we get:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{P}\,y \wedge y\,\mathsf{S}\,z \wedge \neg\, z\,\mathsf{S}\,x \implies x\,\mathsf{P}\,z). \tag{2.4}$$

Secondly, using (df P), $(\mathtt{s}_{\mathsf{S}})$ and $(\mathtt{r}_{\mathsf{S}})$, we get the following facts:

$$\forall_{x,y\in\mathsf{U}}(\exists_{z\in\mathsf{U}}(x\,\mathsf{S}\,z \wedge y\,\mathsf{P}\,z) \iff \exists_{z\in\mathsf{U}}(y\,\mathsf{P}\,z \wedge \neg\, x\,\mathsf{P}\,z)), \tag{2.5}$$

$$\forall_{x,y\in\mathsf{U}}(\exists_{z\in\mathsf{U}}(y\,\mathsf{S}\,z \wedge z\,\mathsf{P}\,x) \iff \exists_{z\in\mathsf{U}}(z\,\mathsf{P}\,x \wedge \neg\, z\,\mathsf{P}\,y)), \tag{2.6}$$

$$\forall_{x,y\in\mathsf{U}}(\exists_{z\in\mathsf{U}}(x\,\mathsf{S}\,z \wedge z\,\mathsf{P}\,y) \iff \exists_{z\in\mathsf{U}}(z\,\mathsf{P}\,y \wedge \neg\, z\,\mathsf{P}\,x)), \tag{2.7}$$

$$\forall_{x,y\in\mathsf{U}}(\exists_{z\in\mathsf{U}}(x\,\mathsf{P}\,z \wedge z\,\mathsf{S}\,y) \iff \exists_{z\in\mathsf{U}}(x\,\mathsf{P}\,z \wedge \neg\, y\,\mathsf{P}\,z)), \tag{2.8}$$

i.e. we have $\mathsf{S} \mid \breve{\mathsf{P}} = \bar{\mathsf{P}} \mid \breve{\mathsf{P}}$, $\breve{\mathsf{P}} \mid \mathsf{S} = \breve{\mathsf{P}} \mid \breve{\mathsf{S}} = \breve{\mathsf{P}} \mid \bar{\mathsf{P}}$, $\mathsf{S} \mid \mathsf{P} = (\breve{\mathsf{P}})^{-} \mid \mathsf{P}$ and $\mathsf{P} \mid \mathsf{S} = \mathsf{P} \mid (\breve{\mathsf{P}})^{-}$.

Remark 2.3. We get some "existentially neutral" theory of events from axioms a1–a4 and $(\star_{\mathsf{E}}^{\Leftarrow})$, $(\mathtt{ct}_{\mathsf{E}})$ (instead of a5). In this "truncated theory", we will obtain all theses for which we do not use $(\star_{\mathsf{E}}^{\Rightarrow})$; for example: $(\mathtt{t}_{\mathsf{P}}^{\bar{\mathsf{E}}})$, $(\mathtt{t}_{\mathsf{P}}^{\mathsf{E}})$, $(\mathtt{t}_{\mathsf{P}})$, $(\mathtt{t}{+}_{\mathsf{P}})$, $(\mathrm{Th}_{\mathsf{P}})$, (2.5)–(2.8). ◇

Finally, from (2.6) and the $(\star_{\mathsf{L}})$ we have the equality $\mathsf{L} = \breve{\mathsf{P}} \mid \bar{\mathsf{P}}$, i.e.:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{L}\,y \iff \exists_{z\in\mathsf{U}}(z\,\mathsf{P}\,x \wedge \neg\, z\,\mathsf{P}\,y)), \tag{$\mathrm{df}_{\mathsf{P}}\,\mathsf{L}$}$$

Moreover, from (2.7) and $(\star_{\mathsf{E}})$ we have the equality $\mathsf{S} \mid \mathsf{P} = (\breve{\mathsf{P}})^{-} \mid \mathsf{P}$, i.e.:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{E}\,y \iff \exists_{z\in\mathsf{U}}(z\,\mathsf{P}\,y \wedge \neg\, z\,\mathsf{P}\,x)). \tag{$\mathrm{df}_{\mathsf{P}}\,\mathsf{E}$}$$

2.5. **Other auxiliary binary relations.** In this subsection, we will define some auxiliary binary relations. Initially, we do not introduce any new axioms for these new concepts. We only want to show what new exciting facts can be obtained using only the assumptions given in Section 2. Let us add that some of these relations were mentioned but not defined by Russell (1914). In the last point of this subsection, we will introduce the second existentially involved axiom.

2.5.1. **The relation of *lasts after*.** When introducing the relation of *temporal enclosure* (see next), Russell used the relation of *lasts after*, which he has not defined anywhere. We say that one event *lasts after* (in short: *is after*) another when it is simultaneous with some event which is wholly after the other (see p. 4). Let us denote the relation *lasts after* by 'A'. So $\mathsf{A} := \mathsf{S} \mid \breve{\mathsf{P}}$, i.e. for all $x, y \in \mathsf{U}$ we put:

$$x \mathrel{\mathsf{A}} y \iff \exists_{z \in \mathsf{U}}(x \mathrel{\mathsf{S}} z \wedge y \mathrel{\mathsf{P}} z). \tag{df A}$$

Remark 2.4. (i) With reference to Remark 2.1, the relation A corresponds with the relation *ends after* $:= \mathsf{S} \mid \breve{\mathsf{P}}$ from (Russell, 1936, p. 348).

(ii) Even though (df A) has an existential quantifier, its use will preserve the "existential neutrality" of the "truncated theory" based on axioms a1–a4, ($\star^{\Leftarrow}_{\mathsf{E}}$) and ($\mathtt{ct}_{\mathsf{E}}$). It would be different if we applied the relation A in some new, adequately constructed axiom. However, we will not introduce any such axiom in this section. ◇

Directly from (df A) and ($\mathtt{r}_{\mathsf{S}}$) we obtain that $\mathsf{P} \subseteq \breve{\mathsf{A}}$, i.e:

$$\forall_{x,y \in \mathsf{U}}(y \mathrel{\mathsf{P}} x \implies x \mathrel{\mathsf{A}} y). \tag{P⊆Ă}$$

Now notice that, by (df A) and (2.5), we have that $\mathsf{A} = \bar{\mathsf{P}} \mid \breve{\mathsf{P}}$, i.e.:

$$\forall_{x,y \in \mathsf{U}}(x \mathrel{\mathsf{A}} y \iff \exists_{z \in \mathsf{U}}(y \mathrel{\mathsf{P}} z \wedge \neg\, x \mathrel{\mathsf{P}} z)). \tag{df$_{\mathsf{P}}$A}$$

Moreover, directly from (df A) and (df P) it follows that A is irreflexive:

$$\forall_{x \in \mathsf{U}} \neg\, x \mathrel{\mathsf{A}} x, \tag{irr$_{\mathsf{A}}$}$$

and using (df A), ($\mathtt{t}_{+\mathsf{P}}$) and ($\mathtt{s}_{\mathsf{S}}$) we can prove that A is asymmetric:

$$\forall_{x,y \in \mathsf{U}}(x \mathrel{\mathsf{A}} y \implies \neg\, y \mathrel{\mathsf{A}} x). \tag{as$_{\mathsf{A}}$}$$

Now, using (df A), ($\mathtt{t}^{\bar{\mathsf{E}}}_{\mathsf{P}}$) and ($\mathtt{r}_{\mathsf{S}}$) we can prove that A is co-transitive:

$$\forall_{x,y,z \in \mathsf{U}}(x \mathrel{\mathsf{A}} y \implies x \mathrel{\mathsf{A}} z \vee z \mathrel{\mathsf{A}} y). \tag{ct$_{\mathsf{A}}$}$$

But notice that, generally:

Lemma 2.1. *Every asymmetric and co-transitive binary relation is transitive.*

Thus, by ($\mathtt{as}_{\mathsf{A}}$), ($\mathtt{ct}_{\mathsf{A}}$) and Lemma 2.1, we have the transitivity of A:

$$\forall_{x,y,z \in \mathsf{U}}(x \mathrel{\mathsf{A}} y \wedge y \mathrel{\mathsf{A}} z \implies x \mathrel{\mathsf{A}} z). \tag{t$_{\mathsf{A}}$}$$

Moreover, from ($\mathtt{ct}_{\mathsf{A}}$) and ($\mathtt{irr}_{\mathsf{A}}$) for all $x, y \in \mathsf{U}$ we obtain:

$$\neg\, x \mathrel{\mathsf{A}} y \iff \overrightarrow{\mathsf{A}}`y \subseteq \overrightarrow{\mathsf{A}}`x \iff \overleftarrow{\mathsf{A}}`x \subseteq \overleftarrow{\mathsf{A}}`y\,. \tag{2.9}$$

Now, we will get conditions that are useful later in the paper. Firstly, directly from (df A), ($\mathtt{s}_{\mathsf{S}}$) and (2.4) we have:

$$\forall_{x,y \in \mathsf{U}}(x \mathrel{\mathsf{A}} y \wedge \neg\, x \mathrel{\mathsf{S}} y \implies y \mathrel{\mathsf{P}} x). \tag{2.10}$$

Secondly, using (df A), (P⊆Ă), ($\mathtt{con}^{\mathsf{S}}_{\mathsf{P}}$) and ($\mathtt{t}_{\mathsf{P}}$), we get:

$$\forall_{x,y,z \in \mathsf{U}}(\neg\, z \mathrel{\mathsf{A}} x \wedge x \mathrel{\mathsf{P}} y \implies z \mathrel{\mathsf{P}} y). \tag{t$^{\bar{\mathsf{A}}}_{\mathsf{P}}$}$$

Thirdly, using ($\mathtt{s}_{\mathsf{S}}$), ($\mathtt{con}^{\mathsf{S}}_{\mathsf{P}}$), ($\star^{\Leftarrow}_{\mathsf{E}}$) and (df A) we obtain:

$$\forall_{x,y,z \in \mathsf{U}}(z \mathrel{\mathsf{S}} x \wedge \neg\, z \mathrel{\mathsf{S}} y \implies x \mathrel{\mathsf{E}} y \vee x \mathrel{\mathsf{A}} y). \tag{2.11}$$

Hence, by ($\mathtt{r}_{\mathsf{S}}$), we have:

$$\forall_{x,y \in \mathsf{U}}(\neg\, x \mathrel{\mathsf{E}} y \wedge \neg\, x \mathrel{\mathsf{A}} y \implies x \mathrel{\mathsf{S}} y). \tag{2.12}$$

Finally, notice that by (df A), ($\mathtt{con}^{\mathsf{S}}_{\mathsf{A}}$) and (P⊆Ă) we have:

$$\forall_{x,y,z \in \mathsf{U}}(x \mathrel{\mathsf{S}} z \wedge \neg\, x \mathrel{\mathsf{A}} y \wedge \neg\, z \mathrel{\mathsf{S}} y \implies z \mathrel{\mathsf{P}} y),$$
$$\forall_{x,y,z \in \mathsf{U}}(x \mathrel{\mathsf{S}} z \wedge \neg\, x \mathrel{\mathsf{A}} y \wedge \neg\, z \mathrel{\mathsf{S}} y \implies y \mathrel{\mathsf{A}} x).$$

Therefore, by $(\mathtt{s}_{\mathsf{S}})$ and $(\mathtt{as}_{\mathsf{A}})$, we obtain:

$$\forall_{x,y,z\in\mathsf{U}}(x\,\mathsf{S}\,z \wedge \neg\, x\,\mathsf{A}\,y \wedge x\,\mathsf{S}\,y \wedge \neg\, x\,\mathsf{A}\,z \implies y\,\mathsf{S}\,z). \tag{2.13}$$

2.5.2. **The relations: *beginning together*, *ending together*, *complete simultaneity*.** Although Russell (1914) mentions the relation of *complete simultaneity*, he neither uses nor defines it. Only when Russell (1914, p. 125) discusses examples of transitive relations he writes: "The symmetrical relations mentioned just now are also transitive [...] provided, in the case of simultaneity, we mean complete simultaneity, i.e. beginning and ending together." However, Russell does not write anywhere what he means by "beginning and ending together". We will use three binary relations $\mathsf{BT} := \bar{\mathsf{E}} \setminus \breve{\mathsf{E}}$, $\mathsf{ET} := \bar{\mathsf{A}} \setminus \breve{\mathsf{A}}$ and $\equiv_{\mathsf{t}} := \mathsf{BT} \cap \mathsf{ET}$ as interpretations of the phrases 'begins together', 'ends together' and 'is completely simultaneous with'. So for all $x, y \in \mathsf{U}$ we put:

$$x\,\mathsf{BT}\,y \iff \neg\, x\,\mathsf{E}\,y \wedge \neg\, y\,\mathsf{E}\,x\,, \tag{df BT}$$
$$x\,\mathsf{ET}\,y \iff \neg\, x\,\mathsf{A}\,y \wedge \neg\, y\,\mathsf{A}\,x\,, \tag{df ET}$$
$$x \equiv_{\mathsf{t}} y \iff x\,\mathsf{BT}\,y \wedge x\,\mathsf{ET}\,y\,. \tag{df $\equiv_{\mathsf{t}}$}$$

Notice that from our definitions, $(\mathtt{as}_{\mathsf{E}})$ and $(\mathtt{as}_{\mathsf{A}})$ we obtain:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{BT}\,y \vee x\,\mathsf{ET}\,y \implies x \sqsubseteq_{\mathsf{t}} y \vee y \sqsubseteq_{\mathsf{t}} x).$$

Directly from the pairs {(df BT), (2.3)}, {(df BT), (df$_{\mathsf{P}}$E)}, {(df ET), (2.9)}, {(df ET), (df$_{\mathsf{P}}$A)}, for all $x, y \in \mathsf{U}$ we have, respectively:

$$x\,\mathsf{BT}\,y \iff \overrightarrow{\mathsf{E}}\text{‘}x = \overrightarrow{\mathsf{E}}\text{‘}y \iff \overleftarrow{\mathsf{E}}\text{‘}x = \overleftarrow{\mathsf{E}}\text{‘}y\,, \tag{df$_{\mathsf{E}}$BT}$$
$$x\,\mathsf{BT}\,y \iff \overrightarrow{\mathsf{P}}\text{‘}x = \overrightarrow{\mathsf{P}}\text{‘}y\,, \tag{df$_{\mathsf{P}}$BT}$$
$$x\,\mathsf{ET}\,y \iff \overrightarrow{\mathsf{A}}\text{‘}x = \overrightarrow{\mathsf{A}}\text{‘}y \iff \overleftarrow{\mathsf{A}}\text{‘}x = \overleftarrow{\mathsf{A}}\text{‘}y\,, \tag{df$_{\mathsf{A}}$ET}$$
$$x\,\mathsf{ET}\,y \iff \overleftarrow{\mathsf{P}}\text{‘}x = \overleftarrow{\mathsf{P}}\text{‘}y\,. \tag{df$_{\mathsf{P}}$ET}$$

The above gives the reflexivity, the symmetry and the transitivity of BT, ET, and $\equiv_{\mathsf{t}}$.

Notice that both $(\mathtt{con}^{\mathsf{S}}_{\mathsf{E}})$ and $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$ are definitionally equivalent to:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{BT}\,y \implies x\,\mathsf{S}\,y). \tag{BT$\subseteq$S}$$

Hence, because $\equiv_{\mathsf{t}} \subseteq \mathsf{BT}$, we obtain:

$$\forall_{x,y\in\mathsf{U}}(x \equiv_{\mathsf{t}} y \implies x\,\mathsf{S}\,y). \tag{$\equiv_{\mathsf{t}}\subseteq$S}$$

Moreover, from (df A), $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$ and $(\mathsf{P}\subseteq\breve{\mathsf{A}})$ we have:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{ET}\,y \implies x\,\mathsf{S}\,y). \tag{ET$\subseteq$S}$$

But the above condition is definitionally equivalent to:

$$\forall_{x,y\in\mathsf{U}}(\neg\, x\,\mathsf{S}\,y \implies x\,\mathsf{A}\,y \vee y\,\mathsf{A}\,x). \tag{$\mathtt{con}^{\mathsf{S}}_{\mathsf{A}}$}$$

From (df$_{\mathsf{E}}$BT), $(\mathtt{s}_{\mathsf{S}})$ and (2.1) we have:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{BT}\,y \implies \forall_{z\in\mathsf{U}}(z\,\mathsf{E}\,x \wedge z\,\mathsf{S}\,x \Leftrightarrow z\,\mathsf{E}\,y \wedge z\,\mathsf{S}\,y)).$$

But using (df BT), $(\mathtt{ct}_{\mathsf{E}})$, $(\mathtt{con}^{\mathsf{S}}_{\mathsf{E}})$, (2.1), $(\mathtt{s}_{\mathsf{S}})$, $(\mathtt{irr}_{\mathsf{E}})$ and $(\mathtt{r}_{\mathsf{S}})$, we obtain:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{BT}\,y \iff \forall_{z\in\mathsf{U}}(\neg\, x\,\mathsf{E}\,z \wedge x\,\mathsf{S}\,z \Leftrightarrow \neg\, y\,\mathsf{E}\,z \wedge y\,\mathsf{S}\,z)). \tag{2.14}$$

Moreover, using (df$_{\mathsf{A}}$ET), (2.11), $(\mathtt{as}_{\mathsf{E}})$, (df$'\sqsubseteq_{\mathsf{t}}$) and (df$_{\mathsf{S}}\sqsubseteq_{\mathsf{t}}$), we obtain:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{ET}\,y \implies \forall_{z\in\mathsf{U}}(z\,\mathsf{A}\,x \wedge z\,\mathsf{S}\,x \Leftrightarrow z\,\mathsf{A}\,y \wedge z\,\mathsf{S}\,y)). \tag{2.15}$$

But using (df ET), $(\mathtt{ct}_{\mathsf{A}})$, $(\mathtt{con}^{\mathsf{S}}_{\mathsf{A}})$, (2.10), (df A), $(\mathtt{irr}_{\mathsf{A}})$ and $(\mathtt{r}_{\mathsf{S}})$, we get:

$$\forall_{x,y\in\mathsf{U}}(x\,\mathsf{ET}\,y \iff \forall_{z\in\mathsf{U}}(\neg\, x\,\mathsf{A}\,z \wedge x\,\mathsf{S}\,z \Leftrightarrow \neg\, y\,\mathsf{A}\,z \wedge y\,\mathsf{S}\,z)). \tag{2.16}$$

2.5.3. **The relation of *temporal enclosure*.** Russell defines this relation by the following:

> One object will be temporally enclosed by another when it is simultaneous with the other but not before or after it. Whatever encloses temporally or is enclosed temporally we shall call an "event." (Russell, 1914, p. 121)

In addition to the relation S, Russell used two relations *is before* and *is after*. We can consider the relation E (*is earlier than*) to be the first, and the relation A (*lasts after*) to be the second.

Let us denote the relation of *temporal enclosure* by '$\sqsubseteq_{\mathrm{t}}$'. Its definition given by Russell can be formally written as follows: $\sqsubseteq_{\mathrm{t}} := \mathsf{S} \cap \bar{\mathsf{E}} \cap \bar{\mathsf{A}}$. So for all $x, y \in \mathsf{U}$ we put:

$$x \sqsubseteq_{\mathrm{t}} y \iff x \,\mathsf{S}\, y \wedge \neg\, x \,\mathsf{E}\, y \wedge \neg\, x \,\mathsf{A}\, y. \tag{df$\sqsubseteq_{\mathrm{t}}$}$$

Condition (2.12) shows that we can simplify the above definition: one object will be temporally enclosed by another when it is not before or after the other. That is, we have the equality $\sqsubseteq_{\mathrm{t}} = \bar{\mathsf{E}} \cap \bar{\mathsf{A}}$:

$$\forall_{x,y\in\mathsf{U}}(x \sqsubseteq_{\mathrm{t}} y \iff \neg\, x \,\mathsf{E}\, y \wedge \neg\, x \,\mathsf{A}\, y). \tag{df$'\sqsubseteq_{\mathrm{t}}$}$$

Hence, by ($\mathtt{irr}_{\mathsf{E}}$) and ($\mathtt{irr}_{\mathsf{A}}$), we obtain the reflexivity of $\sqsubseteq_{\mathrm{t}}$:

$$\forall_{x\in\mathsf{U}}\, x \sqsubseteq_{\mathrm{t}} x. \tag{$\mathtt{r}_{\sqsubseteq_{\mathrm{t}}}$}$$

By ($\mathtt{ct}_{\mathsf{E}}$) and ($\mathtt{ct}_{\mathsf{A}}$), the complements of E and A are transitive. From this and (df$'\sqsubseteq_{\mathrm{t}}$), we have the transitivity of $\sqsubseteq_{\mathrm{t}}$:

$$\forall_{x,y,z\in\mathsf{U}}(x \sqsubseteq_{\mathrm{t}} y \wedge y \sqsubseteq_{\mathrm{t}} z \implies x \sqsubseteq_{\mathrm{t}} z). \tag{$\mathtt{t}_{\sqsubseteq_{\mathrm{t}}}$}$$

Moreover, we get: one event is temporally enclosed by another iff every event simultaneous with the first is also simultaneous with the other. Formally, for all $x, y \in \mathsf{U}$:

$$x \sqsubseteq_{\mathrm{t}} y \iff \overrightarrow{\mathsf{S}}`x \subseteq \overrightarrow{\mathsf{S}}`y \iff \overleftarrow{\mathsf{S}}`x \subseteq \overleftarrow{\mathsf{S}}`y\,. \tag{df$_{\mathsf{S}}\sqsubseteq_{\mathrm{t}}$}$$

We will further show (see (3.10)) that one event will be temporally enclosed by another when the second exists at any instant at which the first exists.

From (df$\equiv_{\mathrm{t}}$), (df BT), (df$'\sqsubseteq_{\mathrm{t}}$), (df$_{\mathsf{S}}\sqsubseteq_{\mathrm{t}}$), (df$_{\mathsf{E}}$BT), (df$_{\mathsf{P}}$BT), (df$_{\mathsf{P}}$ET), for all $x, y \in \mathsf{U}$:

$$\begin{aligned} x \equiv_{\mathrm{t}} y &\iff x \sqsubseteq_{\mathrm{t}} y \wedge y \sqsubseteq_{\mathrm{t}} x\,, && (\mathrm{df}_{\sqsubseteq_{\mathrm{t}}} \equiv_{\mathrm{t}})\\ &\iff \overrightarrow{\mathsf{S}}`x = \overrightarrow{\mathsf{S}}`y \iff \overleftarrow{\mathsf{S}}`x = \overleftarrow{\mathsf{S}}`y, && (\mathrm{df}_{\mathsf{S}}\equiv_{\mathrm{t}})\\ &\iff \overrightarrow{\mathsf{P}}`x = \overrightarrow{\mathsf{P}}`y \wedge \overleftarrow{\mathsf{P}}`x = \overleftarrow{\mathsf{P}}`y. && (\mathrm{df}_{\mathsf{P}}\equiv_{\mathrm{t}}) \end{aligned}$$

By the above, we again obtain that $\equiv_{\mathrm{t}}$ is reflexive, symmetric, transitive, and it is a congruence with respect to the relations S, E and P. We will further show (see (3.11)) that two events are completely simultaneous iff they exist at the same instants.

Remark 2.5. Russell (1914, p. 121) claimed: "[...] if one event encloses another different event, then the other does not enclose the one"; that is, according to him, the relation $\sqsubseteq_{\mathrm{t}}$ is supposed to be antisymmetric. However, (df$_{\sqsubseteq_{\mathrm{t}}} \equiv_{\mathrm{t}}$) shows that $\sqsubseteq_{\mathrm{t}}$ need not be antisymmetric because the condition $x \equiv_{\mathrm{t}} y$ does not entail the identity $x = y$. A simple model can illustrate it (see model 1 in Appendix B). ◇

Finally, notice that directly from our definitions and ($\mathtt{as}_{\mathsf{A}}$) we have:

$$\forall_{x,y\in\mathsf{U}}(x \,\mathsf{A}\, y \implies y \,\mathsf{E}\, x \vee y \sqsubseteq_{\mathrm{t}} x).$$

In addition to $\sqsubseteq_{\mathrm{t}}$, we can introduce the relation of *non-tangential temporal enclosure*, which we will label by '$\Subset_{\mathrm{t}}$'. For arbitrary $x, y \in \mathsf{U}$ we put:

$$x \Subset_{\mathrm{t}} y \iff y \,\mathsf{E}\, x \wedge y \,\mathsf{A}\, x. \tag{df$\Subset_{\mathrm{t}}$}$$

Directly from ($\mathtt{irr}_{\mathsf{E}}$), ($\mathtt{irr}_{\mathsf{A}}$), ($\mathtt{t}_{\mathsf{E}}$) and ($\mathtt{t}_{\mathsf{A}}$), the relation $\Subset_{\mathrm{t}}$ is irreflexive and transitive. From our definitions, ($\mathtt{as}_{\mathsf{E}}$) and ($\mathtt{as}_{\mathsf{A}}$) we have:

$$\forall_{x,y\in\mathsf{U}}(x \Subset_{\mathrm{t}} y \implies x \sqsubseteq_{\mathrm{t}} y \wedge x \not\equiv_{\mathrm{t}} y). \tag{2.17}$$

Furthermore, using ($\mathtt{ct}_{\mathsf{E}}$) and ($\mathtt{ct}_{\mathsf{A}}$) we obtain:

$$\forall_{x,y,z\in \mathsf{U}}(x \Subset_{\mathsf{t}} y \wedge y \sqsubseteq_{\mathsf{t}} z \implies x \Subset_{\mathsf{t}} z),$$
$$\forall_{x,y,z\in \mathsf{U}}(x \sqsubseteq_{\mathsf{t}} y \wedge y \Subset_{\mathsf{t}} z \implies x \Subset_{\mathsf{t}} z).$$

Thus, by (2.17) and ($\mathtt{t}_{\sqsubseteq_{\mathsf{t}}}$), we have the transitivity of $\Subset_{\mathsf{t}}$.

2.5.4. **The relation of *is an initial contemporary of***. Russell (1914, p. 119) writes: "all the events which are simultaneous with a given event, and do not begin later [...] [w]e will call [...] the "initial contemporaries" of the given event." The concept, which is indicated in the quotation, determines the binary relation $\mathsf{IC} := \mathsf{S} \setminus \mathsf{L} = \mathsf{S} \setminus \breve{\mathsf{E}}$ (*is an initial contemporary of*). So for all $x, y \in \mathsf{U}$ we put:

$$x\, \mathsf{IC}\, y \iff x\, \mathsf{S}\, y \wedge \neg\, x\, \mathsf{L}\, y \iff x\, \mathsf{S}\, y \wedge \neg\, y\, \mathsf{E}\, x\,. \tag{df IC}$$

From the reflexivity of S and the irreflexivity of L, we obtain that IC is reflexive. Furthermore, from (df BT), (df IC), (BT⊆S) and ($\mathtt{s}_{\mathsf{S}}$) we have:

$$\forall_{x,y\in \mathsf{U}}(x\, \mathsf{BT}\, y \iff x\, \mathsf{IC}\, y \wedge y\, \mathsf{IC}\, x). \tag{2.18}$$

3. Instants

3.1. **The definition of instants.** From events as "components of the real world", Russell (1914, 1936) constructed abstract objects called *instants* as certain distributive sets of events. A given instant is to be a set of all simultaneous events, i.e., to put it succinctly—those with at least one common moment of duration. To paraphrase the quotation on p. 2 concerning the points created by Whitehead, we can say that we define a given instant as a set of all events that we will naturally say that all of them occur in it. Russell (1914, p. 119) describes the creation of an instant informally in the following words:[4]

> Let us take a group of events of which any two overlap, so that there is some time, however short, when they all exist. If there is any other event which is simultaneous with all of these, let us add it to the group; let us go on until we have constructed a group such that no event outside the group is simultaneous with all of them but all the events inside the group are simultaneous with each other. Let us define this whole group as an instant of time.

Thus, for Russell, a set α of events is an *instant* iff α meets the following two conditions:

(c1) $\neg\exists_{u\in \mathsf{U}}(u \notin \alpha \wedge \forall_{x\in\alpha}\, u\, \mathsf{S}\, x)$,
(c2) $\forall_{x,y\in\alpha}\, x\, \mathsf{S}\, y$.

The pair of the above conditions is logically equivalent to the following:

$$\forall_{u\in \mathsf{U}}(u \in \alpha \Leftrightarrow \forall_{x\in\alpha}\, u\, \mathsf{S}\, x), \qquad \text{i.e.,}\quad \alpha = \{u \in \mathsf{U} : \forall_{x\in\alpha}\, u\, \mathsf{S}\, x\}. \tag{In}$$

Let **In** be the family of all instants. Since $\{u \in \mathsf{U} : \forall_{x\in\alpha} : u\, \mathsf{S}\, x\} = \bigcap\{\vec{\mathsf{S}}`x : x \in \alpha\}$ holds, any instant α shall meet the following equality:

$$\alpha = \bigcap\{\vec{\mathsf{S}}`x : x \in \alpha\}. \tag{In$'$}$$

The above corresponds to the definition used by Russell (1936, p. 351):

$$\alpha \in \mathbf{In} \iff \alpha = \bigcap\{X \in 2^{\mathsf{U}} : \exists_{x\in\alpha}\, X = \vec{\mathsf{S}}`x\} \iff \bigcap\{\vec{\mathsf{S}}`x : x \in \alpha\},$$

which he expressed in the words: "an instant is a class of events which is identical with the common contemporaries of all the members of the class."

Using (c1) and (c2), we obtain the following criterion of the identity of instants:

$$\forall_{\alpha,\beta\in\mathbf{In}}(\alpha = \beta \iff \forall_{x\in\alpha}\forall_{y\in\beta}\, x\, \mathsf{S}\, y). \tag{3.1}$$

Moreover, the quote in this section shows that Russell's intention was for the instants to be maximal sets of events (w.r.t. inclusion). Indeed, using (3.1), we can prove:

[4] Russell more formally expresses the same at the beginning of (Russell, 1936).

Fact 3.1. **In** *is identical with the family of all maximal sets of events among sets satisfying* (c2). *More precisely, we get*:

1. *All instants are maximal in* **In**, *i.e., for all* $\alpha, \beta \in \mathbf{In}$, *if* $\alpha \subseteq \beta$ *then* $\alpha = \beta$.
2. *Every set of events which is maximal among sets satisfying* (c2) *belong to* **In**.

Notice that since there is at least one event, by (c1), no instant is empty. Now for any event x, let $\mathbb{I}_x$ be the family of all instants at which x exists. Using the axiom of choice or some of its equivalent, we can prove that:

$$\forall_{x,y \in \mathsf{U}} (x \mathrel{\mathsf{S}} y \iff \exists_{\alpha \in \mathbf{In}}\, x, y \in \alpha). \tag{3.2}$$

Hence, by the reflexivity of S, we have $\mathbb{I}_x \neq \emptyset \neq \mathbf{In}$. As Russell (1914, pp. 119–120) pointed out, it can also be demonstrated without using the axiom of choice, using axiom $(\mathtt{e})$ he introduced (see the next section).

3.2. **The relation *is before* on the family of instants.** Russell (1914, p. 119) accepted that "one instant is before another if the group which is the one instant contains an event which is earlier than but not simultaneous with, some event in the group which is the other instant." Therefore, we formally introduce the binary relation $\prec$ (*is before*) on the family **In**:

$$\alpha \prec \beta \iff \exists_{x \in \alpha} \exists_{y \in \beta} (x \mathrel{\mathsf{E}} y \wedge \neg\, x \mathrel{\mathsf{S}} y) \iff \exists_{x \in \alpha} \exists_{y \in \beta}\, x \mathrel{\mathsf{P}} y\,. \tag{df $\prec$}$$

Russell makes instants three requirements. The first one is that the relation $\prec$ has to be a strict total order over **In** (see Russell, 1914, pp. 118–119). We can get it using our definitions, $(\mathtt{t}{+}_{\mathsf{P}})$, (3.1) and $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$, respectively.

Theorem 3.2. *The relation* $\prec$ *is irreflexive, transitive and connex, i.e.*:

$$\forall_{\alpha \in \mathbf{In}}\ \alpha \not\prec \alpha, \tag{$\mathtt{irr}_\prec$}$$
$$\forall_{\alpha,\beta,\gamma \in \mathbf{In}} (\alpha \prec \beta \wedge \beta \prec \gamma \implies \alpha \prec \gamma), \tag{$\mathtt{t}_\prec$}$$
$$\forall_{\alpha,\beta \in \mathbf{In}} (\alpha \neq \beta \implies \alpha \prec \beta \vee \beta \prec \alpha). \tag{$\mathtt{con}_\prec$}$$

Lemma 3.3. *Every connex and transitive binary relation is co-transitive.*

So from $(\mathtt{irr}_\prec)$ and $(\mathtt{t}_\prec)$ we obtain that $\prec$ is asymmetric, i.e.:

$$\forall_{\alpha,\beta \in \mathbf{In}} (\alpha \prec \beta \implies \beta \not\prec \alpha). \tag{$\mathtt{as}_\prec$}$$

Moreover, by $(\mathtt{con}_\prec)$, $(\mathtt{t}_\prec)$ and Lemma 3.3, $\prec$ is co-transitive, i.e.:

$$\forall_{\alpha,\beta,\gamma \in \mathbf{In}} (\alpha \prec \beta \implies \alpha \prec \gamma \vee \gamma \prec \beta). \tag{$\mathtt{ct}_\prec$}$$

Remark 3.1. Russell (1914, p. 119) wrote that the conditions $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$, $(\mathtt{as}_{\mathsf{P}})$ and $(\mathtt{t}_{\mathsf{P}})$ are sufficient to obtain the conditions $(\mathtt{irr}_\prec)$, $(\mathtt{t}_\prec)$ and $(\mathtt{con}_\prec)$. In fact, we can read the same in point I, in footnote 1, where Russell lists four assumptions (*a*)–(*d*) that have to give us that $\prec$ is a strict total order. Assumptions (*a*) and (*b*) state $(\mathtt{irr}_{\mathsf{P}})$ and $(\mathtt{t}_{\mathsf{P}})$, respectively (previously Russell mentioned as assumption $(\mathtt{as}_{\mathsf{P}})$ instead of $(\mathtt{irr}_{\mathsf{P}})$, but it is irrelevant as he also assumed $(\mathtt{t}_{\mathsf{P}})$). Assumption (*c*) states that the relations P and S are disjoint (which we get from their definition). Assumption (*d*) is condition $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$. Russell, however, does not prove it. We will show below that this is not the case.

Note that, indeed, the proofs of the conditions $(\mathtt{irr}_\prec)$ and $(\mathtt{con}_\prec)$ are based on conditions (*c*) and (*d*), respectively. However, to prove $(\mathtt{t}_\prec)$, one has to use $(\mathtt{t}{+}_{\mathsf{P}})$, for the derivation of which we used $(\star^{\Leftarrow}_{\mathsf{E}})$, which does not exist among the conditions mentioned by Russell. We can show that conditions $(\mathtt{as}_\prec)$ and $(\mathtt{t}_\prec)$ do not follow from conditions: $(\mathtt{irr}_{\mathsf{P}})$, $(\mathtt{t}_{\mathsf{P}})$, $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$ and $\mathsf{P} \cap \mathsf{S} = \emptyset$ (see model 2 in Appendix B). ⋄

Standardly, for all $\alpha, \beta \in \mathbf{In}$ we put:

$$\alpha \preceq \beta \iff \alpha \prec \beta \vee \alpha = \beta\,. \tag{df $\preceq$}$$

In the light Theorem 3.2, the relation $\preceq$ is a total order over **In**.

3.3. **An axiom for existing instants.** Russell (1914, pp. 119–120) writes:

> We have next to show that every event is "at" at least one instant, i.e. that, given any event, there is at least one class, such as we used in defining instants, of which it is a member. For this purpose, consider all the events which are simultaneous with a given event, and do not begin later, i.e. are not wholly after anything simultaneous with it. We will call these the "initial contemporaries" of the given event. It will be found that this class of events is the first instant at which the given event exists, provided every event wholly after some contemporary of the given event is wholly after some initial contemporary of it.

So for any event x, Russell picks the following set:

$$\phi_x := \{u \in \mathrm{U} : u \,\mathrm{IC}\, x\} \tag{df ϕ_x}$$

The last quote assumes that Russell used to show that ϕ_x is an instant. This assumption is also presented in footnote 1 as point (e): "An event wholly after some contemporary of a given event is wholly after some initial contemporary of the given event." So Russell accepted the following axiom:

$$\text{a6} \qquad \forall_{x,y\in \mathrm{U}}(\exists_{z\in \mathrm{U}}(z \,\mathrm{P}\, y \wedge z \,\mathrm{S}\, x) \implies \exists_{z\in \mathrm{U}}(z \,\mathrm{P}\, y \wedge z \,\mathrm{IC}\, x)). \tag{e}$$

Now let us note some facts about ϕ_x sets. Firstly, from the reflexivity of IC, we have:

$$\forall_{x\in \mathrm{U}}\ x \in \phi_x\,. \tag{3.3}$$

Secondly, directly from (df ϕ_x), (df IC) and (2.2), we get that ϕ_x satisfies the equivalent of condition (c2): $\forall_{x\in \mathrm{U}}\forall_{y,z\in\phi_x}\ y \,\mathrm{S}\, z$. Hence, directly from the definition of **In**, we obtain:

Lemma 3.4. *For any event x: $\phi_x \in \mathbf{In}$ if and only if ϕ_x satisfies the equivalent of condition* (c1): $\neg\exists_{u\in \mathrm{U}}(u \notin \phi_x \wedge \forall_{v\in\phi_x}\ u \,\mathrm{S}\, v)$.

Thirdly, by the above lemma, (s_{S}), (2.3), (df IC) and (df ϕ_x), we can prove:

Theorem 3.5. *For any $x \in \mathrm{U}$, $\phi_x \in \mathbf{In}$ if and only if x satisfies the following condition*:

$$\forall_{y\in \mathrm{U}}(x \,\mathrm{E}\, y \implies \exists_{z\in \mathrm{U}}(z \,\mathrm{P}\, y \wedge z \,\mathrm{IC}\, x)). \tag{%}$$

The "$\Leftarrow$"-part of the above theorem is a formal equivalent to what Russell writes on the first page of (Russell, 1936): "I have shown [(1914)] that every event x hast a first instant if every event that begins after x has begun in wholly after some event which exists when x begins."

From (e), (s_{S}) and ($\star_{\mathrm{E}}^{\Rightarrow}$), we obtain that condition (%) holds for any event x. Thus, in virtue of Theorem 3.5, having axiom (e) and (3.3), the fact that $x \in \mathbb{I}_x \neq \varnothing$ we can obtain constructively:

Fact 3.6. *For any event x, we have $\phi_x \in \mathbb{I}_x \in \mathbf{In}$.*

Notice that using (2.14), (s_{S}) and (2.18) we obtain:

$$\forall_{x,y\in \mathrm{U}}(x \,\mathrm{BT}\, y \iff \phi_x = \phi_y). \tag{3.4}$$

3.4. **The first instant at which a given event exists.** Having the order $\prec$ on the set **In**, we can show that ϕ_x "is the first instant at which [x] exists". For this, let us note that in virtue of (c2) and ($\star_{\mathrm{E}}^{\Leftarrow}$), we obtain that if one of two events that have at least one instant in common wholly precedes a third, the second is earlier than the third:

$$\forall_{\alpha\in\mathbf{In}}\forall_{x,y\in\alpha}\forall_{u\in \mathrm{U}}(x \,\mathrm{P}\, u \implies y \,\mathrm{E}\, u). \tag{3.5}$$

Using the above, we can prove that:

$$\forall_{\alpha\in\mathbf{In}}\forall_{x\in\alpha}\ \alpha \nprec \phi_x\,, \tag{3.6}$$

$$\forall_{x\in \mathrm{U}}\neg\exists_{\alpha\in\mathbb{I}_x}\ \alpha \prec \phi_x\,. \tag{3.6$'$}$$

So ϕ_x is a minimal element in $\mathbb{I}_x$ since $\phi_x \in \mathbb{I}_x$. Hence, by (3.3), ($\mathtt{con}_{\prec}$) and (3.6′), we obtain that ϕ_x is the first instant at which x exists. Formally:

$$\forall_{x\in \mathrm{U}}\forall_{\alpha\in\mathbb{I}_x}\ \phi_x \preceq \alpha. \tag{3.7}$$

Finally, notice that we get:

$$\forall_{x,y\in\mathrm{U}}(x \mathrel{\mathsf{E}} y \iff \phi_x \prec \phi_y). \tag{3.8}$$

Hence, by ($\mathtt{con}_{\prec}$), we get:

$$\forall_{x,y\in\mathrm{U}}(\neg\, x \mathrel{\mathsf{E}} y \iff \phi_y \preceq \phi_x). \tag{3.9}$$

Russell's second requirement to **In** and $\prec$ consisted of three parts:

> [...] every event must be at a certain number of instants; two events are simultaneous if they are at the same instant, and one is before the other if there is an instant, at which the one is, which is earlier than some instant at which the other is. (Russell, 1914, p. 119)

So far, we have only shown that ϕ_x is the first instant at which the given event x exists. However, we did not find a "certain number of instants" at which the event occurs. Notice that, having (e), also without the axiom of choice, we get (3.2), which, as Russell wrote, shows that: "two events are simultaneous if they are at the same instant".

Fact 3.7. *From* (e) *follows* (3.2)*.*

From (3.8) and Fact 3.6, we obtain: if one event is before the other, then "there is an instant, at which the one is, which is earlier than some instant at which the other is." Formally:

$$\forall_{x,y\in\mathrm{U}}(x \mathrel{\mathsf{E}} y \implies \exists_{\alpha\in\mathbb{I}_x}\exists_{\beta\in\mathbb{I}_y}\ \alpha \prec \beta).$$

Moreover, having (e), also without the axiom of choice, we can prove:

$$\forall_{x,y\in\mathrm{U}}(x \sqsubseteq_{\mathrm{t}} y \iff \mathbb{I}_x \subseteq \mathbb{I}_y). \tag{3.10}$$

Hence, by ($\mathrm{df}_{\sqsubseteq_{\mathrm{t}}} \equiv_{\mathrm{t}}$), we have:

$$\forall_{x,y\in\mathrm{U}}(x \equiv_{\mathrm{t}} y \iff \mathbb{I}_x = \mathbb{I}_y). \tag{3.11}$$

3.5. **The last instant at which a given event exists.** In (1914), Russell did not deal with the problem of the existence of the last instant for a given event. However, Russell (1936, p. 363) gives the condition that "ensures every event has a last instant". Figuratively speaking, we can get this condition as a "mirror image" of condition (e), i.e., in the latter, we will replace the relation E with the relation $\mathsf{A} := \mathsf{S} \mid \breve{\mathsf{P}}$ =: *ends after* in (Russell, 1936, p. 348). Namely, the condition $x \mathrel{\mathsf{IC}} y$ we defined by: $x \mathrel{\mathsf{S}} y \wedge \neg\, y \mathrel{\mathsf{E}} x$. Instead of the latter, we will use: $x \mathrel{\mathsf{S}} y \wedge \neg\, y \mathrel{\mathsf{A}} x$, which says that x is a *final contemporary of* y, i.e., x is contemporaneous with the end of y; in short: $x \mathrel{\mathsf{FC}} y$. That is, for all $x, y \in \mathrm{U}$ we put:

$$x \mathrel{\mathsf{FC}} y \iff x \mathrel{\mathsf{S}} y \wedge \neg\, y \mathrel{\mathsf{A}} x. \tag{df FC}$$

In (1936, p. 363) Russell accepted the following axiom:

$$\forall_{x,y\in\mathrm{U}}(\exists_{z\in\mathrm{U}}(x \mathrel{\mathsf{S}} z \wedge y \mathrel{\mathsf{P}} z) \implies \exists_{z\in\mathrm{U}}(y \mathrel{\mathsf{P}} z \wedge z \mathrel{\mathsf{FC}} x)). \tag{s}$$

Now for any $x \in \mathrm{U}$ we put:

$$\lambda_x := \{u \in \mathrm{U} : u \mathrel{\mathsf{FC}} x\}. \tag{df λ_x}$$

Let us note some facts about λ_x sets. Firstly, from the reflexivity of FC, we have

$$\forall_{x\in\mathrm{U}}\ x \in \lambda_x\,. \tag{3.12}$$

Secondly, directly from (df λ_x), (df FC) and (2.13), we get that λ_x satisfies the equivalent of condition (c2): $\forall_{x\in\mathrm{U}}\forall_{y,z\in\lambda_x}\ y \mathrel{\mathsf{S}} z$. Hence, directly from the definition of **In**, we obtain:

Lemma 3.8. *For any event* x: $\lambda_x \in \mathbf{In}$ *if and only if* λ_x *satisfies the equivalent of condition* (c1): $\neg\exists_{u\in\mathrm{U}}(u \notin \lambda_x \wedge \forall_{v\in\lambda_x}\ u \mathrel{\mathsf{S}} v)$.

Thirdly, by the above lemma, $(\mathtt{s}_{\mathrm{S}})$, (df FC), (df λ_x), $(\mathtt{ct}_{\mathrm{A}})$, (2.12), we can prove:

Theorem 3.9. *For any* $x \in \mathsf{U}$, $\lambda_x \in \mathbf{In}$ *if and only if* x *satisfies the following condition*:

$$\forall_{y\in\mathsf{U}}(x \mathrel{\mathsf{A}} y \implies \exists_{z\in\mathsf{U}}(z \mathrel{\mathsf{P}} y \wedge z \mathrel{\mathsf{FC}} x)). \tag{‰}$$

From $(\mathtt{s})$, $(\mathtt{s}_{\mathrm{S}})$ and $(\star_{\mathrm{E}}^{\rightrightarrows})$ we obtain that condition (‰) holds for any event x. Thus, in virtue of Theorem 3.9 and (3.12), we obtain:

Fact 3.10. *For any event x, we have* $\lambda_x \in \mathbb{I}_x \in \mathbf{In}$.

Now notice that from our definitions, $(\mathsf{ET}{\subseteq}\mathsf{S})$ and $(\mathtt{s}_{\mathrm{S}})$ we have:

$$\forall_{x,y\in\mathsf{U}}(x \mathrel{\mathsf{ET}} y \iff x \mathrel{\mathsf{FC}} y \wedge y \mathrel{\mathsf{FC}} x). \tag{3.13}$$

Hence, using (2.16) and $(\mathtt{s}_{\mathrm{S}})$, analogous to (3.4), we get:

$$\forall_{x,y\in\mathsf{U}}(x \mathrel{\mathsf{ET}} y \iff \lambda_x = \lambda_y). \tag{3.14}$$

Furthermore, using (3.5), analogous to (3.6) and (3.6′), we can prove that:

$$\forall_{\alpha\in\mathbf{In}}\forall_{x\in\alpha}\, \lambda_x \not\prec \alpha, \tag{3.15}$$

$$\forall_{x\in\mathsf{U}}\neg\exists_{\alpha\in\mathbb{I}_x}\, \lambda_x \prec \alpha. \tag{3.15′}$$

So λ_x is a maximal element in $\mathbb{I}_x$ since $\lambda_x \in \mathbb{I}_x$. From (3.7), $(\mathtt{con}_{\prec})$ and (3.15′) we have that λ_x is the last instant at which the an event x exists and moreover for any $x \in \mathsf{U}$:

$$\mathbb{I}_x = \{\alpha \in \mathbf{In} : \phi_x \preceq \alpha \preceq \lambda_x\}. \tag{3.16}$$

So we see that having $(\mathtt{s})$, without the axiom of choice, we can prove:

$$\forall_{x,y\in\mathsf{U}}(x \mathrel{\mathsf{A}} y \iff \lambda_y \prec \lambda_x). \tag{3.17}$$

Hence, by $(\mathtt{con}_{\prec})$, we get:

$$\forall_{x,y\in\mathsf{U}}(\neg\, x \mathrel{\mathsf{A}} y \iff \lambda_x \preceq \lambda_y). \tag{3.18}$$

Furthermore, from our definitions, $(\mathtt{t}_{\mathrm{P}}^{\bar{\mathrm{E}}})$, $(\mathtt{t}_{\mathrm{P}}^{\bar{\mathrm{A}}})$ and $(\mathtt{t}_{\mathrm{P}})$ we get:

$$\forall_{x,y\in\mathsf{U}}(x \mathrel{\mathsf{P}} y \iff \lambda_x \prec \phi_y).$$

Finally, directly from our definitions and one of the pairs {(3.9), (3.18)}, {(3.4), (3.14)}, {(3.8), (3.17)}, respectively, for all $x, y \in \mathsf{U}$ we get:

$$x \sqsubseteq_{\mathrm{t}} y \iff \phi_y \preceq \phi_x \wedge \lambda_x \preceq \lambda_y\,,$$

$$x \equiv_{\mathrm{t}} y \iff \phi_y = \phi_x \wedge \lambda_x = \lambda_y\,,$$

$$x \Subset_{\mathrm{t}} y \iff \phi_y \prec \phi_x \wedge \lambda_x \prec \lambda_y\,.$$

Of course, for the second condition, we could also use the first, $(\mathrm{df}_{\sqsubseteq_{\mathrm{t}}} \equiv_{\mathrm{t}})$ and the antisymmetry of $\preceq$.

Notice that $(\mathtt{e})$ and $(\mathtt{s})$ do not follow from a1–a5 and accepted definitions (see model 3 in Appendix B). In (1936), on the first page, Russell wrote that "the existence of instants requires hypotheses which there is no reason to suppose true", and further he adds: "There is, however, no reason, either logical or empirical, for supposing these assumptions [i.e. $(\mathtt{e})$ and $(\mathtt{s})$] to be true." However, Theorems 3.5 and 3.9 show that these assumptions are not only sufficient but also necessary to obtain the first and the last instant at which a given event exists, respectively.

Furthermore, it is not clear why in (1936) Russell challenges conditions $(\mathtt{e})$ and $(\mathtt{s})$, in (1914) he accepts the condition $(\star_{\mathrm{E}}^{\rightrightarrows})$, and in (1936) definitions of relations *begins before* $(:= \mathsf{S} \mid \mathsf{P})$ and *ends after* $(:= \mathsf{S} \mid \breve{\mathsf{P}})$. Namely, condition $(\star_{\mathrm{E}}^{\rightrightarrows})$ says that if $x \mathrel{\mathsf{E}} y$, there should exist an event z such that $z \mathrel{\mathsf{S}} x$ and $z \mathrel{\mathsf{P}} y$. Also, the conditions $(\star_{\mathrm{E}}^{\rightrightarrows})$ and $(\mathtt{e})$ together say that if $x \mathrel{\mathsf{E}} y$, there is an event u such that $u \mathrel{\mathsf{IC}} x$ and $u \mathrel{\mathsf{P}} y$. So if we accept $(\star_{\mathrm{E}}^{\rightrightarrows})$ and reject $(\mathtt{e})$, then there are events x, y and z such that $x \mathrel{\mathsf{E}} y$, $z \mathrel{\mathsf{S}} x$, $x \mathrel{\mathsf{E}} z$ and $z \mathrel{\mathsf{P}} y$ hold. After all, in such a situation, it seems more natural be exist an initial fragment z of x which wholly precedes y, i.e. we have $z \mathrel{\mathsf{BT}} x$ and $z \mathrel{\mathsf{P}} y$. The initial fragment satisfies condition $(\mathtt{e})$. Similar considerations can be made for condition $(\mathtt{e})$ and the definition of the relation *begins before*, as well for condition $(\mathtt{s})$ and the definition of the relation *ends after*, which are considered in (Russell, 1936, p. 348).

3.6. **The density of $\prec$.** Russell's third requirement for $\prec$ says that it is dense on $\mathbf{In}$ (see 1914, 119), i.e.:

$$\forall_{\alpha,\beta\in\mathbf{In}}(\alpha \prec \beta \implies \exists_{\gamma\in\mathbf{In}}(\alpha \prec \gamma \wedge \gamma \prec \beta)). \tag{$\mathtt{d}_\prec$}$$

To obtain this, Russell (1914, p. 120) assumed a specific condition which he reiterated in paragraph III of footnote 1 as: "(f) If one event wholly precedes another, there is an event wholly after the one and simultaneous with something wholly before the other." Formally:

$$\text{a7} \qquad \forall_{x,y\in\mathsf{U}}(x \mathrel{\mathsf{P}} y \implies \exists_{z\in\mathsf{U}}(x \mathrel{\mathsf{P}} z \wedge \exists_{u\in\mathsf{U}}(z \mathrel{\mathsf{S}} u \wedge u \mathrel{\mathsf{P}} y))). \tag{f}$$

To make (f) easier to express, we can use $(\star^{\Leftarrow}_{\mathsf{E}})$, obtaining:

$$\forall_{x,y\in\mathsf{U}}(x \mathrel{\mathsf{P}} y \implies \exists_{z\in\mathsf{U}}(x \mathrel{\mathsf{P}} z \wedge z \mathrel{\mathsf{E}} y)). \tag{f'}$$

We can prove that $(\mathtt{d}_\prec)$ follows from (f'), (3.7), (3.8), $(\mathtt{t}_\prec)$ and Fact 3.6. Moreover, note that as an axiom we could take (f') because (f) follows from $(\star^{\Rightarrow}_{\mathsf{E}})$ and (f').

About assumption (f), Russell, however, notes:

> Whether this is the case or not, is an empirical question; but if it is not, there is no reason to expect the time-series to be compact [dense]. (Russell, 1914, p. 120)

The above shows that Russell believed that, in a specific sense, axiom (f) is not only sufficient but also necessary for the density of $\prec$ in the set of instants. It turns out that this is so.

Theorem 3.11. *If conditions* a1–a5, $(\mathtt{e})$, $(\mathtt{s})$ *hold and* $\prec$ *is dense, then* (f) *holds.*

Thus, the following comes as a surprise:

> In KEW [i.e. (1914)] Russell adopts postulates II and III [i.e. axioms $(\mathtt{e})$ and (f), respectively], which entail the compactness [i.e. density] of the series of instants [...]. But these postulates are seriously a posteriori. I agree with the Russell of OT [i.e. (1936)] that there are no obvious reasons for accepting either as epistemically basic. So we should seek more evident sufficient conditions for compactness.
> (Anderson, 1989, p. 256)
>
> Alas, the best condition I can think of that entails compactness is just:
>
> E. $P(x,y) \supset (\exists z)[P(x,z).P(z,y)]$,
>
> i.e., there is an event between x and y if x wholly precedes y. As far as I can see, Russell's two postulates from KEW [...] are no better epistemically than this. Worse, I think. (Anderson, 1989, p. 257)

Anderson's words above are surprising for two reasons. Firstly, his condition E expresses the density of P:

$$\forall_{x,y\in\mathsf{U}}(x \mathrel{\mathsf{P}} y \implies \exists_{z\in\mathsf{U}}(x \mathrel{\mathsf{P}} z \wedge z \mathrel{\mathsf{P}} y)). \tag{$\mathtt{d}_{\mathsf{P}}$}$$

The density of P entails (f) since S is reflexive. Thus, each epistemic model of $(\mathtt{d}_{\mathsf{P}})$ is also a model of (f). Thus, it is easier to find the epistemic model of (f) than for $(\mathtt{d}_{\mathsf{P}})$. Second, all assumptions of Theorem 3.11 apply in theory from (Russell, 1936). Thus we find no "more evident sufficient conditions" for density.

4. Russell's second theory of events and instants

As we remember (see Remarks 2.1 and 2.4), Russell's second theory of events and instants is a theory of structures of the form $\langle \mathsf{U}, \mathsf{P}\rangle$, where U is a non-empty universe (composed of events) and P is a binary relation on U. In such structures Russell (1936, p. 348) defines the relation $\mathsf{S} := \overline{\mathsf{P}} \setminus \breve{\mathsf{P}}$, i.e., in elementary notation, he used condition $(\mathrm{df}_{\mathsf{P}}\mathsf{S})$. The first three axioms of Russell's second theory express the irreflexivity of P and the transitivity of P and $\mathrm{S} \mid \mathrm{P}$. So, in elementary notation, he used conditions ($\mathtt{t}_{\mathrm{P}}$) and the following:

$$\forall_{x,y,z\in\mathrm{U}}(\exists_{u\in\mathrm{U}}(x\,\mathrm{S}\,u \wedge u\,\mathrm{P}\,y) \wedge \exists_{u\in\mathrm{U}}(y\,\mathrm{S}\,u \wedge u\,\mathrm{P}\,z) \implies \exists_{u\in\mathrm{U}}(y\,\mathrm{S}\,u \wedge u\,\mathrm{P}\,z)). \tag{$\mathtt{t}_{\mathrm{S}|\mathrm{P}}$}$$

In (1936, p. 348), Russell also introduced the relations: *begins before* $:= \mathrm{S} \mid \mathrm{P}$, *begins after* $:= \breve{\mathrm{P}} \mid \mathrm{S}$, *ends after* $:= \mathrm{S} \mid \breve{\mathrm{P}}$ and *ends before* $:= \mathrm{P} \mid \breve{\mathrm{S}}$. Since S is symmetric, i.e. $\breve{\mathrm{S}} = \mathrm{S}$, *begins after* and *ends before* are converse to *begins before* and *ends after*, respectively. Indeed, (*begins before*)$\breve{}$ $= (\mathrm{S} \mid \mathrm{P})\breve{} = \breve{\mathrm{P}} \mid \mathrm{S} =$: *begins after* and (*ends after*)$\breve{}$ $= (\mathrm{S} \mid \breve{\mathrm{P}})\breve{} = \mathrm{P} \mid \mathrm{S} =$: *ends before*. The relations *begins before* and *ends after* correspond to the relations E and A, respectively. In (Russell, 1936, p. 350), the set **In** is defined in the same way as in (Russell, 1914).

4.1. **Russell's second theory vs his first theory of events.** To compare the theory from (Russell, 1936) with the theory from (Russell, 1914), we need to define the relation E in the former. For this, let us assume that $\mathrm{E} := \mathrm{S} \mid \mathrm{P} =$: *begins before*, i.e., in elementary notation, we use condition ($\star_{\mathrm{E}}$). We can prove that the fragment of Russell's second theory based on the first three of its axioms is definitionally equivalent to the fragment of his first theory based on axioms a1–a5:

Theorem 4.1. *The theory based on* ($\mathtt{irr}_{\mathrm{P}}$), ($\mathtt{t}_{\mathrm{P}}$), ($\mathrm{df}_{\mathrm{P}}\mathrm{S}$), ($\mathtt{t}_{\mathrm{S}|\mathrm{P}}$), ($\star_{\mathrm{E}}$) *is equivalent to the theory based on* a1–a5, (df P).

By the above theorem and model 3 from Appendix B, condition ($\mathtt{e}$) does not follow from the first three axioms of Russell's second theory plus definitions (df IC). So it is interesting how Russell obtained in (1936) that for any event x, the set ϕ_x belong to **In**.

We will show that Russell (1936, p. 353) accepts ($\mathtt{e}$) as an additional assumption. Firstly, we read: "The *initial contemporaries* of an event x are the events which exists when x begins, i.e. $\vec{\mathrm{S}}$‘x – $\breve{\mathrm{P}}$“$\vec{\mathrm{S}}$‘x." So there is an event y that satisfies the condition: $x\ \mathrm{S}\ y \wedge \neg\exists_{z\in\mathrm{U}}(x\ \mathrm{S}\ z \wedge z\ \mathrm{P}\ y)$. Given ($\star_{\mathrm{E}}$), we simplify it to the condition: $x\ \mathrm{S}\ y \wedge \neg\, x\ \mathrm{E}\ y$. However, the latter defines $y\ \mathrm{IC}\ x$. Secondly, we read: "The *subsequent contemporaries* of an event x are the events which overlap with x but begin later, i.e. $\vec{\mathrm{S}}$‘$x \cap \breve{\mathrm{P}}$“$\vec{\mathrm{S}}$‘$x$." So there are events y that satisfy the condition: $x\ \mathrm{S}\ y \wedge \exists_{z\in\mathrm{U}}(x\ \mathrm{S}\ z \wedge z\ \mathrm{P}\ y)$. Given ($\star_{\mathrm{E}}$), we simplify it to the condition: $x\ \mathrm{S}\ y \wedge x\ \mathrm{E}\ y$. Third, we read: "Then our condition for the existence of a first instant of x is: Every subsequent contemporary of x begins after the end of some initial contemporary of x." So, literally speaking, the condition given by Russell formally has the form:

$$\begin{aligned}&\forall_{x,y\in\mathrm{U}}(x\ \mathrm{S}\ y \wedge \exists_{z\in\mathrm{U}}(x\ \mathrm{S}\ z \wedge z\ \mathrm{P}\ y) \implies \exists_{z\in\mathrm{U}}(z\ \mathrm{P}\ y \wedge z\ \mathrm{IC}\ x)),\\ &\forall_{x,y\in\mathrm{U}}(x\ \mathrm{S}\ y \wedge x\ \mathrm{E}\ y \implies \exists_{z\in\mathrm{U}}(z\ \mathrm{P}\ y \wedge z\ \mathrm{IC}\ x)).\end{aligned}$$

However, these notations can be simplified to ($\mathtt{e}$). Indeed, each of the conditions $\neg\, x\ \mathrm{S}\ y \wedge \exists_{z\in\mathrm{U}}(x\,\mathrm{S}\,z \wedge z\,\mathrm{P}\,y)$ and $\neg\, x\,\mathrm{S}\,y \wedge x\,\mathrm{E}\,y$ entails $x\,\mathrm{P}\,y$. Hence we get $\exists_{z\in\mathrm{U}}(z\,\mathrm{P}\,y \wedge z\,\mathrm{IC}\,x)$ because IC is reflexive.

4.2. **Are there events lasting only for an instant?** The adopted axioms do not exclude that there may be an event that lasts only for an instant. Notice that in model 1 from Appendix B, all events take place for only one, the same instant. Therefore, the fact that no event lasts only for an instant also does not follow from assumptions ($\mathtt{irr}_{\mathrm{P}}$), ($\mathtt{t}_{\mathrm{P}}$), ($\mathtt{t}_{\mathrm{S}|\mathrm{P}}$), ($\mathtt{e}$) and ($\mathtt{s}$) adopted in (Russell, 1936).

In connection with the study of the density of the set of instants, Russell (1936, p. 351) writes:

> [...] in general we shall have $x \,\mathrm{S} \mid \mathrm{P} \mid \mathrm{S}\, x$; this is only falls if
>
> $$\vec{\mathrm{S}}‘x \cap \mathrm{P}“\vec{\mathrm{S}}‘x = \Lambda\ .\ \vec{\mathrm{S}}‘x \cap \breve{\mathrm{P}}“\vec{\mathrm{S}}‘x = \Lambda, \qquad [\text{where } \Lambda := \varnothing]$$

i.e. if no contemporary of x ends before x or begins after x begins. In that case, $\vec{S}‘x \in \mathbf{In}$, i.e. x lasts only for one instant.

However, by ($\mathtt{s}_S$), each of the above two conditions is equivalent to $\neg\, x \mathrel{S} | \mathrel{P} | \mathrel{S} x$, i.e.:

$$\neg\,\exists_{u,v\in \mathbb{U}}(x \mathrel{S} u \wedge u \mathrel{P} v \wedge v \mathrel{S} x). \tag{\$}$$

Indeed, firstly, we have: $\neg\,\exists_{u\in \mathbb{U}}(x \mathrel{S} u \wedge u \mathrel{P} | \mathrel{S} x)$, i.e. "no contemporary of x ends before x." Secondly, we have: $\neg\,\exists_{v\in \mathbb{U}}(x \mathrel{S} v \wedge v \mathrel{\breve{P}} | \mathrel{S} x)$, i.e. "no contemporary of x begins after x begins." Notice that using Theorems 3.5 and 3.9 we obtain:

Lemma 4.2. *For any $x \in \mathbb{U}$, from condition (\$) follow conditions (‰) and (‱), and so we obtain that ϕ_x and λ_x are instants.*

Now note that $x \in \phi_x \subseteq \vec{S}‘x$, $x \in \lambda_x \subseteq \vec{S}‘x$ and $\vec{S}‘x$ satisfies the equivalent of (c1): $\neg\exists_{u\in\mathbb{U}}(u \notin \vec{S}‘x \wedge \forall_{v\in\vec{S}‘x}\, u \mathrel{S} v)$. Hence, by ($\mathtt{In}$), we obtain:

Lemma 4.3. *For any event x: $\vec{S}‘x \in \mathbf{In}$ iff $\vec{S}‘x$ satisfies the condition $\forall_{u,v\in\vec{S}‘x}\, u \mathrel{S} v$.*

Using Lemmas 4.2 and 4.3, Fact 3.1(1), we can prove (without using the axiom of choice and axioms ($\mathtt{e}$), ($\mathtt{s}$)):

Fact 4.4. *for any $x \in \mathbb{U}$, condition (\$) is equivalent to each of the following:*

(i) *there are no events u and v such that $u \mathrel{S} x$, $v \mathrel{S} x$ and $\neg\, u \mathrel{S} v$, i.e. $\forall_{u,v\in\vec{S}‘x}\, u \mathrel{S} v$;*
(ii) *$\vec{S}‘x$ is an instant;*
(iii) *$\mathbb{I}_x = \{\vec{S}‘x\}$;*
(iv) *$\vec{S}‘x \subseteq \phi_x$, and so $\vec{S}‘x = \phi_x$;*
(v) *$\vec{S}‘x \subseteq \lambda_x$, and so $\vec{S}‘x = \lambda_x$.*

Obviously, condition (iii) implies the following:

(vi) $\mathbb{I}_x$ is a singleton, i.e. x lasts only for an instant.

Thus, without using the axiom of choice, it follows that each of equivalent conditions (\$), (i), ..., (v) implies (vi). But, using this axiom, we obtain that

Fact 4.5. *All conditions of* (\$), (i), ..., (vi) *are equivalent to each other.*

Moreover, without using the axiom of choice and axioms ($\mathtt{e}$) and ($\mathtt{s}$), we can derive:

Fact 4.6. *From* (\$) *entail the following conditions*:

(vii) $\mathbb{I}_x = \{\phi_x\}$;
(viii) $\lambda_x = \phi_x \in \mathbf{In}$;
(ix) $\mathbb{I}_x = \{\lambda_x\}$.

However, using axioms ($\mathtt{e}$) and ($\mathtt{s}$) but without the use of the axiom of choice, we obtain:

Fact 4.7. *All conditions of* (\$), (i), ..., (ix) *are equivalent to each other.*

4.3. **A sufficient condition for the existence of instants.** (Russell, 1936, p. 358) states that: "a sufficient condition for the existence of instants is":

$$\exists_{x,y\in\mathbb{U}}(x \mathrel{S} y \wedge \neg\,\exists_{u,v\in\mathbb{U}}(u \mathrel{S} x \wedge u \mathrel{P} v \wedge v \mathrel{S} y)).$$

Next, Russell (1936) concludes: "If this hypothesis is satisfied, there are two events a, x such that [...] and the last instant of a is the first instant of x." Indeed, by applying the facts previously proved in this work but without ($\mathtt{e}$), ($\mathtt{s}$) and the axiom of choice, it can be shown that:

Fact 4.8. *Let x and y be any events such that*

$$x \mathrel{S} y \wedge \neg\,\exists_{u,v\in\mathbb{U}}(u \mathrel{S} x \wedge u \mathrel{P} v \wedge v \mathrel{S} y). \tag{£}$$

Then:

1. $\phi_x = \lambda_y$ *and* $\phi_x \in \mathbf{In}$.
2. *Neither* $x \mathrel{\mathsf{E}} y$ *or* $y \mathrel{\mathsf{A}} x$.
3. $\neg\, x \mathrel{\mathsf{A}} y$ *iff* $x \sqsubseteq_{\mathrm{t}} y$ *and* $\mathbb{I}_x = \{\lambda_x\} = \{\phi_x\} = \{\lambda_y\} \subseteq \mathbb{I}_y$.
4. $\neg\, y \mathrel{\mathsf{E}} x$ *iff* $y \sqsubseteq_{\mathrm{t}} x$ *and* $\mathbb{I}_y = \{\phi_y\} = \{\lambda_y\} = \{\phi_x\} \subseteq \mathbb{I}_x$.
5. $x \equiv_{\mathrm{t}} y$ *iff* $\mathbb{I}_x = \{\phi_x\} = \{\phi_y\} = \{\lambda_x\} = \{\lambda_y\} = \mathbb{I}_y$ *and* x *satisfies condition* ($)*.*

4.4. **The problem of the density of $\prec$ in (Russell, 1936).** In (Russell, 1914, p. 120), the assumption (f), i.e. axiom a7, was controversial for Russell (cf. the quote on p. 15). In (1936, p. 351), Russell writes that condition $(\mathtt{d}_\prec)$ holds "if (a) no event lasts only for an instant, (b) any two overlapping events have at least one instant in common. These conditions are sufficient; the necessary conditions are slightly less stringent." In the light of the previous point, Russell's conditions (a) and (b) can be written as:

$$\forall_{x\in \mathrm{U}}\exists_{u,v\in \mathrm{U}}(x \mathrel{\mathsf{S}} u \wedge u \mathrel{\mathsf{P}} v \wedge v \mathrel{\mathsf{S}} x), \tag{$\mathfrak{a}$}$$

$$\forall_{x,y\in \mathrm{U}}(x \mathrel{\mathsf{S}} y \implies \exists_{\alpha\in\mathbf{In}}\, x, y \in \alpha). \tag{$\mathfrak{b}$}$$

Notice that condition $(\mathfrak{b})$ is the "⇒"-part of (3.2). Moreover, in the light of Theorem 3.11, if indeed conditions $(\mathfrak{a})$ and $(\mathfrak{b})$ was to be sufficient for the density of the relation $\prec$, their conjunction should be equivalent to condition $(\mathfrak{f})$.

However, Russell was wrong, as Anderson (1989, p. 257) demonstrated. Namely, Anderson made a diagram showing a model in which $(\mathtt{irr}_{\mathsf{P}})$, $(\mathtt{t}_{\mathsf{P}})$, $(\mathrm{df}_{\mathsf{P}}\mathsf{S})$, $(\mathtt{t}_{\mathsf{S}\mid\mathsf{P}})$, $(\mathfrak{a})$ and $(\mathfrak{b})$ hold but $(\mathtt{d}_\prec)$ does not hold. In this diagram, Anderson only outlined a fragment of the model by adding: "And if the diagram is imagined to continue in the same way [...]."[5]

Remark 4.1. Anderson (1989, p. 256) wrote:

> Unfortunately, there's an error in Russell's proof. {endnote 10: The error occurs on line 10 from the bottom, p. 219 (line 7 from the bottom of p. 351 of the reprint of OT [i.e. (1936)] in *Logic and Knowledge*, [...]). The stated further condition is not sufficient as claimed [whereby, the "further condition" is to be $(\mathfrak{b})$].}

However, Anderson did not indicate where Russell went wrong in his argument. Let us explain where Russell made a mistake.

Russell (1936, p. 351) notes that the density of $\prec$

> requires [we will be using our symbolism]
>
> $$\alpha,\gamma \in \mathbf{In} \wedge x \in \alpha \wedge z \in \gamma \wedge x \mathrel{\mathsf{P}} y \implies$$
> $$\exists_{\beta,a,c,y,y'}\, a \in \alpha \wedge c \in \gamma \wedge \beta \in \mathbf{In} \wedge y, y' \in \beta \wedge a \mathrel{\mathsf{P}} y \wedge y' \mathrel{\mathsf{P}} c\,.$$
>
> For this it is necessary (not sufficient) to have
>
> $$\mathsf{P} \subseteq \mathsf{S} \mid \mathsf{P} \mid \mathsf{S} \mid \mathsf{P} \mid \mathsf{S}$$
>
> since, in the above, $x \mathrel{\mathsf{S}} a \wedge a \mathrel{\mathsf{P}} y \wedge y \mathrel{\mathsf{S}} y' \wedge y' \mathrel{\mathsf{P}} c \wedge c \mathrel{\mathsf{S}} z$ and $x \mathrel{\mathsf{P}} z$.

Next, Russell aptly points out that assuming $(\mathfrak{a})$ we get:

$$\mathsf{P} \mid \mathsf{S} \subseteq \mathsf{S} \mid \mathsf{P} \mid \mathsf{S} \mid \mathsf{P} \mid \mathsf{S}.$$

And since, as Russell notices, "$\mathsf{P} \subseteq \mathsf{P} \mid \mathsf{S}$ is always true", we get $\mathsf{P} \subseteq \mathsf{S} \mid \mathsf{P} \mid \mathsf{S} \mid \mathsf{P} \mid \mathsf{S}$. Next Russell adds: "The only further condition required for [the density of $\prec$] is, by the above, [$(\mathfrak{b})$]." Russell, however, does not explain this. Indeed, the condition "$\mathsf{P} \subseteq \mathsf{S} \mid \mathsf{P} \mid \mathsf{S} \mid \mathsf{P} \mid \mathsf{S}$" says that for arbitrary $x, z \in \mathrm{U}$ we have:

$$x \mathrel{\mathsf{P}} z \implies \exists_{a,y,c,y'\in \mathrm{U}}(x \mathrel{\mathsf{S}} a \wedge a \mathrel{\mathsf{P}} y \wedge y \mathrel{\mathsf{S}} y' \wedge y' \mathrel{\mathsf{P}} c \wedge c \mathrel{\mathsf{S}} z).$$

But using $(\mathfrak{b})$, from the above, we only get:

$$\forall_{\alpha,\gamma\in\mathbf{In}}\forall_{x\in\alpha}\forall_{z\in\gamma}(x \mathrel{\mathsf{P}} z \implies \exists_{\alpha',\gamma',\beta\in\mathbf{In}}(x \in \alpha' \wedge \alpha' \prec \beta \prec \gamma' \wedge z \in \gamma'),$$

[5] Model 4 in Appendix B formally presents Anderson's diagram. See also model 5 in this appendix.

which is even true in Anderson's diagram (see model 4, p. 27), where $\alpha' \prec \alpha$, $\beta = \alpha$ and $\gamma \prec \gamma'$, and in model 5. We also do not get ($\mathtt{d}_{\prec}$) starting with the condition:

$$x \mathrel{\mathsf{P}} z \implies \exists_{a,y,c\in\mathsf{U}}(x \mathrel{\mathsf{S}} a \wedge a \mathrel{\mathsf{P}} y \wedge y \mathrel{\mathsf{S}} x \wedge x \mathrel{\mathsf{P}} c \wedge c \mathrel{\mathsf{S}} z),$$

which is stronger than "$\mathsf{P} \subseteq \mathsf{S} \mid \mathsf{P} \mid \mathsf{S} \mid \mathsf{P} \mid \mathsf{S}$". From the above and ($\mathfrak{b}$) we only get:

$$\forall_{\alpha,\gamma\in\mathbf{In}}\forall_{x\in\alpha,\, z\in\gamma}(x \mathrel{\mathsf{P}} z \implies \exists_{\alpha',\gamma',\beta\in\mathbf{In}}(x \in \alpha' \wedge x \in \beta \wedge \alpha' \prec \beta \prec \gamma' \wedge z \in \gamma')),$$

which is also true in models 4 and 5 on p. 27.

Finally, note that Russell unnecessarily complicated the notation because it was enough to note that from ($\mathfrak{a}$) for arbitrary $x, y \in \mathsf{U}$ we have a condition:

$$x \mathrel{\mathsf{P}} z \implies \exists_{a,y\in\mathsf{U}}(x \mathrel{\mathsf{S}} a \wedge a \mathrel{\mathsf{P}} y \wedge y \mathrel{\mathsf{S}} x \wedge x \mathrel{\mathsf{P}} z),$$

which is stronger than previously considered. Hence, by ($\mathfrak{b}$), we obtain:

$$\forall_{\alpha,\gamma\in\mathbf{In}}\forall_{x\in\alpha}\forall_{z\in\gamma}(x \mathrel{\mathsf{P}} z \implies \exists_{\alpha',\beta\in\mathbf{In}}(x \in \alpha' \wedge x \in \beta \wedge \alpha' \prec \beta \prec \gamma)),$$

which is not true in Anderson's diagram (model 4 but it is true in model 5. Unfortunately, in neither of the three considered cases, we get $\alpha \preceq \alpha'$ (see model 5). ⋄

5. Thomason's theory of events vs Russell's theories

Thomason's theory from (1989), like Russell's theory from (1936), has a primitive relation P characterized by two axioms ($\mathtt{irr}_{\mathsf{P}}$) and ($\mathrm{Th}_{\mathsf{P}}$). Thus, we obtain both axioms of Thomason's theory in the theory we reconstruct.

Furthermore, Thomason (1989, p. 49) defined the relations *begins before* and *ends before* as $\mathsf{bb} := (\breve{\mathsf{P}})^{-} \mid \mathsf{P}$ and $\mathsf{eb} := \mathsf{P} \mid (\breve{\mathsf{P}})^{-}$, respectively. So, by ($\mathrm{df}_{\mathsf{P}}\mathsf{E}$), bb corresponds with E. Moreover, in virtue of (2.8), we have $\mathsf{eb} = \mathsf{P} \mid \mathsf{S}$; so eb corresponds with $\breve{\mathsf{A}}$ since $\breve{\mathsf{A}} = (\mathsf{S} \mid \breve{\mathsf{P}})\breve{} = \mathsf{P} \mid \breve{\mathsf{S}} = \mathsf{P} \mid \mathsf{S}$.

From (Th_{P}) we obtain '$x \mathsf{P} y \wedge y \mathsf{P} x \Rightarrow x \mathsf{P} x \vee y \mathsf{P} y$' and '$x \mathsf{P} y \wedge y \mathsf{P} u \Rightarrow x \mathsf{P} u \vee y \mathsf{P} y$'. From them and ($\mathtt{irr}_{\mathsf{P}}$), we obtain ($\mathtt{as}_{\mathsf{P}}$) and ($\mathtt{t}_{\mathsf{P}}$), respectively.

To compare Russell's theories with the theory from (Thomason, 1989), we must define the relations E and S in the later one. For this, we put $\mathsf{E} := (\breve{\mathsf{P}})^{-} \mid \mathsf{P} =: \mathsf{bb}$, i.e., we use ($\mathrm{df}_{\mathsf{P}}\mathsf{E}$); and $\mathsf{S} := \bar{\mathsf{P}} \cap (\breve{\mathsf{P}})^{-}$, i.e., we use ($\mathrm{df}_{\mathsf{P}}\mathsf{S}$). We can prove that both the fragment of Russell's first theory based on axioms a1–a5 and the fragment of Russell's second theory based on the first three of its axioms are definitionally equivalent to Thomason's theory based on ($\mathtt{irr}_{\mathsf{P}}$) and ($\mathrm{Th}_{\mathsf{P}}$), i.e. we can prove that (cf. Theorem 4.1):

Theorem 5.1. *Let* U *be a non-empty set and* P, S, E *be binary relations on* U. *Then the following theories of structures of the form* $\langle \mathsf{U}, \mathsf{P}, \mathsf{S}, \mathsf{E}\rangle$ *are equivalent*:

(1) *the theory based on* ($\mathtt{irr}_{\mathsf{P}}$), ($\mathtt{t}_{\mathsf{P}}$), ($\mathtt{t}_{\mathsf{S}\mid\mathsf{P}}$), ($\mathrm{df}_{\mathsf{P}}\mathsf{S}$), ($\star_{\mathsf{E}}$);
(2) *the theory based on* a1–a5, (df P);
(3) *the theory based on* ($\mathtt{irr}_{\mathsf{P}}$), ($\mathrm{Th}_{\mathsf{P}}$), ($\mathrm{df}_{\mathsf{P}}\mathsf{E}$), ($\mathrm{df}_{\mathsf{P}}\mathsf{S}$).

6. Russell's and Whitehead's influence on the works of Leśniewski, Tarski and Lejewski

When Russell wrote "one spatial object may be contained within another, and entirely enclosed by the other" (see quote on p. 2), he meant the relation of *being a part of* on which Leśniewski (1991a,b) based his theory called *mereology*. With this concept, Leśniewski defines the concept of *being a collective class of* (*some objects*). Collective classes (sets) are certain wholes composed of parts. Of course, Leśniewski did not invent the concept of a *collective class*. It is discussed, for example, by Whitehead and Russell in their comments in *Principia Mathematica* (1910–3) concerning the theory of classes developed in that work. Whitehead made use of such sets in his thoughts on the philosophy of space-time (see, e.g., 1919; 1920; 1929). However, Whitehead and Russell used these terms informally. They were formalized only by Leśniewski, who created mereology.

In constructing instants, Russell used Whitehead's *method of extensive abstraction* (which he knew before Whitehead published it). The operation of this method can be seen in Tarski's (1929; 1956) construction of points in his geometry of solids. By presenting this construction, it is easier to explain what Russell meant when in the quote mentioned above from p. 2, he wrote about the point as a specific class of spatial objects which contain a point. The primary concepts of Tarski's theory are the concept of *solid* (as an appropriate piece of space), the concept of *a ball* (or *a sphere*; as a solid of a particular kind), and the notion of *being a part of*. Points are identified with distributive sets of concentric balls. Intuitively, the point is common to all balls concentric with a given ball. We abstract from everything that is not common to concentric balls, i.e. we also ignore their "extensiveness". Only their middle remains. Of course, this centre is not in the universe of considerations, i.e. among solids. We treat it as an abstract entity. To put it succinctly, a set of concentric balls "imitates" a point (which, as a distributive set, is an abstract entity). The main idea of Tarski's theory is that one can define the relation *concentricity of balls* using only the notion *being a ball* and the notion *being a part of*.

Let $\mathbb{S}$ be the set of all solids and $\mathbb{B}$ be the set of all balls ($\mathbb{B} \subsetneq \mathbb{S}$). In the set $\mathbb{B}$, we define a concentric relation $\circledcirc$, which is reflexive, symmetric and transitive (see Gruszczyński and Pietruszczak, 2008, 2009). For any ball b, we define a distributive set π_b of all concentric balls from b, i.e. $\pi_b := \{x \in \mathbb{B} : x \circledcirc b\}$. We identify the set π_b with the point defined by b. Generally, a *point* is a distributive set of all balls concentric with a given ball. Let Π be the family of all points, i.e. we put $\Pi := \{\pi \in 2^{\mathbb{B}} : \exists_{b \in \mathbb{B}}\, \pi = \pi_b\} = \{\pi_b : b \in \mathbb{B}\}$. Notice that for the points, we get the following conditions:

(p1) $\neg\exists_{b\in\mathbb{B}}(b \notin \pi \wedge \forall_{x\in\pi}\, b \circledcirc x)$,

(p2) $\forall_{x,y\in\pi}\, x \circledcirc y$.

The pair of the above conditions is logically equivalent to the following:

$$\forall_{b\in\mathbb{B}}(b \in \pi \Leftrightarrow \forall_{x\in\alpha}\, b \circledcirc x), \qquad \text{i.e., } \pi = \{b \in \mathbb{B} : \forall_{x\in\pi}\, b \circledcirc x\}. \tag{Pt}$$

The above conditions correspond to the conditions (c1), (c2) and (In) that Russell used to construct instants from events. However, there is a fundamental difference between Russell's solution and Tarski's solution. For any instant α, Russell gets $\alpha = \bigcap\{\vec{\mathrm{S}}`x : x \in \alpha\}$. However, in Tarski's theory, there is a trivial condition to this. Namely, for any point $\pi$ we have $\{\pi_b : b \in \pi\} = \{\pi\}$, where $\pi_b = \vec{\circledcirc}`b$, so $\pi = \bigcap\{\pi\}$. Indeed, for any $x, y \in \pi$ $\pi_x = \pi = \pi_y$ holds since $\circledcirc$ is an equivalence relation. For a set of instants, such a trivial case does not hold because S is not an equivalence relation, i.e. for different instants x and y, we can have $\vec{\mathrm{S}}`x \neq \vec{\mathrm{S}}`y$.

In comparing Russell's construction of instants with Tarski's construction of points, it is irrelevant that in the former, we used events and in the latter, only balls, not all solids. Namely, Tarski's construction of points can be modified in such a way that—as in the presented quote—a given point is the set of all solids that can be said to contain it. However, the points are still designated by balls. Thus, for any ball b, to the point π_b belongs every and only such a solid whose part is some sphere concentric with the sphere b, i.e., for any $s \in \mathbb{S}$: $s \in \pi_b$ iff there is an $x \in \mathbb{B}$ such that $x \circledcirc b$ and x is part of s. With this approach, we still get $\{\pi_b : b \in \pi\} = \{\pi\}$.

Tarski's geometry of solids based on mereology is classified as point-free geometries, in which the primary notion of *spatial region* is used instead of the primary notion of *dimensionless point*. In it, points are defined as abstract formations obtained based on regions. Let us add that the universe of the theory, i.e. the set of all regions, is not space. The space is the largest region; the other regions are its mereological parts.

It can be assumed that Russell's paper also influenced Lejewski's work, who created a theory, *chronology*, which is based directly on mereology, and uses two primitives: *is* *wholly earlier than* and *is an object whose duration is shorter than that of* (see Lejewski, 1982, 1986). Lejewski (1986) also planned to create a theory of objects extended and distributed in space, which he called *stereology*.

Finally, note that Peter Simons (1987, pp. 81, 82, 92, 93) expressed similar views on the topic discussed in this section.

7. Future work – intents again

Russell (1914) outlines a different "solution" to obtain a dense set of instants. It is supposed be to define instants by the relation $\sqsubseteq_{\mathrm{t}}$ of *temporal enclosure*:

> Instants may also be defined by means of the enclosure-relation, exactly as was done in the case of points. [...] In order that the relation of temporal enclosure may be a "point-producer," we require (1) that it should be transitive; (2) that every event encloses itself but if one event encloses another different event, then the other does not enclose the one; (3) that given any set of events such that there is at least one event enclosed by all of them, then there is an event enclosing all that they all enclose, and itself enclosed by all of them; (4) that there is at least one event. To ensure infinite divisibility, we require also that every event should enclose events other than itself. (Russell, 1914, p. 121)

Regarding the requirements of (1) and (2), we showed that the relation $\sqsubseteq_{\mathrm{t}}$ is transitive and reflexive. In Remark 2.5, however, we explained that Russell's requirement for it to be antisymmetric ("if one event encloses another different event, then the other does not enclose the one") is impracticable. By ($\mathrm{df}_{\sqsubseteq_{\mathrm{t}}}\ \equiv_{\mathrm{t}}$), we only have: if $x \sqsubseteq_{\mathrm{t}} y$ and $x \not\equiv_{\mathrm{t}} y$, then $y \not\sqsubseteq_{\mathrm{t}} x$. We will formally express requirement (3) as follows:

$$\forall_{X\in 2^{\mathrm{U}}}\,(\exists_{u\in\mathrm{U}}\forall_{x\in X}\,u \sqsubseteq_{\mathrm{t}} x \implies \exists_{y\in\mathrm{U}}(\forall_{u\in\mathrm{U}}(\forall_{x\in X}(u \sqsubseteq_{\mathrm{t}} x \Rightarrow u \sqsubseteq_{\mathrm{t}} y) \wedge \forall_{x\in X}\, y \sqsubseteq_{\mathrm{t}} x).$$

Requirement (4) is our assumption that $\mathrm{U} \neq \emptyset$. Since $\sqsubseteq_{\mathrm{t}}$ may not be antisymmetric, for the last additional requirement, it should rather have the form: *every event should enclose an event which is not completely simultaneous with it*. Formally:

$$\forall_{x\in\mathrm{U}}\exists_{y\in\mathrm{U}}(y \sqsubseteq_{\mathrm{t}} x \wedge x \not\equiv_{\mathrm{t}} y). \tag{7.1}$$

Continuing with the last quote, Russell writes:

> Assuming these characteristics, temporal enclosure is an infinitely divisible point-producer. We can now form an "enclosure-series" of events, by choosing a group of events such that of any two there is one which encloses the other; this will be a "punctual enclosure-series" if, given any other enclosure-series such that every member of our first series encloses some member of our second, then every member of our second series encloses some member of our first. Then an "instant" is the class of all events which enclose members of a given punctual enclosure-series.

So an *"enclosure-series" of events* is supposed to be any non-empty set Q of events satisfying the following condition:

$$\forall_{x,y\in Q}(x \sqsubseteq_{\mathrm{t}} y \vee y \sqsubseteq_{\mathrm{t}} x). \tag{7.2}$$

Let **ES** be the family of all "enclosure-series" of events. Moreover, a set Q from **ES** is a "punctual enclosure-series" iff Q satisfies the following condition:

$$\forall_{Y\in\mathbf{ES}}(\forall_{x\in Q}\exists_{y\in Y}\, y \sqsubseteq_{\mathrm{t}} x \Rightarrow \forall_{y\in Y}\exists_{x\in Q}\, x \sqsubseteq_{\mathrm{t}} y).$$

Let **PES** be the family of all "punctual enclosure-series". Finally, for any $Q \in \mathbf{PES}$ we put $\mathrm{F}_Q := \{u \in \mathrm{U} : \exists_{x\in Q}\, x \sqsubseteq_{\mathrm{t}} u\}$. In the light (7.2) we can consider F_Q to be a filter generated by Q. So by an instant, Russell means any filter $X = \mathrm{F}_Q$ for $Q \in \mathbf{PES}$.

The way shown by Russell resembles the one chosen by Grzegorczyk (1950, 1960) to define points in mereological fields.[6] It would resemble it more if we had chosen the relation

[6] Grzegorczyk's theory (1960) is examined in detail in (Gruszczyński and Pietruszczak, 2018, 2019).

of *non-tangential temporal enclosure*, $\Subset_{\mathrm{t}}$, instead of the relation $\sqsubseteq_{\mathrm{t}}$. Then, in virtue of (2.17), condition (7.1) can be replaced by the following:

$$\forall_{x\in\mathrm{U}}\exists_{y\in\mathrm{U}}\ y \Subset_{\mathrm{t}} x$$

and condition (7.2) can be replaced by the condition below:

$$\forall_{x,y\in Q}(x \Subset_{\mathrm{t}} y \vee x \equiv_{\mathrm{t}} y \vee y \Subset_{\mathrm{t}} x).$$

It is a counterpart of the first of the three conditions to be met by Grzegorczyk's pre- points in mereological fields (see Grzegorczyk, 1960; Gruszczyński and Pietruszczak, 2018). Counterparts of the second and third of these conditions would respectively say that for any Q from **PES**: $\forall_{x\in Q}\exists_{y\in Q}\ y \Subset_{\mathrm{t}} x$ and $\forall_{x,y\in\mathrm{U}}(\forall_{u\in Q}(u \mathrel{\mathrm{S}} x \wedge u \mathrel{\mathrm{S}} y) \;\Rightarrow\; x \mathrel{\mathrm{S}} y)$. Grzegorczyk (1960), similar to Russell, as a point takes any filter generated by some pre-point. It seems that the sets from **PES** should meet all counterparts of conditions for pre-points.

In future works, we will investigate whether Russell's goals can be achieved in an above-presented way.

Appendix A. Proofs

Proof of $(\mathtt{con}^{\mathrm{S}}_{\mathrm{E}})$. "$(\mathtt{con}^{\mathrm{S}}_{\mathrm{P}}) \Rightarrow (\mathtt{con}^{\mathrm{S}}_{\mathrm{E}})$" In the light of (df P), we make use of $\mathrm{P} \subseteq \mathrm{E}$. "$(\mathtt{con}^{\mathrm{S}}_{\mathrm{E}}) \Rightarrow (\mathtt{con}^{\mathrm{S}}_{\mathrm{P}})$" Suppose that $\neg\, x \mathrel{\mathrm{P}} y$ and $\neg\, y \mathrel{\mathrm{P}} x$. Then, by (df P), we have: $\neg\, x \mathrel{\mathrm{E}} y \vee x \mathrel{\mathrm{S}} y$ and $\neg\, y \mathrel{\mathrm{E}} x \vee y \mathrel{\mathrm{S}} x$. Hence, by $(\mathtt{s}_{\mathrm{S}})$, we get: $x \mathrel{\mathrm{S}} y \vee \neg(x \mathrel{\mathrm{E}} y \vee y \mathrel{\mathrm{E}} x)$. This and $(\mathtt{con}^{\mathrm{S}}_{\mathrm{E}})$ give us $x \mathrel{\mathrm{S}} y$.

Proof of $(\mathtt{ct}_{\mathrm{E}})$. Suppose that (a) $x \mathrel{\mathrm{E}} y$ and (b) $\neg\, x \mathrel{\mathrm{E}} z$. Then, by (df P) and $(\star_{\mathrm{E}})$ we for some $u_0 \in \mathrm{U}$ we have (a$_1$) $x \mathrel{\mathrm{S}} u_0$, (a$_2$) $u_0 \mathrel{\mathrm{P}} y$ and (b′) $\forall_{u\in\mathrm{U}}(x \mathrel{\mathrm{S}} u \;\Rightarrow\; \neg\, u \mathrel{\mathrm{P}} z)$. If $u_0 \mathrel{\mathrm{S}} z$, then $z \mathrel{\mathrm{E}} y$, by $(\mathtt{s}_{\mathrm{S}})$, (a$_2$), (df P) and $(\star^{\Leftarrow}_{\mathrm{E}})$. So we assume that $\neg\, u_0 \mathrel{\mathrm{S}} z$. Then, by $(\mathtt{con}^{\mathrm{S}}_{\mathrm{P}})$, either $u_0 \mathrel{\mathrm{P}} z$ or $z \mathrel{\mathrm{P}} u_0$. But, by (a$_1$) and (b′), we have $\neg\, u_0 \mathrel{\mathrm{P}} z$. So $z \mathrel{\mathrm{P}} u_0$; so also $z \mathrel{\mathrm{P}} u_0$. Moreover, by (a$_2$), also $u_0 \mathrel{\mathrm{E}} y$. Therefore, by $(\mathtt{t}_{\mathrm{E}})$, we have $z \mathrel{\mathrm{E}} y$.

Proof of $(\mathrm{Th}_{\mathrm{P}})$. Suppose that (a) $x \mathrel{\mathrm{P}} y$, (b) $z \mathrel{\mathrm{P}} u$ and (c) $\neg\, x \mathrel{\mathrm{P}} u$. Then, by (c) and (df P), either $\neg\, x \mathrel{\mathrm{E}} u$ or $x \mathrel{\mathrm{S}} u$. In the first case, by (b) and $(\mathtt{t}^{\bar{\mathrm{E}}}_{\mathrm{P}})$, we have $z \mathrel{\mathrm{P}} y$. Hence, by (a) and $(\mathtt{t}_{\mathrm{P}})$, we have $z \mathrel{\mathrm{P}} y$. In the second case, by $(\mathtt{s}_{\mathrm{S}})$ and $(\mathtt{t}{+}_{\mathrm{P}})$, we also have $z \mathrel{\mathrm{P}} x$.

Proof of (2.4). Let (a) $x \mathrel{\mathrm{P}} y$, (b) $y \mathrel{\mathrm{S}} z$, and (c) $\neg\, z \mathrel{\mathrm{S}} x$. Then, by (c) and $(\mathtt{con}^{\mathrm{S}}_{\mathrm{P}})$, either $z \mathrel{\mathrm{P}} x$ or $x \mathrel{\mathrm{P}} z$. However, in the first case, by (a) and $(\mathtt{t}_{\mathrm{P}})$, we have $z \mathrel{\mathrm{P}} y$, which contradicts (b).

Proof of $(\mathtt{as}_{\mathrm{A}})$. Assume for a contradiction that $x \mathrel{\mathrm{A}} y$ and $y \mathrel{\mathrm{A}} x$. Then for some z_1 we have: (a$_1$) $x \mathrel{\mathrm{S}} z_1$ and (b$_1$) $y \mathrel{\mathrm{P}} z_1$; and for some z_2 we have: (a$_2$) $y \mathrel{\mathrm{S}} z_2$ and (b$_2$) $x \mathrel{\mathrm{P}} z_2$. Then from (b$_2$), (a$_2$), (b$_1$), $(\mathtt{s}_{\mathrm{S}})$ and $(\mathtt{t}{+}_{\mathrm{P}})$ we have $x \mathrel{\mathrm{P}} z_1$, which contradicts (a$_1$).

Proof of $(\mathtt{ct}_{\mathrm{A}})$. Suppose that $x \mathrel{\mathrm{A}} y$ and $\neg\, x \mathrel{\mathrm{A}} z$. Then for some u we have (a) $x \mathrel{\mathrm{S}} u$, (b) $y \mathrel{\mathrm{P}} u$ and (c) $\forall_v(x \mathrel{\mathrm{S}} v \Rightarrow \neg\, z \mathrel{\mathrm{P}} v)$. So $\neg\, z \mathrel{\mathrm{P}} u$, by (a) and (c). Hence either $z \mathrel{\mathrm{S}} u$ or $\neg\, z \mathrel{\mathrm{E}} u$. In the first case, by (b), we have $z \mathrel{\mathrm{A}} y$. In the second case, by (b) and $(\mathtt{t}^{\bar{\mathrm{E}}}_{\mathrm{P}})$, we have $y \mathrel{\mathrm{P}} z$. Hence we also have $z \mathrel{\mathrm{A}} y$ since $z \mathrel{\mathrm{S}} z$.

Proof of Lemma 2.1. Let a relation R be asymmetric and co-transitive. Assume that $x \mathrel{R} y$ and $y \mathrel{R} z$. Then $\neg\, z \mathrel{R} y$ since R is asymmetric. So $x \mathrel{R} z$ since R is co-transitive.

Proof of $(\mathtt{t}^{\bar{\mathrm{A}}}_{\mathrm{P}})$. Suppose that (a) $x \mathrel{\mathrm{P}} y$ and (b) $\neg\, z \mathrel{\mathrm{A}} x$, i.e. $\forall_{u\in\mathrm{U}}(u \mathrel{\mathrm{S}} z \;\Rightarrow\; \neg\, x \mathrel{\mathrm{P}} u)$. Then, by (b) and $(\mathtt{r}_{\mathrm{S}})$, we have $\neg\, x \mathrel{\mathrm{P}} z$. But, by (a) and (b), we have $\neg\, y \mathrel{\mathrm{S}} z$. Hence, by $(\mathtt{con}^{\mathrm{S}}_{\mathrm{P}})$, either $y \mathrel{\mathrm{P}} z$ or $z \mathrel{\mathrm{P}} y$. In the first case, by (a) and $(\mathtt{t}_{\mathrm{P}})$, we have a contradiction: $x \mathrel{\mathrm{P}} z$.

Proof of (2.11). Assume that (a) $z \mathrel{\mathrm{S}} x$, (b) $\neg\, z \mathrel{\mathrm{S}} y$ and (c) $\neg\, x \mathrel{\mathrm{A}} y$, i.e. $\forall_{u\in\mathrm{U}}(x \mathrel{\mathrm{S}} u \;\Rightarrow\; y \mathrel{\mathrm{P}} u)$. Then, by $(\mathtt{s}_{\mathrm{S}})$, (a) and (c), we have $\neg\, y \mathrel{\mathrm{P}} z$. Hence, by (b) and $(\mathtt{con}^{\mathrm{S}}_{\mathrm{P}})$, we have $z \mathrel{\mathrm{P}} y$. From this, (a) and $(\star^{\Leftarrow}_{\mathrm{E}})$ we have $x \mathrel{\mathrm{E}} y$.

Proof of (2.14). "$\Rightarrow$" Let (a) $x \mathrel{\mathrm{BT}} y$, i.e. (a$_1$) $\neg\, x \mathrel{\mathrm{E}} y$, (a$_2$) $\neg\, y \mathrel{\mathrm{E}} x$. Now suppose that (b) $\neg\, x \mathrel{\mathrm{E}} z$ and (c) $x \mathrel{\mathrm{S}} z$. Then, by $(\mathtt{ct}_{\mathrm{E}})$, (a$_2$) and (b), we have (d) $\neg\, y \mathrel{\mathrm{E}} z$. Assume for a contradiction that (e) $\neg\, y \mathrel{\mathrm{S}} z$. Then, by (d) and $(\mathtt{con}^{\mathrm{S}}_{\mathrm{E}})$, we have (f) $z \mathrel{\mathrm{E}} y$. However, (2.1), $(\mathtt{s}_{\mathrm{S}})$, (a$_1$), (c), (e) and (f) give a contradiction. So we have $y \mathrel{\mathrm{S}} z$. Moreover, we can get the converse implication similarly. "$\Leftarrow$" Suppose that $\forall_{z\in\mathrm{U}}(\neg\, x \mathrel{\mathrm{E}} z \wedge x \mathrel{\mathrm{S}} z \Leftrightarrow \neg\, y \mathrel{\mathrm{E}} z \wedge y \mathrel{\mathrm{S}} z)$. Then, by $(\mathtt{irr}_{\mathrm{E}})$ and $(\mathtt{r}_{\mathrm{S}})$, we obtain $\neg\, x \mathrel{\mathrm{E}} y$ and $\neg\, y \mathrel{\mathrm{E}} x$, i.e., $x \mathrel{\mathrm{BT}} y$.

Proof of (2.15). Let $x \mathrel{\mathsf{ET}} y$, i.e., $\neg\, x \mathrel{\mathsf{A}} y$ and $\neg\, y \mathrel{\mathsf{A}} x$. Moreover, suppose that $z \mathrel{\mathsf{A}} x$ and $z \mathrel{\mathsf{S}} x$. Then $z \mathrel{\mathsf{A}} y$, by (df$_{\mathsf{A}}$ ET). Now assume for a contradiction that $\neg\, z \mathrel{\mathsf{S}} y$. Then, by (2.11), either $y \mathrel{\mathsf{E}} x$ or $x \mathrel{\mathsf{A}} y$. So $x \mathrel{\mathsf{E}} y$. Hence $\neg\, y \mathrel{\mathsf{E}} x$, by ($\mathtt{as}_{\mathsf{E}}$). So, by (df$'\sqsubseteq_{\mathrm{t}}$), we have $x \sqsubseteq_{\mathrm{t}} y$. Therefore, by (df$_{\mathsf{S}}\sqsubseteq_{\mathrm{t}}$), we obtain a contradiction: $\neg\, z \mathrel{\mathsf{S}} x$. Moreover, we can get the converse implication similarly.

Proof of (2.16). "$\Rightarrow$" Let (a) $x \mathrel{\mathsf{ET}} y$, i.e. (a$_1$) $\neg\, x \mathrel{\mathsf{A}} y$, (a$_2$) $\neg\, y \mathrel{\mathsf{A}} x$. Now suppose that (b) $\neg\, x \mathrel{\mathsf{A}} z$ and (c) $x \mathrel{\mathsf{S}} z$. Then, by ($\mathtt{ct}_{\mathsf{A}}$), (a$_2$) and (b), we have (d) $\neg\, y \mathrel{\mathsf{A}} z$. Assume for a contradiction that (e) $\neg\, y \mathrel{\mathsf{S}} z$. Then, by (d) and ($\mathtt{con}^{\mathsf{S}}_{\mathsf{A}}$), we have (f) $z \mathrel{\mathsf{A}} y$. But, by (2.10), (e) and (f), we have $y \mathrel{\mathsf{P}} z$. From this, (c) and (df A), we obtain a contradiction: $x \mathrel{\mathsf{A}} y$. Moreover, we can get the converse implication similarly. "$\Leftarrow$" Suppose that $\forall_{z\in\mathsf{U}}(\neg\, x \mathrel{\mathsf{A}} z \wedge x \mathrel{\mathsf{S}} z \Leftrightarrow \neg\, y \mathrel{\mathsf{A}} z \wedge y \mathrel{\mathsf{S}} z)$. Then, by ($\mathtt{irr}_{\mathsf{A}}$) and ($\mathtt{r}_{\mathsf{S}}$), we obtain $\neg\, x \mathrel{\mathsf{A}} y$ and $\neg\, y \mathrel{\mathsf{A}} x$, i.e. $x \mathrel{\mathsf{ET}} y$.

Proof of (df$_{\mathsf{S}}\sqsubseteq_{\mathrm{t}}$). Firstly, from (df$'\sqsubseteq_{\mathrm{t}}$) and (2.11) we have: $\forall_{x,y\in\mathsf{U}}(x \sqsubseteq_{\mathrm{t}} y \implies \forall_{z\in\mathsf{U}}(z \mathrel{\mathsf{S}} x \Rightarrow z \mathrel{\mathsf{S}} y))$.

Secondly, from (df A), (df P) and ($\mathtt{s}_{\mathsf{S}}$) we obtain: $\forall_{x,y}(\forall_{z\in\mathsf{U}}(z \mathrel{\mathsf{S}} x \Rightarrow z \mathrel{\mathsf{S}} y) \implies \neg\, x \mathsf{A} y)$; and from ($\star^{\rightleftarrows}_{\mathsf{E}}$), ($\mathtt{s}_{\mathsf{S}}$) and (df P) we have: $\forall_{x,y}(\forall_{z\in\mathsf{U}}(z\mathsf{S}x \Rightarrow z\mathsf{S}y) \implies \neg\, x\mathsf{E}y)$. Hence, by (df$'\sqsubseteq_{\mathrm{t}}$), we have: $\forall_{x,y}(\forall_{z\in\mathsf{U}}(z \mathrel{\mathsf{S}} x \Rightarrow z \mathrel{\mathsf{S}} y) \implies x \sqsubseteq_{\mathrm{t}} y)$.

Proof of Fact 3.1. 1. Assume for a contradiction that for some $\alpha, \beta \in \mathbf{In}$ we have $\alpha \subseteq \beta$ and $\alpha \neq \beta$. Then, by (3.1), for some $x \in \alpha$ and $y \in \beta$ we have $\neg\, x \mathrel{\mathsf{S}} y$. However, we also have $x \in \beta$. So $x \mathrel{\mathsf{S}} y$, by (c2).

2. Let us take any set X of events which is maximal in the family of sets satisfying (c2). We will show that X also satisfies (c1). For this, assume for a contradiction that there is $u \in \mathsf{U}$ such that $u \notin X$ and $\forall_{x\in X}\, u \mathrel{\mathsf{S}} x$. Then $Y := X \cup \{u\}$ satisfies (c2). So X is not maximal among sets satisfying (c2).

Proof of (3.2). "$\Rightarrow$" For any chain C of sets satisfying the condition (c2), the sum $\bigcup C$ also satisfies this condition. Thus, in any partially ordered (by the relation of inclusion) family of sets that satisfy (c2), each chain C has a supremum, which is $\bigcup C$. Moreover, for any $x, y \in \mathsf{U}$ such that $x \mathrel{\mathsf{S}} y$, the set $\{x, y\}$ satisfies (c2). So let us take the family $\mathcal{F}_{x,y}$ of all sets that satisfy (c2) and include $\{x, y\}$. The Kuratowski-Zorn lemma shows that there is a maximal set in $\mathcal{F}_{x,y}$. By Fact 3.1(2), we get that this set belongs to $\mathbb{I}_x \cap \mathbb{I}_y$. "$\Leftarrow$" Directly from (c2).

Proof of Theorem 3.2. Ad ($\mathtt{irr}_{\prec}$): Assume for a contradiction that for some $\alpha \in \mathbf{In}$ we have $\alpha \prec \alpha$. Then for some $x, y \in \alpha$ we have $x \mathrel{\mathsf{P}} y$. But, by (c2), we obtain a contradiction: $x \mathrel{\mathsf{S}} y$.

Ad ($\mathtt{t}_{\prec}$): Suppose that $\alpha \prec \beta$ and $\beta \prec \gamma$. Then for some $x \in \alpha$, $y, z \in \beta$ and $u \in \gamma$ we have $x \mathrel{\mathsf{P}} y$ and $z \mathrel{\mathsf{P}} u$. Hence $y \mathrel{\mathsf{S}} z$, by (c2). Therefore, by ($\mathtt{t}{+}_{\mathsf{P}}$), we obtain $x \mathrel{\mathsf{P}} u$. Hence we have: $\alpha \prec \gamma$.

Ad ($\mathtt{con}_{\prec}$): Suppose that $\alpha \neq \beta$. Then, by (3.1), for some $x \in \alpha$ and $y \in \beta$ we have $\neg\, x \mathrel{\mathsf{S}} y$. Hence, by ($\mathtt{con}^{\mathsf{S}}_{\mathsf{P}}$), either $x \mathrel{\mathsf{P}} y$ or $y \mathrel{\mathsf{P}} x$. So either $\alpha \prec \beta$ or $\beta \prec \alpha$.

Proof of Lemma 3.3. Let a relation R be connex and transitive. Suppose that $x \mathrel{R} y$. If $y = z$ then we have $x \mathrel{R} z$. If $y \neq z$, then either $y \mathrel{R} z$ or $z \mathrel{R} y$ since R is connex. In the first case, we have $x \mathrel{R} z$, by the transitivity of R.

Proof of Theorem 3.5. "$\Rightarrow$" Suppose that (%) does not hold, i.e., for some y_0 we have (a) $x \mathrel{\mathsf{E}} y_0$ and (b) for any z, if $z \mathrel{\mathsf{S}} x$ and $\neg\, x \mathrel{\mathsf{E}} z$, then either $z \mathrel{\mathsf{S}} y_0$ or $\neg\, z \mathrel{\mathsf{E}} y_0$. But, by (2.3), if $\neg\, x \mathrel{\mathsf{E}} z$ and $\neg\, z \mathrel{\mathsf{E}} y_0$, then $\neg\, x \mathrel{\mathsf{E}} y_0$. From this, (a), (df ϕ_x) and (b) we obtain: for any $z \in \phi_x$ we have $z \mathrel{\mathsf{S}} y_0$. Since, by (a), $y_0 \notin \phi_x$, from Lemma 3.4 we obtain that $\phi_x \notin \mathbf{In}$. "$\Leftarrow$" In virtue of Lemma 3.4, we need to check that ϕ_x satisfies the equivalent of condition (c1): $\neg\exists_{u\in\mathsf{U}}(u \notin \phi_x \wedge \forall_{v\in\lambda_x}\, u \mathrel{\mathsf{S}} v)$. Assume for a contradiction that for some $u_0 \in \mathsf{U}$ we have $u_0 \notin \phi_x$ and $\forall_{v\in\phi_x}\, u_0 \mathrel{\mathsf{S}} v$. Then $\neg\, u_0 \mathrel{\mathsf{IC}} x$ and $u_0 \mathrel{\mathsf{S}} x$. Hence $x \mathrel{\mathsf{E}} u_0$. Moreover, by (%), for some z_0 we have (i) $z_0 \mathrel{\mathsf{P}} u_0$ and (ii) $z_0 \mathrel{\mathsf{IC}} x$. By (ii), we have $z_0 \in \phi_x$. Therefore, $z_0 \mathrel{\mathsf{S}} u_0$, by ($\mathtt{s}_{\mathsf{S}}$). But it is contrary to (i).

Proof of (3.4). "$\Rightarrow$" Suppose that $x \mathrel{\mathsf{BT}} y$. Then for any $u \in \mathsf{U}$: $u \in \phi_x$ iff $u \mathrel{\mathsf{IC}} x$ iff $u \mathrel{\mathsf{S}} x$ and $\neg\, x \mathrel{\mathsf{E}} u$ iff (by ($\mathtt{s}_{\mathsf{S}}$) and (2.14)) $u \mathrel{\mathsf{S}} y$ and $\neg\, y \mathrel{\mathsf{E}} u$ iff $u \mathrel{\mathsf{IC}} y$ iff $u \in \phi_y$. "$\Leftarrow$" Let

$\phi_x = \phi_y$. Then, since $x \in \phi_x$ and $y \in \phi_y$, we have $x \in \phi_y$ and $y \in \phi_x$, i.e, $x \mathrel{\mathsf{IC}} y$ and $y \mathrel{\mathsf{IC}} x$. So, by (2.18), we have $x \mathrel{\mathsf{BT}} y$.

Proof of (3.6). Assume for a contradiction that for some event x and instant α, we have (a) $x \in \alpha$ and (b) $\alpha \prec \phi_x$. From (b) for some $y, z \in \mathsf{U}$ we have (c) $y \in \alpha$, (d) $z \in \phi_x$ and (e) $y \mathrel{\mathsf{P}} z$. From (a), (c), (e) and (3.5) we have $x \mathrel{\mathsf{E}} z$. Furthermore, from (d), we have $z \mathrel{\mathsf{IC}} x$. So we obtain a contradiction: $\neg\, x \mathrel{\mathsf{E}} z$.

Proof of (3.8). "⇒" Let $x \mathrel{\mathsf{E}} y$. Then (a) $\neg\, y \mathrel{\mathsf{E}} x$, by $(\mathtt{as}_{\mathsf{E}})$, and (b) $\neg\, x \mathrel{\mathsf{BT}} y$. From (b) and (3.4) we have (c) $\phi_x \neq \phi_y$. If $x \mathrel{\mathsf{S}} y$, then $x \mathrel{\mathsf{IC}} y$, by (a). Hence $x \in \phi_y$. So $\phi_x \prec \phi_y$, by (c), (3.6) and $(\mathtt{con}_{\prec})$. If $\neg\, x \mathrel{\mathsf{S}} y$, then $x \mathrel{\mathsf{P}} y$, by (a). So we obtain $\phi_x \prec \phi_y$ since $x \in \phi_x$ and $y \in \phi_y$, by (3.3). "⇐" Suppose that $\phi_x \prec \phi_y$, Then there are events u and v such that (a) $u \mathrel{\mathsf{S}} x$, (b) $\neg\, y \mathrel{\mathsf{E}} v$ and (c) $u \mathrel{\mathsf{P}} v$. Firstly, notice that (d) $x \mathrel{\mathsf{E}} v$. Indeed, if $\neg\, x \mathrel{\mathsf{E}} v$ then, by $(\mathtt{t}^{\bar{\mathsf{E}}}_{\mathsf{P}})$ and (c), we have $u \mathrel{\mathsf{P}} x$, which contradicts (a). Secondly, by $(\mathtt{ct}_{\mathsf{E}})$, (d) and (b), we have $x \mathrel{\mathsf{E}} y$.

Proof of Fact 3.7. "⇒" Let $x \mathrel{\mathsf{S}} y$. Note that one of the following cases occurs: $x \mathrel{\mathsf{BT}} y$, $x \mathrel{\mathsf{E}} y$, $y \mathrel{\mathsf{E}} x$. In the first case, by (3.3) and (3.4), we have $x, y \in \phi_x = \phi_y$. In the second case, by $(\mathtt{as}_{\mathsf{E}})$, we have $\neg\, x \mathrel{\mathsf{E}} y$. Hence, by our assumption, we have $x \mathrel{\mathsf{IC}} y$. Therefore, $x \in \phi_y$; and so we have $x, y \in \phi_y$, by (3.3). Similarly, in the third case, we show $x, y \in \phi_y$.

Proof of (3.10). "⇒" Assume that $\mathbb{I}_x \nsubseteq \mathbb{I}_y$, i.e., for some $\alpha_0 \in \mathbf{In}$ we have $x \in \alpha_0$ and $y \notin \alpha_0$. Then, by (c1), for some $z_0 \in \alpha_0$ we have $\neg\, z_0 \mathrel{\mathsf{S}} y$. Furthermore, by (c2), we have $z_0 \mathrel{\mathsf{S}} x$. Therefore, $x \not\sqsubseteq_{\mathrm{t}} y$, by $(\mathrm{df}_{\mathsf{S}}{\sqsubseteq_{\mathrm{t}}})$. "⇐" Assume that $x \not\sqsubseteq_{\mathrm{t}} y$. Then, by $(\mathrm{df}_{\mathsf{S}}{\sqsubseteq_{\mathrm{t}}})$, for some z_0 we have $z_0 \mathrel{\mathsf{S}} x$ and $\neg\, z_0 \mathrel{\mathsf{S}} y$. Hence, by (3.2), for some $\alpha_0 \in \mathbf{In}$ we have $z_0, x \in \alpha_0$. Moreover, by (c2), we have $y \notin \alpha_0$. So we obtain that $\mathbb{I}_x \nsubseteq \mathbb{I}_y$.

Proof of (‰). "⇒" Suppose that (‰) does not hold, i.e., for some y_0 we have (a) $x \mathrel{\mathsf{A}} y_0$ and (b) for any z, if $z \mathrel{\mathsf{S}} x$ and $\neg\, x \mathrel{\mathsf{A}} z$, then either $z \mathrel{\mathsf{S}} y_0$ or $\neg\, z \mathrel{\mathsf{E}} y_0$. Now notice that, by $(\mathtt{ct}_{\mathsf{A}})$, if $x \mathrel{\mathsf{A}} y_0$ and $\neg\, x \mathrel{\mathsf{A}} z$, then $z \mathrel{\mathsf{A}} y_0$. Moreover, by (2.12), if $\neg\, z \mathrel{\mathsf{E}} y_0$ and $\neg\, z \mathrel{\mathsf{A}} y_0$, then $z \mathrel{\mathsf{S}} y_0$. From this, (a), $(\mathrm{df}\,\lambda_x)$ and (b) we obtain: for any $z \in \lambda_x$ we have $z \mathrel{\mathsf{S}} y_0$. Since, by (a), $y_0 \notin \lambda_x$, from Lemma 3.8, we obtain that $\lambda_x \notin \mathbf{In}$. "⇐" In virtue of Lemma 3.8, we need to check that λ_x satisfies the equivalent of (c1): $\neg\exists_{u \in \mathsf{U}}(u \notin \lambda_x \wedge \forall_{v \in \lambda_x} u \mathrel{\mathsf{S}} v)$. Assume for a contradiction that for some $u_0 \in \mathsf{U}$ we have $u_0 \notin \lambda_x$ and $\forall_{v \in \lambda_x} u_0 \mathrel{\mathsf{S}} v$. Then $\neg\, u_0 \mathrel{\mathsf{FC}} x$ and $u_0 \mathrel{\mathsf{S}} x$. Hence $x \mathrel{\mathsf{A}} u_0$. Moreover, by (‰), for some z_0 we have (i) $z_0 \mathrel{\mathsf{P}} u_0$ and (ii) $z_0 \mathrel{\mathsf{FC}} x$. By (ii), we have $z_0 \in \lambda_x$. Therefore, $z_0 \mathrel{\mathsf{S}} u_0$, by $(\mathtt{s}_{\mathsf{S}})$. However, it is contrary to (i).

Proof of (3.14). "⇒" Suppose that $x \mathrel{\mathsf{ET}} y$. Then for any $u \in \mathsf{U}$: $u \in \lambda_x$ iff $u \mathrel{\mathsf{FC}} x$ iff $u \mathrel{\mathsf{S}} x$ and $\neg\, x \mathrel{\mathsf{A}} u$ iff (by $(\mathtt{s}_{\mathsf{S}})$ and (2.16)) $u \mathrel{\mathsf{S}} y$ and $\neg\, y \mathrel{\mathsf{A}} u$ iff $u \mathrel{\mathsf{FC}} y$ iff $u \in \lambda_y$. "⇐" Let $\lambda_x = \lambda_y$. Then, since $x \in \lambda_x$ and $y \in \lambda_y$, we have $x \in \lambda_y$ and $y \in \lambda_x$, i.e, $x \mathrel{\mathsf{FC}} y$ and $y \mathrel{\mathsf{FC}} x$. So, by (3.13), we have $x \mathrel{\mathsf{BT}} y$.

Proof of (3.16). "⊆" From (3.7), $(\mathtt{con}_{\prec})$ and (3.15). "⊇" Suppose that $\alpha \notin \mathbb{I}_x$, i.e., $x \notin \alpha$. Then for some $y_0 \in \alpha$ we have $\neg\, x \mathrel{\mathsf{S}} y_0$. So, by $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$, either $y_0 \mathrel{\mathsf{P}} x$ or $x \mathrel{\mathsf{P}} y_0$. In the first case, $\alpha \prec \phi_x$. In the second case, $\lambda_x \prec \alpha$.

Proof of (3.17). "⇒" Let $x \mathrel{\mathsf{A}} y$. If $\neg\, x \mathrel{\mathsf{S}} y$, then $y \mathrel{\mathsf{P}} x$, by (2.10). So, by Fact 3.10, we obtain $\lambda_y \prec \lambda_x$. If $x \mathrel{\mathsf{S}} y$, then $x \in \lambda_y$ since $\neg\, y \mathrel{\mathsf{A}} x$, by $(\mathtt{as}_{\mathsf{A}})$. Moreover, $y \notin \lambda_x$; and so $\lambda_x \neq \lambda_y$ since $y \in \lambda_y$. So $\lambda_y \prec \lambda_x$, by (3.16). "⇐" Suppose that $\lambda_y \prec \lambda_x$, Then there are events u and v such that (a) $u \mathrel{\mathsf{S}} x$, (b) $\neg\, y \mathrel{\mathsf{A}} v$ and (c) $v \mathrel{\mathsf{P}} u$. From $(\mathtt{t}^{\bar{\mathsf{A}}}_{\mathsf{P}})$, (b) and (c), we have $y \mathrel{\mathsf{P}} u$. Hence, by (a) and $(\mathtt{s}_{\mathsf{S}})$, we have $x \mathrel{\mathsf{A}} y$.

Proof of $(\mathtt{d}_{\prec})$. Let $\alpha \prec \beta$, i.e., for some $x, y \in \mathsf{U}$ we have: (i) $x \in \alpha$, (ii) $y \in \beta$ and (iii) $x \mathrel{\mathsf{P}} y$. Then, by (iii) and (f'), for some z we have (iv) $x \mathrel{\mathsf{P}} z$ and (v) $z \mathrel{\mathsf{E}} y$. Therefore, $\alpha \prec \phi_z$, by (i) and (iv). Moreover, (vi) $\phi_z \prec \phi_y$, by (v) and (3.8). Furthermore, by (ii) and (3.7), we have $\phi_y \leq \beta$. If $\phi_y = \beta$ then $\alpha \prec \phi_z \prec \beta$. If $\phi_y \prec \beta$ then $\phi_z \prec \beta$, by (vi) and $(\mathtt{t}_{\prec})$. So also $\alpha \prec \phi_z \prec \beta$.

Proof of Theorem 3.11. Suppose that (f) does not hold, i.e., there are events x and y such that $x \mathrel{\mathsf{P}} y$ and for all events z and u such that $x \mathrel{\mathsf{P}} z$ and $u \mathrel{\mathsf{P}} y$ we have $\neg\, z \mathrel{\mathsf{S}} u$. Then $\lambda_x \prec \phi_y$ since $x \in \lambda_x$ and $y \in \phi_y$. Assume for a contradiction that for some instant α_0, we have $\lambda_x \prec \alpha_0 \prec \phi_y$. Then there are z_1 and z_2 such that $z_1 \in \lambda_x$, $z_2 \in \alpha_0$ and $z_1 \mathrel{\mathsf{P}} z_2$, and there are u_1 and u_2 such that $u_1 \in \alpha_0$, $u_2 \in \phi_y$ and $u_1 \mathrel{\mathsf{P}} u_2$. Since $\neg\, x \mathrel{\mathsf{A}} z_1$ and $z_1 \mathrel{\mathsf{P}} z_2$, we have $x \mathrel{\mathsf{P}} z_2$, by $(\mathtt{t}^{\bar{\mathsf{A}}}_{\mathsf{P}})$. Moreover, since $u_1 \mathrel{\mathsf{P}} u_2$ and $\neg\, y \mathrel{\mathsf{E}} u_2$, we have $u_1 \mathrel{\mathsf{P}} y$, by $(\mathtt{t}^{\bar{\mathsf{E}}}_{\mathsf{P}})$. Therefore, $\neg\, z_2 \mathrel{\mathrm{S}} u_2$, which is contrary to that $z_2, u_1 \in \alpha_0$. Thus, $\prec$ is not dense.

Proof of Theorem 4.1. "$\Rightarrow$" a5 = $(\star_{\mathsf{E}})$. *Ad* a3 and a4, i.e. $(\mathtt{s}_{\mathrm{S}})$ and $(\mathtt{con}^{\mathrm{S}}_{\mathsf{P}})$: From $(\mathrm{df}_{\mathsf{P}}\mathrm{S})$. *Ad* a2, i.e. $(\mathtt{t}_{\mathsf{E}})$: From $(\star_{\mathsf{E}})$ and $(\mathtt{t}_{\mathrm{S}|\mathsf{P}})$. *Ad* a1, i.e. $(\mathtt{as}_{\mathsf{E}})$: From $(\mathtt{t}_{\mathsf{E}})$ and $(\mathtt{irr}_{\mathsf{E}})$, which we obtain from $(\star_{\mathsf{E}})$ and $(\mathrm{df}_{\mathsf{P}}\mathrm{S})$. *Ad* (df P): Firstly, by $(\mathtt{irr}_{\mathsf{P}})$, we have (a): $x \mathrel{\mathsf{P}} y \implies (x \mathrel{\mathsf{P}} y \vee y \mathrel{\mathsf{P}} x) \wedge \exists_z(\neg\, x \mathrel{\mathsf{P}} z \wedge \neg\, z \mathrel{\mathsf{P}} x \wedge z \mathrel{\mathsf{P}} y)$. Secondly, by $(\mathtt{t}_{\mathsf{P}})$, we have (b): $\neg(y \mathrel{\mathsf{P}} x \wedge \exists_z(\neg\, x \mathrel{\mathsf{P}} z \wedge \neg\, z \mathrel{\mathsf{P}} x \wedge z \mathrel{\mathsf{P}} y))$. Hence we obtain (c): $(x \mathrel{\mathsf{P}} y \vee y \mathrel{\mathsf{P}} x) \wedge \exists_z(\neg\, x \mathrel{\mathsf{P}} z \wedge \neg\, z \mathrel{\mathsf{P}} x \wedge z \mathrel{\mathsf{P}} y) \implies x \mathrel{\mathsf{P}} y$. Thus, from (a), (c), $(\mathrm{df}_{\mathsf{P}}\mathrm{S})$ and $(\star_{\mathsf{E}})$ we obtain: $x \mathrel{\mathsf{P}} y \iff \neg\, x \mathrel{\mathrm{S}} y \wedge x \mathrel{\mathsf{E}} y$.

"$\Leftarrow$" $(\mathtt{irr}_{\mathsf{P}})$, $(\mathtt{t}_{\mathsf{P}})$ and $(\mathrm{df}_{\mathsf{P}}\mathrm{S})$ are proved on pages 4, 6 and 5, respectively. Moreover, $(\mathtt{t}_{\mathrm{S}|\mathsf{P}})$ follows from $(\star_{\mathsf{E}})$ and $(\mathtt{t}_{\mathsf{P}})$.

Proof of Lemma 4.2. For (%): Suppose that $x \mathrel{\mathsf{E}} y$. Hence, by $(\star_{\overset{\rightrightarrows}{\mathsf{E}}})$ and $(\mathtt{s}_{\mathrm{S}})$, for some z_1: $z_1 \mathrel{\mathrm{S}} x$ and $z_1 \mathrel{\mathsf{P}} y$. Now suppose that $\neg\, z_1 \mathrel{\mathsf{IC}} x$. Then $x \mathrel{\mathsf{E}} z_1$. Hence, applying $(\star_{\overset{\rightrightarrows}{\mathsf{E}}})$ and $(\mathtt{s}_{\mathrm{S}})$ again, for some z_2: $z_2 \mathrel{\mathrm{S}} x$ and $z_1 \mathrel{\mathsf{P}} z_2$. Thus, $(\$)$ does not hold. For (‰): Similarly, we use (df A) instead of $(\star_{\overset{\rightrightarrows}{\mathsf{E}}})$. Therefore, we can use Theorems 3.5 and 3.9.

Proof of Fact 4.4. "$(\$) \dashv\vdash$ (i)" From $(\mathrm{df}_{\mathsf{P}}\mathrm{S})$. "(i) $\dashv\vdash$ (ii)" From Lemma 4.3. "(iii) $\vdash$ (ii)" Obvious.

"(ii) $\vdash$ (iii)" Assume for a contradiction that $\vec{\mathrm{S}}\text{‘}x \in \mathbf{In}$ and for some $\alpha_0 \in \mathbf{In}$ we have $x \in \alpha_0 \neq \vec{\mathrm{S}}\text{‘}x$. Then, by Fact 3.1(1), we have $\vec{\mathrm{S}}\text{‘}x \not\subseteq \alpha_0$; i.e., for some $u \in \vec{\mathrm{S}}\text{‘}x$: $u \notin \alpha_0$. So $u \mathrel{\mathrm{S}} x$. Hence, by (i), we have (a): for each $z \in \vec{\mathrm{S}}\text{‘}x$ we have $z \mathrel{\mathrm{S}} u$. Furthermore, since $x \in \alpha_0$, we obtain (b): for any $y \in \alpha_0$, we have $y \mathrel{\mathrm{S}} x$. Since $u \notin \alpha_0$, for some $v \in \alpha_0$ we have $\neg\, v \mathrel{\mathrm{S}} u$. Hence, by (b), we have $v \mathrel{\mathrm{S}} x$. Hence, by (a), we obtain a contradiction: $v \mathrel{\mathrm{S}} u$.

"$(\$) \vdash$ (iv), (v)" From $(\$)$ and Lemma 4.2, we have $\phi_x, \lambda_x \in \mathbf{In}$. Hence, by (3.3) and (3.12), we have $\phi_x, \lambda_x \in \mathbb{I}_x$. But from $(\$)$ we obtain (iii). So $\phi_x = \vec{\mathrm{S}}\text{‘}x = \lambda_x$.

"(iv) $\vdash (\$)$" Suppose that there are events u and v such that $u \mathrel{\mathrm{S}} x$, $v \mathrel{\mathrm{S}} x$ and $u \mathrel{\mathsf{P}} v$. Then, by $(\mathtt{s}_{\mathrm{S}})$ and $(\star_{\overset{\leftleftarrows}{\mathsf{E}}})$, we have $x \mathrel{\mathsf{E}} v$. Hence $\neg\, v \mathrel{\mathsf{IC}} x$. Therefore, we obtain $\vec{\mathrm{S}}\text{‘}x \not\subseteq \phi_x$.

Proof of Fact 4.5. "(vi) $\vdash$ (i)" Let $\mathbb{I}_x$ be a singleton, i.e., $\mathbb{I}_x = \{\alpha_0\}$, for some $\alpha_0 \in \mathbf{In}$. Suppose that $u \mathrel{\mathrm{S}} x$ and $v \mathrel{\mathrm{S}} y$. Then, by (3.2) (which we obtain from the axiom of choice), for some $\alpha_1, \alpha_2 \in \mathbf{In}$ we have $x, u \in \alpha_1$ and $x, v \in \alpha_2$. Therefore, $\alpha_1 = \alpha_0 = \alpha_2$; and so $u, v \in \alpha_0$. Hence $u \mathrel{\mathrm{S}} v$. Thus, we obtain (i).

Proof of Fact 4.6. "$(\$) \vdash$ (vii), (viii), (ix)" From $(\$)$ and Lemma 4.2, we have $\phi_x, \lambda_x \in \mathbf{In}$. Hence, by (3.3) and (3.12), we have $\phi_x \in \mathbb{I}_x \ni \lambda_x$. But from $(\$)$ we obtain (iii). So $\phi_x = \vec{\mathrm{S}}\text{‘}x = \lambda_x$ and $\mathbb{I}_x = \{\phi_x\} = \{\lambda_x\}$.

Proof of Fact 4.7. "(vi) $\vdash$ (vii), (viii), (ix)" By Facts 3.6 and 3.10, which we obtain from $(\mathtt{e})$ and (ɘ), $x \in \phi_x, \lambda_x \in \mathbf{In}$, respectively. Hence, by our assumption, $\lambda_x = \phi_x$ and $\mathbb{I}_x = \{\phi_x\} = \{\lambda_x\}$.

"(viii) $\vdash$ (vii), (ix)" From (3.16), (3.3) and (3.12).

"(vii) $\vdash (\$)$" Suppose that there are u and v such that $u \mathrel{\mathrm{S}} x$, $v \mathrel{\mathrm{S}} x$ and $u \mathrel{\mathsf{P}} v$. Then, by $(\mathtt{s}_{\mathrm{S}})$ and $(\star_{\overset{\leftleftarrows}{\mathsf{E}}})$, we have $x \mathrel{\mathsf{E}} v$. So $\neg\, v \mathrel{\mathsf{IC}} x$, i.e. $v \notin \phi_x$. Furthermore, by $(\mathtt{as}_{\mathsf{E}})$, we have $\neg\, v \mathrel{\mathsf{E}} x$. Hence $x \mathrel{\mathsf{IC}} v$, i.e. $x \in \phi_v$. Moreover, since $x \in \phi_x$, we have $\phi_x \neq \phi_v$. By Fact 3.6, $\phi_x, \phi_v \in \mathbf{In}$. So $\{\phi_x, \phi_v\} \subseteq \mathbb{I}_x \neq \{\phi_x\}$.

"(ix) $\vdash (\$)$" Suppose that there are u and v such that $u \mathrel{\mathrm{S}} x$, $v \mathrel{\mathrm{S}} x$ and $u \mathrel{\mathsf{P}} v$. Then, by $(\mathtt{s}_{\mathrm{S}})$ and (df A), we have $x \mathrel{\mathsf{A}} u$. So $\neg\, u \mathrel{\mathsf{FC}} x$, i.e. $v \notin \lambda_x$. Furthermore, by $(\mathtt{as}_{\mathsf{A}})$, we have $\neg\, u \mathrel{\mathsf{A}} x$. Hence $x \mathrel{\mathsf{FC}} v$, i.e. $x \in \lambda_u$. Moreover, since $x \in \lambda_x$, we have $\lambda_x \neq \lambda_u$. By Fact 3.10, $\lambda_x, \lambda_u \in \mathbf{In}$. So $\{\lambda_x, \lambda_u\} \subseteq \mathbb{I}_x \neq \{\lambda_x\}$.

Proof of Fact 4.8 1. *Ad* $\phi_x = \lambda_y$: We show that for any u we have: $u \mathrel{\mathsf{IC}} a$ iff $u \mathrel{\mathsf{FC}} b$. "$\Rightarrow$" Firstly, $u \mathrel{\mathrm{S}} x$ entails $\neg\, x \mathrel{\mathsf{A}} u$. Indeed, if $y \mathrel{\mathsf{A}} u$, then, by (df A) and (£), for some v we have a contradiction: $u \mathrel{\mathrm{S}} x$, $u \mathrel{\mathsf{P}} v$ and $v \mathrel{\mathrm{S}} y$. Secondly, $u \mathrel{\mathrm{S}} x$ and $\neg\, x \mathrel{\mathsf{E}} u$ entail $u \mathrel{\mathrm{S}} y$. Indeed, if $\neg\, u \mathrel{\mathrm{S}} y$ then, by $(\mathtt{con}^{\mathrm{S}}_{\mathsf{P}})$, either $u \mathrel{\mathsf{P}} y$ or $y \mathrel{\mathsf{P}} u$. In the first case, by $(\mathtt{r}_{\mathrm{S}})$ and (£), we obtain a contradiction: $u \mathrel{\mathrm{S}} x$, $u \mathrel{\mathsf{P}} y$ and $y \mathrel{\mathrm{S}} y$. In the second case, by $(\mathtt{t}^{\mathsf{E}}_{\mathsf{P}})$, we also obtain a contradiction: $y \mathrel{\mathsf{P}} x$. "$\Leftarrow$" Firstly, $u \mathrel{\mathrm{S}} y$ entails $\neg\, x \mathrel{\mathsf{E}} u$. Indeed, if $x \mathrel{\mathsf{E}} u$, then, by $(\star_{\overset{\rightrightarrows}{\mathsf{E}}})$ and (£), for some z we have a contradiction: $z \mathrel{\mathrm{S}} x$, $z \mathrel{\mathsf{P}} u$ and $u \mathrel{\mathrm{S}} y$. Secondly, $u \mathrel{\mathrm{S}} y$ and

$\neg\, y \mathrel{\mathsf{A}} u$ entail $u \mathrel{\mathsf{S}} x$. Indeed, if $\neg\, u \mathrel{\mathsf{S}} x$, then either $u \mathrel{\mathsf{P}} x$ or $x \mathrel{\mathsf{P}} u$. In the second case, by $(\mathtt{r}_{\mathsf{S}})$ and (£), we obtain a contradiction: $x \mathrel{\mathsf{S}} x$, $x \mathrel{\mathsf{P}} u$ and $u \mathrel{\mathsf{S}} y$. In the first case, by $(\mathtt{t}^{\bar{\mathsf{A}}}_{\mathsf{P}})$, we also obtain a contradiction: $y \mathrel{\mathsf{P}} x$.

Ad $\phi_x \in \mathbf{In}$: Since $y \in \lambda_y$, we have $y \in \phi_x$. Assume for a contradiction that ϕ_x does not satisfy condition (c1). Then there is v such that $v \notin \phi_x$ and $\forall_{z\in\phi_x}\ v \mathrel{\mathsf{S}} z$. Therefore, $v \mathrel{\mathsf{S}} y$ and $v \mathrel{\mathsf{S}} x$; and so $x \mathrel{\mathsf{E}} v$. So for some u, by (£), we obtain a contradiction: $u \mathrel{\mathsf{S}} x$, $u \mathrel{\mathsf{P}} v$ and $v \mathrel{\mathsf{S}} y$. Thus, by Lemma 3.4, we have $\phi_x \in \mathbf{In}$.

2. "$\Rightarrow$" If $x \mathrel{\mathsf{E}} y$ then, by $(\mathtt{r}_{\mathsf{S}})$, $(\star^{\Rightarrow}_{\mathsf{E}})$, (£), there is u such that we obtain a contradiction: $u \mathrel{\mathsf{S}} x$, $u \mathrel{\mathsf{P}} y$ and $y \mathrel{\mathsf{S}} y$. If $y \mathrel{\mathsf{A}} x$ then, by $(\mathtt{r}_{\mathsf{S}})$, (df A), (£), there is v such that we obtain a contradiction: $x \mathrel{\mathsf{S}} x$, $x \mathrel{\mathsf{P}} v$ and $v \mathrel{\mathsf{S}} y$.

3. "$\Leftarrow$" Directly from our definitions.

"$\Rightarrow$" *Ad* $x \sqsubseteq_{\mathrm{t}} y$: We will use $(\mathrm{df}_{\mathsf{S}}{\sqsubseteq_{\mathrm{t}}})$, that is, we will show that $\vec{\mathsf{S}}`x \subseteq \vec{\mathsf{S}}`y$. Indeed, if for some z we have $z \mathrel{\mathsf{S}} x$ and $\neg\, z \mathrel{\mathsf{S}} y$, then, by (2.11), our assumption and point 2, we obtain a contradiction. *Ad* $\lambda_x \in \mathbf{In}$: Assume for a contradiction that λ_x does not satisfy condition (c1). Then there is v such that $v \notin \lambda_x$ and $\forall_{z\in\lambda_x}\ v \mathrel{\mathsf{S}} z$. Therefore, $v \mathrel{\mathsf{S}} x$; and so $x \mathrel{\mathsf{A}} v$. Hence, by (2.9) and our assumption, we have $y \mathrel{\mathsf{A}} v$. So for some u, by (df A) and (£), we obtain a contradiction: $u \mathrel{\mathsf{S}} x$, $v \mathrel{\mathsf{P}} u$ and $v \mathrel{\mathsf{S}} y$. Thus, by Lemma 3.8, we have $\lambda_x \in \mathbf{In}$. *Ad* the rest: By our assumption, (3.18), (3.16) and point 1, we have $\lambda_x \preceq \lambda_y = \phi_x \preceq \lambda_x$; and so $\lambda_x = \phi_x = \lambda_y$.

4. "$\Leftarrow$" Directly from our definitions.

"$\Rightarrow$" *Ad* $y \sqsubseteq_{\mathrm{t}} x$: We will use $(\mathrm{df}_{\mathsf{S}}{\sqsubseteq_{\mathrm{t}}})$, that is, we will show that $\vec{\mathsf{S}}`y \subseteq \vec{\mathsf{S}}`x$. Indeed, if for some z we have $z \mathrel{\mathsf{S}} y$ and $\neg\, z \mathrel{\mathsf{S}} x$, then, by (2.11), our assumption and point 2, we obtain a contradiction. *Ad* $\phi_y \in \mathbf{In}$: Assume for a contradiction that ϕ_y does not satisfy condition (c1). Then there is v such that $v \notin \phi_y$ and $\forall_{z\in\phi_y}\ v \mathrel{\mathsf{S}} z$. Therefore, $v \mathrel{\mathsf{S}} y$; and so $y \mathrel{\mathsf{E}} v$. Hence, by (2.3) and our assumption, we have $x \mathrel{\mathsf{E}} v$. So for some u, by $(\star^{\Rightarrow}_{\mathsf{E}})$ and (£), we obtain a contradiction: $u \mathrel{\mathsf{S}} x$, $u \mathrel{\mathsf{P}} v$ and $v \mathrel{\mathsf{S}} y$. Thus, by Lemma 3.4, we have $\phi_y \in \mathbf{In}$. *Ad* the rest: By our assumption, (3.9), (3.16) and point 1, we have $\lambda_y = \phi_x \preceq \phi_y \preceq \lambda_y$; and so $\phi_y = \lambda_y = \phi_x$.

5. "$\Rightarrow$" By points 3 and 4 and our definitions. For $(\$)$, we use $(\mathrm{df}_{\mathsf{S}}{\equiv_{\mathrm{t}}})$ and (£).

"$\Leftarrow$" By (3.11).

Proof of Theorem 5.1. "(1) $\Leftrightarrow$ (2)" Theorem 4.1.

"(2) $\Rightarrow$ (3)" $(\mathtt{irr}_{\mathsf{P}})$, $(\mathrm{Th}_{\mathsf{P}})$, $(\mathrm{df}_{\mathsf{P}}\mathsf{E})$ and $(\mathrm{df}_{\mathsf{P}}\mathsf{S})$ are proved on pages 4, 6, 6 and 5, respectively.

"(3) $\Rightarrow$ (1)" From $(\mathrm{Th}_{\mathsf{P}})$ and $(\mathtt{irr}_{\mathsf{P}})$ we obtain $(\mathtt{t}_{\mathsf{P}})$. By $(\mathrm{df}_{\mathsf{P}}\mathsf{S})$ we have $(*)$ for all x, y: $x \mathrel{\mathsf{S}\mid\mathsf{P}} y$ iff for some u we have $\neg\, x \mathrel{\mathsf{P}} u$, $\neg\, u \mathrel{\mathsf{P}} x$ and $u \mathrel{\mathsf{P}} y$. *Ad* $(\mathtt{t}_{\mathsf{S}|\mathsf{P}})$: Suppose that $x \mathrel{\mathsf{S}\mid\mathsf{P}} y$ and $y \mathrel{\mathsf{S}\mid\mathsf{P}} z$. Then, by $(*)$, for some u_1 and u_2 we have: (a_1) $\neg\, x \mathrel{\mathsf{P}} u_1$, (b_1) $\neg\, u_1 \mathrel{\mathsf{P}} x$, (c_1) $u_1 \mathrel{\mathsf{P}} y$, (a_2) $\neg\, y \mathrel{\mathsf{P}} u_2$, (b_2) $\neg\, u_2 \mathrel{\mathsf{P}} y$, (c_2) $u_2 \mathrel{\mathsf{P}} z$. In the light $(\mathrm{Th}_{\mathsf{P}})$, (c_1), (c_2) and (b_2) we have $u_1 \mathrel{\mathsf{P}} z$. Hence, by (a_1) and (b_1), we have $x \mathrel{\mathsf{S}\mid\mathsf{P}} z$. *Ad* $(\star_{\mathsf{E}})$: "$\Rightarrow$" Suppose that $x \mathrel{\mathsf{E}} y$, i.e., by $(\mathrm{df}_{\mathsf{P}}\mathsf{E})$, for some u we have $\neg\, u \mathrel{\mathsf{P}} x$ and $u \mathrel{\mathsf{P}} y$. Now if $\neg\, x \mathrel{\mathsf{P}} u$, then $x \mathrel{\mathsf{S}\mid\mathsf{P}} y$, by $(*)$. If $x \mathrel{\mathsf{P}} u$ then $x \mathrel{\mathsf{P}} y$, by $(\mathtt{t}_{\mathsf{P}})$. So, by $(\mathtt{irr}_{\mathsf{P}})$, we have: $\neg\, x \mathrel{\mathsf{P}} x$ and $x \mathrel{\mathsf{P}} y$. Thus, by $(*)$, also $x \mathrel{\mathsf{S}\mid\mathsf{P}} y$. "$\Leftarrow$" Directly from $(*)$ and $(\mathrm{df}_{\mathsf{P}}\mathsf{E})$.

Appendix B. Models

Model 1. We take the model of all axioms in which $\mathsf{U} := \{1,2\}$, $1 \neq 2$, $\mathsf{E} := \varnothing$ and $\mathsf{S} := \mathsf{U}\times\mathsf{U}$. We have $\mathsf{L} = \mathsf{P} = \mathsf{A} = \varnothing$ and $\mathsf{BT} = \mathsf{ET} = {\sqsubseteq_{\mathrm{t}}} = {\equiv_{\mathrm{t}}} = \mathsf{U}\times\mathsf{U}$. So $1 \equiv_{\mathrm{t}} 2$, $1 \sqsubseteq_{\mathrm{t}} 2$ and $2 \sqsubseteq_{\mathrm{t}} 1$. Moreover, we have $\mathsf{IC} = \mathsf{U}\times\mathsf{U}$. So the universe $\mathsf{U} := \{1,2\}$ is the only instant in this model; hence, $\lambda_1 = \phi_1 = \{1,2\} = \phi_2 = \lambda_2$.

Model 2. We give the following model: $\mathsf{U} := \{1,2,3,4\}$, $\mathsf{E} := \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 3,2\rangle, \langle 3,4\rangle\}$ and $\mathsf{S} := \mathsf{U}^2 \setminus \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,4\rangle, \langle 4,3\rangle\}$; and so $\mathsf{P} = \{\langle 1,2\rangle, \langle 3,4\rangle\}$ and $\mathsf{IC} = \mathrm{id}_{\mathsf{U}} \cup \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 1,4\rangle, \langle 3,2\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$. Thus, $\mathsf{P}\cap\mathsf{S} = \varnothing$ and $(\mathtt{irr}_{\mathsf{P}})$, $(\mathtt{as}_{\mathsf{P}})$, $(\mathtt{t}_{\mathsf{P}})$, $(\mathtt{con}^{\mathsf{S}}_{\mathsf{P}})$, $(\mathtt{irr}_{\mathsf{E}})$, $(\mathtt{as}_{\mathsf{E}})$, $(\mathtt{t}_{\mathsf{E}})$, $(\mathtt{r}_{\mathsf{S}})$,$(\mathtt{s}_{\mathsf{S}})$, $(\mathtt{con}^{\mathsf{S}}_{\mathsf{E}})$, $(\star^{\Rightarrow}_{\mathsf{E}})$, $(\mathtt{ct}_{\mathsf{E}})$, (e) hold but $(\star^{\Leftarrow}_{\mathsf{E}})$, $(\mathtt{t}^{\bar{\mathsf{E}}}_{\mathsf{P}})$, $(\mathtt{t}^{\mathsf{E}}_{\mathsf{P}})$, $(\mathtt{t}{+}_{\mathsf{P}})$, $(\mathrm{Th}_{\mathsf{P}})$ do not hold. In the model, $\mathbf{In}$ consists of tree sets: $\alpha_1 = \{1,4\}$, $\alpha_2 = \{2,3\}$,

$\alpha_3 = \{1, 3\}$. We have $\alpha_1 \prec \alpha_2$ and $\alpha_2 \prec \alpha_1$ but $\alpha_1 \not\prec \alpha_1$. Thus, ($\mathtt{t}_\prec$) and ($\mathtt{as}_\prec$) do not hold in the model.

Model 3. Let U consist of the following open intervals of real numbers: $(0, 2)$, $(1, 3)$ and $\left(\frac{1}{n}, \frac{2n-1}{2n(n-1)}\right)$, for any $n > 1$. For any $x \in \mathrm{U}$, let $x = (\mathrm{b}_x, \mathrm{e}_x)$. Moreover, for all $x, y \in \mathrm{U}$ we put: $x \mathrel{\mathsf{E}} y$ iff $\mathrm{b}_x < \mathrm{b}_y$; $x \mathrel{\mathsf{S}} y$ iff $x \cap y \neq \emptyset$. So $x \mathrel{\mathsf{P}} y$ iff $\mathrm{e}_x < \mathrm{b}_y$; $x \mathrel{\mathsf{IC}} y$ iff $x \cap y \neq \emptyset$ and $\mathrm{b}_x \leqslant \mathrm{b}_y$. Thus, $(0, 2) \mathrel{\mathsf{E}} y$ and $(0, 2) \mathrel{\mathsf{S}} y$, for any $y \neq (0, 2)$ Moreover, for any $n > 0$: $\left(\frac{1}{n}, \frac{2n-1}{2n(n-1)}\right) \mathrel{\mathsf{P}} (1, 3)$ and $\left(\frac{1}{n+1}, \frac{2n+1}{2n(n+1)}\right) \mathrel{\mathsf{P}} \left(\frac{1}{n}, \frac{2n-1}{2n(n-1)}\right)$. Therefore, $x \mathrel{\mathsf{IC}} y$ iff $x = (0, 2)$ and $y \neq (0, 2)$. Of course, all axioms a1–a5 are true. But we have: $(0, 2) \mathrel{\mathsf{E}} (1, 3)$ and there is no $z \in \mathrm{U}$ such that $z \mathrel{\mathsf{P}} (1, 3)$ and $z \mathrel{\mathsf{IC}} (0, 2)$. So ($e$) is false.

Note that in this model, $\phi_{(1,3)} = \{(1, 3)\}$. So, by (c1), the set $\phi_{(0,2)}$ does not belong to **In** since $(1, 3) \mathrel{\mathsf{S}} (0, 2)$. Moreover, $\phi_{(1,3)} = \{(0, 2), (1, 3)\}$ and $\phi_{(1,3)}$ belongs to **In**. So also from Fact 3.1 we obtain that $\phi_{(0,2)}$ is not an instant.

Model 4. Let us then formally present the model that will correspond to Anderson's diagram (see p. 18). Let the universe U consist of all closed intervals of the form $[2k, 2k+3]$, where k is an integer. For any $x \in \mathrm{U}$, let $x = [\mathrm{b}_x, \mathrm{e}_x]$. For all $x, y \in \mathrm{U}$ we put: $x \mathrel{\mathsf{P}} y$ iff $\mathrm{e}_x < \mathrm{b}_y$, So for all $x, y \in \mathrm{U}$: $x \mathrel{\mathsf{S}} y$ iff $x \cap y \neq \emptyset$; $x \mathrel{\mathsf{S} \mid \mathsf{P}} y$ iff $\mathrm{b}_x < \mathrm{b}_y$ iff $x \mathrel{\mathsf{E}} y$; $x \mathrel{\mathsf{IC}} y$ iff $x \cap y \neq \emptyset$ and $\mathrm{b}_x \leqslant \mathrm{b}_y$. Obviously, a1–a6, ($\mathtt{s}$), ($\mathtt{irr}_{\mathsf{P}}$), ($\mathtt{t}_{\mathsf{P}}$), ($\mathrm{df}_{\mathsf{P}}\mathsf{S}$), ($\mathtt{t}_{\mathsf{S}|\mathsf{P}}$), ($\star_{\mathsf{E}}$), ($a$) and ($b$) are true in the model. But ($\mathtt{d}_\prec$) and (f) do not hold. Indeed, for any $x \in \mathrm{U}$ we have $\phi_x \neq \lambda_x$ and $\mathbb{I}_x = \{\phi_x, \lambda_x\}$. So, by (3.16), there is no $\alpha \in \mathbf{In}$ such that $\phi_x \prec \alpha \prec \lambda_x$. Moreover, notice that for any $\alpha \in \mathbf{In}$ there are $x, y \in \mathrm{U}$ such that $x \neq y$, $\alpha = \{x, y\}$, $\alpha = \phi_x$ and $\alpha = \lambda_y$.

Model 5. We can also give a different model of a1–a6, ($\mathtt{s}$), ($\mathtt{irr}_{\mathsf{P}}$), ($\mathtt{t}_{\mathsf{P}}$), ($\mathrm{df}_{\mathsf{P}}\mathsf{S}$), ($\mathtt{t}_{\mathsf{S}|\mathsf{P}}$), ($a$), ($b$) in which ($f$) and ($\mathtt{d}_\prec$) do not hold; and which has the first and last instant. This model is based on intervals used in the Cantor set construction. Let us enter the sequence $(\mathcal{S}_i)_{i=0}^{\infty}$ of closed intervals included in the interval $[0, 1]$. The set $\mathcal{S}_0$ consists of one interval $[0, 1]$. From this interval we cut the middle open interval $(1/3, 2/3)$ of length $1/3$. There are two closed intervals $[0, 1/3]$ and $[2/3, 1]$ that remain, which make up the set $\mathcal{S}_1$. From these last closed intervals, cut out their middle open intervals $(1/9, 1/6)$ and $(7/9, 8/9)$ of length $1/9$. We get four closed intervals $[0, 1/9]$, $[1/6, 1/3]$, $[0, 7/9]$, $[8/9, 1]$, which form the set $\mathcal{S}_2$. Generally, for $n > 0$, the set $\mathcal{S}_n$ consists of 2^n of disjoint closed intervals that arise from 2^{n-1} closed intervals belonging to $\mathcal{S}_n$ by cutting out each one of their middle open intervals of length $1/2^n$. Let $\mathrm{U} := \bigcup_{=1}^{\infty} \mathcal{S}_i$. For any $x \in \mathrm{U}$, let $x = [\mathrm{b}_x, \mathrm{e}_x]$. Moreover, for all $x, y \in \mathrm{U}$ we put: $x \mathrel{\mathsf{E}} y$ iff $\mathrm{b}_x < \mathrm{b}_y$; $x \mathrel{\mathsf{S}} y$ iff $x \cap y \neq \emptyset$. So $x \mathrel{\mathsf{P}} y$ iff $\mathrm{e}_x < \mathrm{b}_y$; $x \mathrel{\mathsf{IC}} y$ iff $x \cap y \neq \emptyset$ and $\mathrm{b}_x \leqslant \mathrm{b}_y$. Obviously, a1–a6, ($\mathtt{s}$), ($\mathtt{irr}_{\mathsf{P}}$), ($\mathtt{t}_{\mathsf{P}}$), ($\mathrm{df}_{\mathsf{P}}\mathsf{S}$), ($\mathtt{t}_{\mathsf{S}|\mathsf{P}}$), ($\star_{\mathsf{E}}$), ($a$) and ($b$) are true in the model. In general, the set of instants in this model can be assigned to the numbers of the Cantor set. Notice that $\phi_{[\mathrm{b}_x, \mathrm{e}_x]} = \{y \in \mathrm{U} : \mathrm{b}_x \in y\}$ and $\lambda_{[\mathrm{b}_x, \mathrm{e}_x]} = \{y \in \mathrm{U} : \mathrm{e}_x \in y\}$. Therefore, $\lambda_{[0, 1/3]} \prec \phi_{[2/3, 1]}$. But there is no instant α such that $\lambda_{[0, 1/3]} \prec \alpha \prec \phi_{[2/3, 1]}$. Of course, ($f$) is also false in this model. Namely, if it were true, then ($\mathtt{d}_\prec$) would also be true. You can also see that $[0, 1/3] \mathrel{\mathsf{P}} [2/3, 1]$ but there is no $z \in \mathrm{U}$ such that $[0, 1/3] \mathrel{\mathsf{P}} z$ and $z \mathrel{\mathsf{E}} [2/3, 1]$ because for any $u \in \mathrm{U}$: if $[0, 1/3] \mathrel{\mathsf{P}} u$ then $u = [2/3, 1]$.

Notice that in this model for any $\alpha \in \mathbf{In}$ we have $\phi_{[0,1]} \preceq \alpha \preceq \lambda_{[0,1]}$.

Finally, for $x := [0, 1/3]$, $\alpha := \lambda_x$, $y := [2/3, 1]$ and $\gamma := \phi_y$, we have: $\alpha' \prec \beta \preceq \alpha$.

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