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# SIMPLIFIED KRIPKE STYLE SEMANTICS FOR MODAL LOGICS K45, KB4 AND KD45

**Abstract**

In this paper we show that logics K45, KB4 (= KB5) and KD45 are determined by some classes of simplified Kripke frames without binary accessibility relations between possible worlds. These frames are ordered pairs of sets $\langle W, A\rangle$, where $W$ is a non-empty set of *worlds* and $A \subseteq W$ (a set of *common alternatives* to all worlds in $W$). From a frame $\langle W, A\rangle$ we can construct models of the form $\langle W, A, V\rangle$, where $V$ is a standard valuation which to formulae and words assigns truth-values with respect to the set $A$. For K45 we use the class of all simplified frames; for KB4 we have the case that $A = \emptyset$ or $A = W$; and for KD45 we use frames with $A \neq \emptyset$.

Moreover, to each of these logics we also assign a suitable class of finite euclidean relational frames which satisfy conditions for normal extensions of K5 presented by Nagle in [2].

*Keywords*: simplified Kripke style semantics, normal modal logics K45, KB4 and KD45.

## 1. Introduction. Preliminaries

Modal formulae are formed in the standard way from the set At of propositional letters: '$p$', '$q$', '$p_0$', '$p_1$', '$p_2$', ...; the truth-value operators: '$\neg$' and '$\supset$' (negation and material implication); the modal operator '$\Box$' (necessity; the possibility sign '$\Diamond$' is the abbreviation of '$\neg\Box\neg$'); and brackets. By For we denote the set of modal formulae.

We remind that a set $\Lambda$ of modal formulae is a *normal modal logic* iff $\Lambda$ contains all classical tautologies and the following formula

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \tag{K}$$

and $\Lambda$ is closed under the following rules: modus ponens, necessitation and uniform substitution, i.e. for any $\varphi, \psi \in \mathrm{For}$

$$\text{if } \varphi \text{ and } \ulcorner \varphi \supset \psi \urcorner \text{ are members of } \Lambda, \text{ so is } \psi, \tag{MP}$$

$$\text{if } \varphi \in \Lambda \text{ then } \ulcorner \Box\varphi \urcorner \in \Lambda, \tag{RN}$$

$$\text{if } \varphi \in \Lambda \text{ then } \mathrm{s}\,\varphi \in \Lambda, \tag{US}$$

where $\mathrm{s}\,\varphi$ is the result of uniform substitution of formulae for propositional letters in $\varphi$. By (US), all these logics include the set PL of all modal formulae which are instances of classical tautologies.

We remind that K is the smallest normal modal logic. For other modal logics we will make use of the following formulae

$$\Box p \supset \Diamond p \tag{D}$$

$$\Box p \supset p \tag{T}$$

$$p \supset \Box\Diamond p \tag{B}$$

$$\Box p \supset \Box\Box p \tag{4}$$

$$\Diamond p \supset \Box\Diamond p \tag{5}$$

Using the names of the formulae above, to simplify naming normal logics, we write $\mathrm{KX}_1 \ldots \mathrm{X}_n$ to denote the smallest normal logic containing the formulae ($\mathtt{X}_1$), ..., ($\mathtt{X}_n$). Thus, for example, K5, K45, KB4, KB5, KD45, and KT5 are the smallest normal modal logics which contain, respectively, the following formulae: (5); (4) and (5); (B) and (4); (B) and (5); (D), (4) and (5); (T) and (5). We have S5 := KT5 = KTB4 = KDB4 = KDB5 and KB4 = KB5 (cf. e.g. [1], pp. 137 and 139).

For all normal modal logics we may use Kripke *relational frames* of the form $\langle W, R\rangle$, where $W$ is a non-empty set of *worlds* and $R$ is a binary *accessibility* relation in $W$. For a frame $\langle W, R\rangle$ a *relational model* is any triple $\langle W, R, V\rangle$ in which $V\colon \mathrm{For} \times W \to \{0, 1\}$ is a function which, to formulae and words, assigns truth-values with respect to $R$. This function preserves classical conditions for truth-value operators and for any $\varphi \in \mathrm{For}$ and $x \in W$

$(V^R_\Box)$ $\qquad V(\Box\varphi, x) = 1$ iff $\forall_{y \in R(x)}\, V(\varphi, y) = 1$,

where $R(x) := \{y : x\,R\,y\}$, that is $R(x)$ is the set of worlds which are accessible from $x$.[1]

We say that a formula $\varphi$ is *valid in a model* $\langle W, R, V\rangle$ iff $V(\varphi, x) = 1$ for each $x$ from $W$. A formula is *valid in a frame* iff it is valid in every model on this frame. A formula is *valid in a class of models* (resp. *frames*) iff it is valid in all models (resp. frames) from this class. We say that a normal logic is *determined by* a model (resp. frame; class of models; class of frame) iff all and only formulae in this logic are valid there.

We say that a frame $\langle W, R\rangle$ (or a model on this frame) is, respectively: *reflexive*, *serial*, *symmetric*, *transitive*, *Euclidean* iff it satisfies, respectively, the following condition: $\forall_{x \in W}\, x\,R\,x$; $\forall_{x \in W} \exists_{y \in W}\, x\,R\,y$; $\forall_{x,y \in W}(x\,R\,y \iff y R x$; $\forall_{x,y,z \in W}(xRy\ \&\ yRx \implies xRz)$; $\forall_{x,y,z \in W}(xRy\ \&\ xRz \implies yRz)$. The logics K, K5, K45, KB4 (= KB5), KD45 and S5 are determined, respectively, by the classes of (cf. e.g. [1]):

- all relational frames,
- Euclidean frames,
- transitive and Euclidean frames,
- symmetric and transitive (symmetric and Euclidean) frames,
- serial, transitive and Euclidean frames,
- reflexive and Euclidean (reflexive, symmetric and transitive; serial, symmetric and transitive) frames.

The facts mentioned in the first paragraph of the abstract have a connection with the well-known fact concerning the logic S5. This logic is also determined by the class of *universal frames* in which all worlds are accessible from all worlds, i.e. $R = W \times W$ (cf. e.g. [1]; $R$ we also call *universal*). A model $\langle W, W \times W, V\rangle$ on a universal frame may be identified with the pair $\langle W, V\rangle$ in which for any $\varphi \in$ For and $x \in W$ we have

$(V^W_\Box)$ $\qquad V(\Box\varphi, x) = 1$ iff $\forall_{y \in W}\, V(\varphi, y) = 1$.

So for the logic S5 we can use models of the form $\langle W, V\rangle$.

Similarly, for logics K45, KB4 (=KB5) and KD45 – instead of relational frames – we can use *simplified frames* of the form $\langle W, A\rangle$, where $W$ is a non-empty set of worlds and $A \subseteq W$ ($A$ is a set of *common alternatives*

[1] We say that a relational frame (model) is *empty* iff $R = \emptyset$.

to all worlds from $W$). For a frame $\langle W, A\rangle$ a *simplified model* is any triple $\langle W, A, V\rangle$ in which $V\colon \mathrm{For} \times W \to \{0,1\}$ is a function which to formulae and words assigns truth-values with respect to the set $A$. This function preserves classical conditions for truth-value operators and, moreover, for any $\varphi \in \mathrm{For}$ and $x \in W$

$$(V_{\Box}^{A}) \qquad V(\Box\varphi, x) = 1 \text{ iff } \forall_{y\in A}\, V(\varphi, y) = 1.$$

Naturally, also for S5 we can use simplified frames (models) – a special kind in which $A = W$. Of course, a simplified model $\langle W, W, V\rangle$ may be identified with the pair $\langle W, V\rangle$.

Of course, for simplified models (frames) and classes of simplified models (frames) we use notions of *valid* and of *determination*, defined similarly as for relational models (frames) and classes of relational models (frames). For logics K45, KB4 and KD45 in Section 2 we prove the following determination theorems with respect to some classes of simplified frames.

THEOREM 1.1. (i) K45 *is determined by the class of all simplified frames.*
(ii) KB4 *is determined by the class of «empty» or universal simplified frames, i.e. in which $A = \emptyset$ or $A = W$.*
(iii) KD45 *is determined by the class of «non-empty» simplified frames, i.e. in which $A \neq \emptyset$.*

The facts mentioned in the second paragraph of the abstract have a connection with the following fact concerning normal extensions of the logic K5 which is proved by Nagle in [2].

NAGLE'S FACT 1 ([2], p. 325). *Every normal logic containing* (5) *is determined by a set of relational frames $\langle W, R\rangle$ which are finite*[2]*, Euclidean and satisfy one and only one of the following conditions*:
(a) *$W$ is a singleton and $R = \emptyset$,*
(b) *$R$ is total in $W$, i.e. $R = W \times W$,*[3]
(c) *there is a unique "initial" world $w \in W$ such that $R$ is total in $W \setminus \{w\}$ and $w\, R\, x$ for some $x \in W \setminus \{w\}$.*

In Section 3 for each one of the logics K45, KB4 and KD45 we give and prove some special version of Nagle's Fact.

[2] A frame (model) is said to be *finite* just in case the number of points in $W$ is finite.
[3] The relation $R$ is *total* in a set $X$ iff $X \times X \subseteq R$.

## 2. The proof of Theorem 1.1

We begin with the proofs of some facts which will be helpful for the analysis of the logics K45, KB4 and KD45.

LEMMA 2.1. *For any frame* $\langle W, R\rangle$ *and* $x \in W$ *we put* $W^x := \{x\} \cup A^x$, *where* $A^x := \{y \in W : x\,R^n\,y$ *for some* $n > 0\}$, *and* $R^x := R \cap (W^x \times W^x)$.[4]

(i) *If* $R$ *is reflexive then* $A^x = W^x$.
(ii) *If* $R$ *is serial then* $A^x \neq \emptyset$.
(iii) *If* $R$ *is transitive then* $A^x = \{y \in W : x\,R\,y\}$.
(iv) *If* $R$ *is symmetric, then either* $A^x = W^x$ *or* $A^x = \emptyset$.
(v) *If* $R$ *is Euclidean, then* $(W^x \setminus \{x\})^2 \subseteq R^x$ *and either* $R^x = W^x \times W^x$ *or* $R^x \subseteq W^x \times (W^x \setminus \{x\})$.[5]
(vi) *If* $R$ *is symmetric and Euclidean, then either* $R^x = W^x \times W^x$ *or* $R^x = \emptyset$.
(vii) *If* $R$ *is transitive and Euclidean, then*
  (a) *either* $R^x = W^x \times W^x$ *or* $R^x = W^x \times (W^x \setminus \{x\})$,
  (b) $R^x = W^x \times A^x$.

PROOF: Points (i)–(iii) are obvious.

(iv) Let $A^x \neq \emptyset$. Then for some $y \in A^x$ we have $x\,R\,y$. So $y\,R\,x$, since $R$ is symmetric. So $x\,R^2\,x$ and $x \in A^x$.

(v) Firstly, let $y, z \in W^x$ and $y \neq x \neq z$. Then by induction we obtain that $y\,R\,z$. Thus $(W^x \setminus \{x\})^2 \subseteq R^x$.

Secondly, also by induction, we obtain that if $x\,R\,x$, then $y\,R\,z$ for any $y, z \in W^x$, i.e. $R^x = W^x \times W^x$. If $\langle x, x\rangle \notin R$, then $R^x \neq W^x \times W^x$. Thus, if $y\,R^x\,z$, then $z \neq x$, since $R$ is Euclidean. Hence $y \in W^x$ and $z \in W^x \setminus \{x\}$.

(vi) By (iv), either $A^x = W^x$ or $A^x = \emptyset$. If $A^x = W^x$, then $x\,R\,x$; so $R^x = W^x \times W^x$, since $R$ is Euclidean. If $A^x = \emptyset$, then either $R^x = W^x \times W^x$ or $R^x = \emptyset$, by (v).

(viia) Suppose that $R^x \neq W^x \times W^x$. Let $y, z \in W^x$ and $z \neq x$. Then, by (iii), $x\,R\,z$ and either $x = y$ or $x\,R\,y$. Hence $y\,R\,z$, since $R$ is Euclidean. Thus, $W^x \times (W^x \setminus \{x\}) \subseteq R^x$. So $R^x = W^x \times (W^x \setminus \{x\})$, by (v).

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[4] $xR^1y$ iff $xRy$; and for $n > 1$: $xR^ny$ iff there are $y_1, \ldots, y_{n-1} \in W$ such that $xRy_1$, $y_1Ry_2, \ldots, y_{n-1}Ry$. Of course, $\langle W^x, R^x\rangle$ is a frame for generated models.

[5] For this fact see [3], Lemma 1.

(viib) For any $y, z \in A^x$, $x\,R\,y$ and $x\,R\,z$, by (iii). Hence $y\,R\,z$, because $R$ is Euclidean. Thus $A^x \times A^x \subseteq R$.

Moreover, by definition of $A^x$, we have $R^x = (R \cap \{\langle x, x\rangle\}) \cup (\{x\} \times A^x) \cup (A^x \times \{x\}) \cup (A^x \times A^x)$. We consider three cases. Firstly, if $A^x = \emptyset$, then $R^x = R \cap \{\langle x, x\rangle\}$ and $\langle x, x\rangle \notin R$, so $R^x = \emptyset$. Secondly, if $A^x \neq \emptyset$ and $\langle x, x\rangle \notin R$, then there is no $y \in A^x$ such that $y R x$. So $A^x \times \{x\} = \emptyset$. Hence $R^x = (\{x\} \times A^x) \cup (A^x \times A^x) = (\{x\} \cup A^x) \times A^x$. Thirdly, if $A^x \neq \emptyset$ and $\langle x, x\rangle \in R$, then $x \in A^x = W^x$. Hence $R^x = \{\langle x, x\rangle\} \cup (\{x\} \times A^x) \cup (A^x \times \{x\}) \cup (A^x \times A^x) = W^x \times W^x$. In all cases we obtain that $R^x = W^x \times A^x$. ⊣

We say that a relation $R$ in a relational frame $\langle W, R\rangle$ is *semi-universal* iff $R = W \times A$ for some $A \subseteq W$, i.e. all worlds from $A$ are accessible from all worlds from $W$ (in other words, $A$ is a set of *common alternatives* to all worlds from $W$). We also call these frames *semi-universal.* Notice that:

Lemma 2.2.

(i) *All semi-universal relations are transitive and Euclidean.*
(ii) *$R$ is reflexive and semi-universal iff $R = W \times W$.*
(iii) *$R$ is symmetric and semi-universal iff $R = W \times W$ or $R = \emptyset = W \times \emptyset$.*
(iv) *$R$ is serial and semi-universal iff $R = W \times A$ with $A \neq \emptyset$.*

We prove that logics K45, KB4 and KD45 are determined by some classes of semi-universal relational frames. For K45 we may use the class of all semi-universal frames (instead of the class of all transitive-Euclidean frames). For KB4 it must be the case that either $A = W$ or $A = \emptyset$, so we may use the class of universal or empty relational frames (instead of the class of all symmetric-transitive frames). Finally, for KD45 it must be the case that $A \neq \emptyset$, so we may use the class of non-empty semi-universal frames (instead of the class of all serial-transitive-Euclidean frames).[6]

By Lemma 2.1 we obtain

Corollary 2.3. *For any relational frame $\langle W, R\rangle$:*

(i) *If $R$ is transitive and Euclidean, then $R^x$ is semi-universal.*
(ii) *If $R$ is symmetric and transitive, then $R^x$ is universal or empty.*
(iii) *If $R$ is serial, transitive and Euclidean, then $R^x$ is non-empty and semi-universal.*

[6] Thus for these logics we have the like case as for the logic S5 (=KT5 = KTB4), where we may use the class of universal frames instead of the class of all reflexive-Euclidean (i.e. reflexive-transitive-symmetric) frames.

Given the above facts we notice that classes of semi-universal relational models for K45, KB4, and KD45 are connected with some classes of generated models. We make use of models generated from relational models (cf. [1], p. 97).

Let $\mathscr{M} = \langle W, R, V\rangle$ and $x \in W$. Then the model generated by $x$ from $\mathscr{M}$ is the relational model $\mathscr{M}^x = \langle W^x, R^x, V^x\rangle$ in which $W^x$ and $R^x$ are as in Lemma 2.1 and for any $\alpha \in \mathrm{At}$ and $y \in W^x$ we have $V^x(\alpha, y) = V(\alpha, y)$. Of course, $V^x$ preserves classical conditions for truth-value operators and satisfies the condition $(V_{\Box}^{R})$ for $R^x$.

For any class $\boldsymbol{C}$ of relational models we put the following class of generated models $\mathcal{G}(\boldsymbol{C}) := \{\mathscr{M}^x : \mathscr{M} \in \boldsymbol{C} \text{ and } x \text{ is in } \mathscr{M}\}$.

FACT 2.4 ([1], Theorems 3.10–3.12). *For any* $\varphi \in \mathrm{For}$

(i) *for all $x$ from $\mathscr{M}$ and $y \in W^x$, $V^x(\varphi, y) = V(\varphi, y)$;*

(ii) *$\varphi$ is valid in $\mathscr{M}$ iff for every $x$ in $\mathscr{M}$, $\varphi$ is valid in $\mathscr{M}^x$;*

(iii) *$\varphi$ is valid in $\boldsymbol{C}$ iff $\varphi$ is valid in $\mathcal{G}(\boldsymbol{C})$.* ⊣

From the above facts and facts from p. 165 we obtain

THEOREM 2.5. (i) *The logic* K45 *is determined by the class of all semi-universal relational frames.*

(ii) *The logic* KB4 *is determined by the class of universal or empty relational frames.*

(iii) *The logic* KD45 *is determined by the class of semi-universal relational frames which are non-empty.*

Any semi-universal relational model $\langle W, W \times A, V\rangle$ may be identified with the triple $\langle W, A, V\rangle$ in the sense of the following

LEMMA 2.6. *Let $W$ be a non-empty set, $A \subseteq W$ and $v \colon \mathrm{At} \times W \to \{0, 1\}$. Moreover,*

- *let $\langle W, W \times A, V\rangle$ be a semi-universal model in which $V$ is the extension of $v$* (by the condition $(V_{\Box}^{R})$ for $R = W \times A$)*;*
- *let $\langle W, A, V'\rangle$ be a simplified model in which $V'$ is the extension of $v$* (by the condition $(V_{\Box}^{A})$).

*Then $V = V'$, i.e. $\langle W, W \times A, V\rangle$ may be identified with $\langle W, A, V'\rangle$.*

*Remark.* If $A = W$, then by Lemma 2.6 the models $\langle W, W \times A, V\rangle$ and $\langle W, A, V\rangle$ may be identified with the simplified model $\langle W, V\rangle$ for S5. ⊣

By Lemma 2.6, any semi-universal model is, essentially, a simplified model which instead of a relation $W \times A$ has a set $A$ of *common alternatives to all worlds in* $W$. From this fact and Theorem 2.5 we obtain Theorem 1.1.

Notice that K45 (resp. KD45) is also determined by the class of simplified frames which are «non-universal» (resp. both «non-empty» and «non-universal»), i.e. $A \neq W$ (resp. $\emptyset \neq A \neq W$). Indeed, any formula falsifiable in a universal simplified frame $\langle W, W\rangle$ is also falsifiable in a simplified frame $\langle W \cup \{x\}, W\rangle$, where $x \notin W$. Thus, all formulae valid in all frames fulfilling $\emptyset \neq A \neq W$ are also valid in all universal simplified frames.

## 3. Some special version of Nagle's Fact

By the facts from Section 2 and by Nagle's Fact, to each one of the logics K45, KB4 and KD45 we assign a suitable class of finite Euclidean frames which satisfy one and only one of the conditions (a), (b), and a special case of (c) from Nagle's Fact. We have the following

THEOREM 3.1.

(i) K45 *is determined by the class of finite relational frames* $\langle W, R\rangle$ *which satisfy one and only one of the following conditions*: (a) *and* (b) *from Nagle's Fact, and*

(c′) $W$ *is not a singleton and there is a world* $w \in W$ *such that* $R = W \times (W \setminus \{w\})$.[7]

(ii) KB4 *is determined by the class of finite relational frames* $\langle W, R\rangle$ *which satisfy one and only one of the conditions*: (a) *and* (b).

(iii) KD45 *is determined by the class of finite relational frames* $\langle W, R\rangle$ *which satisfy one and only one of the conditions*: (b) *and* (c′).

In all cases (i)–(iii), either $R = \emptyset$, or $R = W \times W$, or $R = W \times (W \setminus \{w\})$; so $R$ is semi-universal, and naturally, $R$ is transitive and Euclidean.

Notice that K45 (resp. KD45) is also determined by the class of finite relational frames $\langle W, R\rangle$ which satisfy one of the conditions (a) and (c′) (resp. (c′) alone). Indeed, any formula falsifiable in a universal frame $\langle W, W \times W\rangle$ is also falsifiable in a frame $\langle W \cup \{x\}, (W \cup \{x\}) \times W\rangle$, where $x \notin W$. Thus, all formulae valid in all (c′)-frames are also valid in all universal frames.

[7] The condition (c′) is a particular case of the condition (c) from Nagle's Fact. Namely, the relation $W \times (W \setminus \{w\})$ is total in the non-empty set $W \setminus \{w\}$ and all worlds from this set are accessible from $w$.

Directly from Theorem 3.1 (since $\emptyset = W \times \emptyset$) we obtain the following corollary, which has a connection with Theorem 1.1.

Corollary 3.2.

(i) K45 *is determined by the class of finite simplified frames* $\langle W, A\rangle$ *which satisfy one and only one of the following conditions*:

(a*) $W$ *is a singleton and* $A = \emptyset$,

(b*) $A = W$.

(c*) $W$ *is not a singleton and* $A = W \setminus \{w\}$ *for some* $w \in W$.

(ii) KB4 *is determined by the class of finite simplified frames* $\langle W, A\rangle$ *which satisfy one and only one of the conditions* (a*) *and* (b*).

(iii) KD45 *is determined by the class of finite simplified frames* $\langle W, A\rangle$ *which satisfy one and only one of the conditions*: (b*) *and* (c*).

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