Bulletin of the **S**ection of **L**ogic
Volume 41:1/2 (2012), pp. 33–50

Andrzej Pietruszczak

SEMANTICAL INVESTIGATIONS ON SOME WEAK MODAL LOGICS. Part I*

Abstract

In this paper we examine weak logics similar to $\mathbf{S0.5}[\Box\Phi]$, where $\Phi \subseteq \mathbf{S0.5}$. We also examine their versions (one of which is $\mathbf{S0.5_{rte}}[\Box\Phi]$) that are closed under replacement of tautological equivalents (rte). We have that: $\mathbf{S0.5_{rte}}[\Box(\mathtt{K}), \Box(\mathtt{T})] \subsetneq \mathbf{S0.9}$, $\mathbf{S0.5_{rte}}[\Box(\mathtt{X}), \Box(\mathtt{T})] \subsetneq \mathbf{S1}$, and in general, if $\Phi \subseteq \mathbf{E1}$, then $\mathbf{S0.5_{rte}}[\Box\Phi] \subsetneq \mathbf{S2}$.

In the second part we shall give simplified semantics for these logics, formulated by means of some Kripke-style models. We shall also prove that the logics in question are determined by some classes of these models.

Key words: Very weak modal logics, simplified Kripke-style semantics.

1. Preliminaries

1.1. Basic notions

Modal formulae are formed in the standard way from the set At of propositional letters: 'p', 'q', 'r', 'p_0', 'p_1', 'p_2', ...; truth-value operators: '$\neg$', '$\vee$', '$\wedge$', '$\supset$', and '$\equiv$' (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); the modal operator '$\Box$' (necessity; the possibility sign '$\Diamond$' is the abbreviation of '$\neg\Box\neg$'); and brackets. Let For be the set of all modal formulae.

*The first version of this work were presented during The Third Conference: *Non-Classical Logic. Theory and Applications*, NCU, Toru, September 16–18, 2010.

In original Lewis' works (see e.g. [5]) the primitive modal operator is the possibility sign '$\Diamond$'. The necessity sign '$\Box$' is the abbreviation of '$\neg\Diamond\neg$'. Moreover, $\ulcorner\varphi \prec \psi\urcorner$ (the strict implication) was used as an abbreviation of $\ulcorner\neg\Diamond(\varphi \wedge \neg\psi)\urcorner$.

In this paper, as in [4], the primitive modal operator is '$\Box$' and $\ulcorner\varphi \prec \psi\urcorner$ is an abbreviation of $\ulcorner\Box(\varphi \supset \psi)\urcorner$. Moreover, similarly as in [5, 4], the strict equivalence $\ulcorner\varphi \succ\!\!\prec \psi\urcorner$ is an abbreviation of $\ulcorner(\varphi \prec \psi) \wedge (\psi \prec \varphi)\urcorner$.

For any formula φ let $\mathrm{sub}(\varphi)$ be the set of all instances of φ. For any set Φ of formulae we put: $\mathrm{sub}(\Phi) := \bigcup_{\varphi\in\Phi} \mathrm{sub}(\varphi)$, $\Box\Phi := \{\ulcorner\Box\varphi\urcorner : \varphi \in \Phi\}$ and $\Diamond\Phi := \{\ulcorner\Diamond\varphi\urcorner : \varphi \in \Phi\}$.

Let Taut be the set of all classical tautologies (without the modal operator). We put $\top :=$ '$p \supset p$'. Moreover, let **PL** be the set of modal formulae which are instances of classical tautologies. Of course, $\mathbf{PL} = \mathrm{sub}(\mathrm{Taut})$.

A formula φ is *propositionally atomic* iff $\varphi \in \mathrm{At}$ or $\varphi \in \Box\mathrm{For}$. Let PAt be the set of all propositionally atomic formulae, i.e. $\mathrm{PAt} := \mathrm{At} \cup \Box\mathrm{For}$.

Let $\mathsf{Val}^{\mathsf{cl}}$ be the set of all valuations $V\colon \mathrm{For} \to \{0,1\}$ which preserve classical truth conditions for truth-value operators.

LEMMA 1.1. 1. $V \in \mathsf{Val}^{\mathsf{cl}}$ *iff for some assignment* $v\colon \mathrm{PAt} \to \{0,1\}$, V *is the unique extension of* v *by classical truth conditions for truth-value operators.*

2. *For any* $\varphi \in \mathrm{For}$: $\varphi \in \mathbf{PL}$ *iff for every assignment* $v\colon \mathrm{PAt} \to \{0,1\}$ *we have that* $V(\varphi) = 1$, *where* V *is the unique extension of* v *by classical truth conditions for truth-value operators.*
3. *For any* $\varphi \in \mathrm{For}$: $\varphi \in \mathbf{PL}$ *iff* $V(\varphi) = 1$, *for any* $V \in \mathsf{Val}^{\mathsf{cl}}$.

For any $\Psi \subseteq \mathrm{For}$ and $\varphi \in \mathrm{For}$ we write $\Psi \models_{\mathbf{PL}} \varphi$ iff for any V from $\mathsf{Val}^{\mathsf{cl}}$: if $V(\Psi) \subseteq \{1\}$, then $V(\varphi) = 1$. Of course, $\Psi \models_{\mathbf{PL}} \varphi$ iff for some $\{\psi_1, \ldots, \psi_n\} \subseteq \Psi$, $n \geqslant 0$, we have that $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \mathbf{PL}$. We also write $\Psi \models_{\mathbf{PL}} \Phi$ iff $\Psi \models_{\mathbf{PL}} \varphi$, for any $\varphi \in \Phi$.

A set $\boldsymbol{\Sigma}$ of modal formulae is a *modal system* iff $\mathbf{PL} \subseteq \boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}$ is closed under the rule of detachment for '$\supset$' (*modus ponens*), i.e., for any $\varphi, \psi \in \mathrm{For}$:

$$\text{if } \varphi \text{ and } \ulcorner\varphi \supset \psi\urcorner \text{ are members of } \boldsymbol{\Sigma}\text{, so is } \psi. \tag{MP}$$

A set of modal formulae is a *logic* iff it is a modal system and it is closed under the rule of uniform substitution. Of course, **PL** is the smallest modal system and it is a logic.

For any modal system $\boldsymbol{\Sigma}$, any $\Psi \subseteq \mathrm{For}$ and any $\varphi \in \mathrm{For}$: φ is *deducible from* Ψ *in* $\boldsymbol{\Sigma}$ (written: $\Psi \vdash_{\boldsymbol{\Sigma}} \varphi$) iff for some $\{\psi_1, \ldots, \psi_n\} \subseteq \Psi$, $n \geqslant 0$, we have that $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \boldsymbol{\Sigma}$. Of course, $\models_{\mathbf{PL}} = \vdash_{\mathbf{PL}} \subseteq \vdash_{\boldsymbol{\Sigma}}$. Moreover, $\boldsymbol{\Sigma} \vdash_{\boldsymbol{\Sigma}} \varphi$ iff $\varphi \in \boldsymbol{\Sigma}$ iff $\emptyset \vdash_{\boldsymbol{\Sigma}} \varphi$.

A system $\boldsymbol{\Sigma}$ is *consistent* iff $\boldsymbol{\Sigma} \neq \mathrm{For}$; equivalently in the light of propositional logic **PL**, iff '$p \wedge \neg p$' does not belong to $\boldsymbol{\Sigma}$.

To simplify notation of logics we use the following code. If $\boldsymbol{\Lambda}$ is a logic and $\Phi \subseteq \mathrm{For}$, then $\boldsymbol{\Lambda}[\Phi]$ denotes the smallest logic which includes the set $\boldsymbol{\Lambda} \cup \Phi$. We write $\boldsymbol{\Lambda}[\varphi_1, \ldots, \varphi_n]$ instead of $\boldsymbol{\Lambda}[\{\varphi_1, \ldots, \varphi_n\}]$, and $\boldsymbol{\Lambda}[\Phi_1, \ldots, \Phi_n]$ instead of $\boldsymbol{\Lambda}[\Phi_1 \cup \cdots \cup \Phi_n]$.

We say that a modal system is *congruential* (or *classical*) iff it is closed under the following rule of congruence, i.e., for any $\varphi, \psi \in \mathrm{For}$:

$$\text{if } \ulcorner\varphi \equiv \psi\urcorner \in \boldsymbol{\Sigma}, \text{ then } \ulcorner\Box\varphi \equiv \Box\psi\urcorner \in \boldsymbol{\Sigma}. \tag{RE}$$

FACT 1.2. *A modal system* $\boldsymbol{\Sigma}$ *is congruential iff it is closed under replacement, i.e., for any* $\varphi, \psi, \chi \in \mathrm{For}$*:*

$$\textit{if } \ulcorner\varphi \equiv \psi\urcorner \in \boldsymbol{\Sigma} \textit{ and } \chi \in \boldsymbol{\Sigma}, \textit{ then } \chi[^{\varphi}/_{\psi}] \in \boldsymbol{\Sigma}, \tag{RRE}$$

or equivalently:

$$\textit{if } \ulcorner\varphi \equiv \psi\urcorner \in \boldsymbol{\Sigma}, \textit{ then } \ulcorner\chi[^{\varphi}/_{\psi}] \equiv \chi\urcorner \in \boldsymbol{\Sigma}, \tag{RRE$'$}$$

where $\chi[^{\varphi}/_{\psi}]$ *is any formula that results from* χ *by replacing zero, one or more occurrences of* φ*, in* χ*, by* ψ*.*

A modal system $\boldsymbol{\Sigma}$ is called *monotonic* iff $\boldsymbol{\Sigma}$ is closed under the following rule of monotonicity, i.e., i.e., for any $\varphi, \psi \in \mathrm{For}$:

$$\text{if } \ulcorner\varphi \supset \psi\urcorner \in \boldsymbol{\Sigma} \text{ then } \ulcorner\Box\varphi \supset \Box\psi\urcorner \in \boldsymbol{\Sigma}. \tag{RM}$$

FACT 1.3. *A modal system is monotonic iff it is congruential and contains all instances of the following formula:*

$$\Box(p \wedge q) \supset (\Box p \wedge \Box q) \tag{M}$$

A modal system $\boldsymbol{\Sigma}$ is called *regular* iff $\boldsymbol{\Sigma}$ is closed under the following regularity rule, i.e., for any $\varphi, \psi, \chi \in \mathrm{For}$:

$$\text{if } \ulcorner(\varphi \wedge \psi) \supset \chi\urcorner \in \boldsymbol{\Sigma}, \text{ then } \ulcorner(\Box\varphi \wedge \Box\psi) \supset \Box\chi\urcorner \in \boldsymbol{\Sigma}. \tag{RR}$$

FACT 1.4. *For any modal system the following conditions are equivalent*:
(a) *the system is regular,*
(b) *it is monotonic and contains all instances of*

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \tag{K}$$

(c) *it is monotonic and contains all instances of*

$$(\Box p \wedge \Box q) \supset \Box(p \wedge q) \tag{C}$$

(d) *it is monotonic and contains all instances of*

$$\big(\Box(p \supset q) \wedge \Box(q \supset r)\big) \supset \Box(p \supset r) \tag{X}$$

(e) *it is congruential and contains all instances of*

$$\Box(p \wedge q) \equiv (\Box p \wedge \Box q) \tag{R}$$

To simplify notation of logics we use the following code. If $\boldsymbol{\Lambda}$ is a regular logic and $\Phi \subseteq$ For, then $\boldsymbol{\Lambda}{\oplus}\Phi$ denotes the smallest regular logic which includes the set $\boldsymbol{\Lambda} \cup \Phi$. We write $\boldsymbol{\Lambda}{\oplus}\varphi_1 \ldots \varphi_n$ instead of $\boldsymbol{\Lambda}{\oplus}\{\varphi_1, \ldots, \varphi_n\}$.

C2 is the smallest regular logic and **E2** is the smallest regular logic which contains (T), i.e. **E2** = **C2**⊕(T).

We say that a modal system $\boldsymbol{\Sigma}$ is *normal* iff it contains all instances of (K) and is closed under the following rule:

$$\text{if } \varphi \in \boldsymbol{\Sigma}, \text{ then } \ulcorner \Box\varphi \urcorner \in \boldsymbol{\Sigma}. \tag{RN}$$

FACT 1.5. *For any modal system the following conditions are equivalent*:
(a) *it is normal,*
(b) *it is regular and contains* $\Box\top$*,*
(c) *it is congruential, contains* $\Box\top$ *and includes* sub(K)*.*

By the above fact, if $\boldsymbol{\Lambda}$ is a normal logic, then $\boldsymbol{\Lambda}{\oplus}\Gamma$ is as well. Indeed, $\boldsymbol{\Lambda}$ is regular and contains $\Box\top$. Hence $\boldsymbol{\Lambda}{\oplus}\Gamma$ is also regular and contains $\Box\top$. So $\boldsymbol{\Lambda}{\oplus}\Gamma$ is normal.

In this paper we investigate some weak modal logics. For these logics we are using the following lemmas.

LEMMA 1.6. *For any modal system $\boldsymbol{\Sigma}$ which includes the following set*

$$\mathrm{E}_{\mathbf{PL}} := \{\ulcorner\Box\varphi \equiv \Box\psi\urcorner \,:\, \ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}\},$$

1. $\Box\top \in \boldsymbol{\Sigma}$ *iff* $\Box\mathbf{PL} \subseteq \boldsymbol{\Sigma}$.
2. *If* $\mathrm{sub}(\mathtt{X}) \subseteq \boldsymbol{\Sigma}$, *then* $\mathrm{sub}(\mathtt{K}) \subseteq \boldsymbol{\Sigma}$.

PROOF: 1. For any $\tau \in \mathbf{PL}$, $\ulcorner\tau \equiv \top\urcorner \in \mathbf{PL}$ and $\ulcorner\Box\tau \equiv \Box\top\urcorner \in \boldsymbol{\Sigma}$, since $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$. Hence, by $\mathbf{PL}$, also $\Box\tau \in \boldsymbol{\Sigma}$, since $\Box\top \in \boldsymbol{\Sigma}$.

2. For any $\varphi, \psi \in \mathrm{For}$, $\ulcorner\varphi \equiv (\top \supset \varphi)\urcorner \in \mathbf{PL}$ and $\ulcorner\psi \equiv (\top \supset \psi)\urcorner \in \mathbf{PL}$. So if $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$, then $\ulcorner\Box\varphi \equiv \Box(\top \supset \varphi)\urcorner$ and $\ulcorner\Box\psi \equiv \Box(\top \supset \psi)\urcorner$ belong to $\boldsymbol{\Sigma}$. Moreover, if $(\mathtt{X}) \in \boldsymbol{\Sigma}$, then $\ulcorner\big(\Box(\top \supset \varphi) \wedge \Box(\varphi \supset \psi)\big) \supset \Box(\top \supset \psi)\urcorner \in \boldsymbol{\Sigma}$. Hence $\ulcorner\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)\urcorner \in \boldsymbol{\Sigma}$, by $\mathbf{PL}$. ⊣

LEMMA 1.7 ([6]). *For any modal system $\boldsymbol{\Sigma}$: $\boldsymbol{\Sigma}$ includes the following set*

$$\mathrm{M}_{\mathbf{PL}} := \{\ulcorner\Box\varphi \supset \Box\psi\urcorner \,:\, \ulcorner\varphi \supset \psi\urcorner \in \mathbf{PL}\}$$

iff $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$ *and* $\mathrm{sub}(\mathtt{M}) \subseteq \boldsymbol{\Sigma}$.

LEMMA 1.8. *For any modal system $\boldsymbol{\Sigma}$ which includes* $\mathrm{M}_{\mathbf{PL}}$:

$$\Box\mathbf{PL} \subseteq \boldsymbol{\Sigma} \textit{ iff } \Box\top \in \boldsymbol{\Sigma} \textit{ iff } \boldsymbol{\Sigma} \textit{ has some formula of the form } \ulcorner\Box\varphi\urcorner.$$

LEMMA 1.9 ([6]). *For any modal system $\boldsymbol{\Sigma}$ the following conditions are equivalent*:

(a) $\boldsymbol{\Sigma}$ *includes the following set*

$$\mathrm{R}_{\mathbf{PL}} := \{\ulcorner(\Box\varphi \wedge \Box\psi) \supset \Box\chi\urcorner \,:\, \ulcorner(\varphi \wedge \psi) \supset \chi\urcorner \in \mathbf{PL}\},$$

(b) $\mathrm{M}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$ *and* $\mathrm{sub}(\mathtt{K}) \subseteq \boldsymbol{\Sigma}$,

(c) $\mathrm{M}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$ *and* $\mathrm{sub}(\mathtt{X}) \subseteq \boldsymbol{\Sigma}$,

(d) $\mathrm{M}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$ *and* $\mathrm{sub}(\mathtt{C}) \subseteq \boldsymbol{\Sigma}$,

(e) $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$ *and* $\mathrm{sub}(\mathtt{R}) \subseteq \boldsymbol{\Sigma}$.

LEMMA 1.10. *Fix any system $\boldsymbol{\Sigma}$*:

1. *If* $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$, *then $\boldsymbol{\Sigma}$ contains all instances of the following formula*

$$\Diamond p \equiv \neg\Box\neg p \qquad (\mathtt{df}\,\Diamond)$$

2. *If* $\mathrm{R}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$*, then* $\boldsymbol{\Sigma}$ *contains all instances of the following formulae*

$$\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q) \tag{$\mathtt{R}^{\Diamond}$}$$

$$\Diamond(p \supset q) \equiv (\Box p \supset \Diamond q) \tag{$\mathtt{R}^{\Diamond\Box}$}$$

Lemma 1.11. *For any modal system* $\boldsymbol{\Sigma}$:

1. *If* $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$*, then* $\boldsymbol{\Sigma}$ *contains all instances of the following formula*

$$(p \prec q) \equiv \neg \Diamond(p \wedge \neg q) \tag{$\mathtt{df}' \prec$}$$

2. *If* $\mathrm{R}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$*, then* $\boldsymbol{\Sigma}$ *contains all instances of*

$$(p \prec q) \equiv \Box(p \equiv q) \tag{$\mathtt{df}' \prec$}$$

Lemma 1.12. *For any modal system* $\boldsymbol{\Sigma}$:

1. *If* $\boldsymbol{\Sigma}$ *contains all instances of the following formula*

$$\Box p \supset p \tag{$\mathtt{T}$}$$

then $\boldsymbol{\Sigma}$ *is closed under the following rule*

$$\textit{if } \ulcorner \Box\varphi \urcorner \in \boldsymbol{\Sigma}, \textit{ then } \varphi \in \boldsymbol{\Sigma}. \tag{RN_*}$$

2. *If* $\boldsymbol{\Sigma}$ *is closed under* (RN_*)*, then* $\boldsymbol{\Sigma}$ *is closed under the following rule of detachment for '*$\prec$*'* (*strict version of modus ponens*)

$$\textit{if } \ulcorner \varphi \prec \psi \urcorner \in \boldsymbol{\Sigma} \textit{ and } \varphi \in \boldsymbol{\Sigma}, \textit{ then } \psi \in \boldsymbol{\Sigma}. \tag{SMP}$$

3. *If* $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$ *and* $\boldsymbol{\Sigma}$ *is closed under* (SMP)*, then* $\boldsymbol{\Sigma}$ *is closed under* (RN_*).

Proof: For 3. Let $\ulcorner \Box\varphi \urcorner \in \boldsymbol{\Sigma}$. Since $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$ and $\ulcorner \varphi \equiv (\top \supset \varphi) \urcorner \in \mathbf{PL}$, we have that $\ulcorner \Box\varphi \equiv \Box(\top \supset \varphi) \urcorner \in \boldsymbol{\Sigma}$. Hence, by **PL**, $\ulcorner \Box(\top \supset \varphi) \urcorner \in \boldsymbol{\Sigma}$. So $\varphi \in \boldsymbol{\Sigma}$, by (SMP) and **PL**. ⊣

1.2. t-regular modal systems

In [6] a modal system is called *t-regular* iff it includes the set $\mathrm{R}_{\mathbf{PL}}$. Thus, the set $\mathrm{R}_{\mathbf{PL}}$ replaces the rule (RR) in the formulation of regular systems.

By definition, any modal system which includes some t-regular system, is also t-regular. So, if $\boldsymbol{\Lambda}$ is a t-regular logic, then $\boldsymbol{\Lambda}[\Phi]$ is. Moreover, every regular system is t-regular.

FACT 1.13. *For any t-regular modal system* $\boldsymbol{\Sigma}$ *the following conditions are equivalent*:

(a) $\Diamond\top \in \boldsymbol{\Sigma}$,

(b) $\boldsymbol{\Sigma}$ *contains all instances of the following formula*

$$\Box p \supset \Diamond p \tag{D}$$

FACT 1.14. *For any t-regular modal system* $\boldsymbol{\Sigma}$*, if* $\boldsymbol{\Sigma}$ *contains one of the following formula, then* $\boldsymbol{\Sigma}$ *contains all the following formulae*:[1]

$$\begin{aligned} \Box p &\supset (p \lor \Box q) \\ \Diamond q &\supset (\Box p \supset p) \\ \Diamond(q \supset q) &\supset (\Box p \supset p) \\ \neg\Box(q \land \neg q) &\supset (\Box p \supset p) \end{aligned} \tag{$\mathtt{T_q}$}$$

The logic **C1** from [7] is the smallest t-regular system. **C1** is a logic and $\mathbf{C1} := \mathbf{PL}[\mathrm{R}_{\mathbf{PL}}]$. The logics **D1** and **E1** from [4] are respectively the smallest t-regular logics which contain (D) and (T), i.e. $\mathbf{D1} := \mathbf{PL}[\mathrm{R}_{\mathbf{PL}}, \mathtt{D}] = \mathbf{C1}[\mathtt{D}] = \mathbf{C1}[\Diamond\top]$ and $\mathbf{E1} := \mathbf{PL}[\mathrm{R}_{\mathbf{PL}}, \mathtt{T}] = \mathbf{C1}[\mathtt{T}]$. We have that $\mathbf{C1} \subsetneq \mathbf{D1} \subsetneq \mathbf{E1}$ and $\mathbf{C1} \subsetneq \mathbf{C1}[\mathtt{T_q}] \subsetneq \mathbf{E1}$ (see [6])

Notice that $\mathbf{E1} = \mathbf{C1}[\mathtt{D},\mathtt{T_q}]$. Indeed, from **C1** and (D) we obtain '$\Diamond(q \supset q)$', and hence (T), by ($\mathtt{T_q}$) and (MP).

1.3. t-normal modal systems

In [6] a modal system is called *t-normal* iff it contains all instances of (K) and includes the set $\Box\mathbf{PL}$. Thus, the set $\Box\mathbf{PL}$ replaces the rule (RN) in the formulation of normal systems. By definitions, any modal system which includes some t-normal system, is also t-normal. So, if $\boldsymbol{\Lambda}$ is a t-normal logic, then $\boldsymbol{\Lambda}[\Phi]$ is. Moreover, every normal system is t-normal.

---

[1]The name '$\mathtt{T_q}$' is an abbreviation for 'quasi-T', because for normal logics with (T) (resp. ($\mathtt{T_q}$)) we use reflexive (resp. quasi- reflexive) standard Kripke models.

By lemmas 1.6–1.9 we obtain:

LEMMA 1.15. *For any system the following conditions are equivalent*:

(a) *it is t-normal,*
(b) *it is t-regular and contains* $\Box\top$*,*
(c) *it is t-regular and contains some formula of the form* $\ulcorner\Box\varphi\urcorner$.

In [4] the logic **S0.5** is the smallest modal logic which includes $\Box$Taut, and contains (K) and (T). The logic $\mathbf{S0.5}^\circ$ is associated with Lemmon's **S0.5**. It is the smallest logic which includes $\Box$Taut and contains (K). Of course, by uniform substitution, **S0.5** and $\mathbf{S0.5}^\circ$ include the set $\Box\mathbf{PL}$; so $\mathbf{S0.5}^\circ$ is the smallest t-normal system, and **S0.5** is the smallest t-normal system which includes sub(T). So we have that $\mathbf{S0.5}^\circ := \mathbf{PL}[\Box\mathrm{Taut},\mathtt{K}] = \mathbf{C1}[\Box\top]$ and $\mathbf{S0.5} := \mathbf{PL}[\Box\mathrm{Taut},\mathtt{K},\mathtt{T}] = \mathbf{S0.5}^\circ[\mathtt{T}] = \mathbf{E1}[\Box\top]$. It is the case that $\mathbf{S0.5}^\circ \subsetneq \mathbf{S0.5}^\circ[\mathtt{D}] \subsetneq \mathbf{S0.5}$, because $(\mathtt{D}) \notin \mathbf{S0.5}^\circ$ and $(\mathtt{T}) \notin \mathbf{S0.5}^\circ[\mathtt{D}]$. Moreover, $\mathbf{S0.5}^\circ \subsetneq \mathbf{S0.5}^\circ[\mathtt{T_q}] \subsetneq \mathbf{S0.5}$, since $(\mathtt{T_q}) \notin \mathbf{S0.5}^\circ$ and $(\mathtt{T}) \notin \mathbf{S0.5}^\circ[\mathtt{T_q}]$ (see e.g. [6] and Corollary 3.5 in the second part). Notice that $\mathbf{S0.5} = \mathbf{S0.5}^\circ[\mathtt{D},\mathtt{T_q}]$.

By Lemma 1.12, the logic **S0.5** is closed under (RN_*) and (SMP). However for any $\varphi \in$ For: $\ulcorner\Box\varphi\urcorner \in \mathbf{S0.5}^\circ$ iff $\varphi \in \mathbf{PL}$ iff $\ulcorner\Box\varphi\urcorner \in \mathbf{S0.5}$ (see Fact 3.8 in the second part). So $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$ and $\mathbf{S0.5}^\circ[\mathtt{T_q}]$ are also closed under (RN_*) and (SMP).[2]

1.4. Replacement for tautologous equivalents

We say that a modal system $\boldsymbol{\Sigma}$ is an *rte-system* iff $\boldsymbol{\Sigma}$ is closed under replacement for tautological equivalents, i.e.:

$$\forall_{\varphi,\psi,\chi\in\mathrm{For}}\colon \text{ if } \ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL} \text{ and } \chi \in \boldsymbol{\Sigma}, \text{ then } \chi[^{\varphi}/_{\psi}] \in \boldsymbol{\Sigma}. \tag{rte}$$

We consider the following sets of formulae:

[2]Notice that the rules (RN_*) and (SMP) are not derivable in $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$ and $\mathbf{S0.5}^\circ[\mathtt{T_q}]$ in the following sense. We can consider $\mathbf{S0.5}^\circ$ (resp. $\mathbf{S0.5}^\circ[\mathtt{D}]$; $\mathbf{S0.5}^\circ[\mathtt{T_q}]$; **S0.5**) as being axiomatized by axioms **PL**, sub(K) (resp. plus sub(D); sub($\mathtt{T_q}$); sub(T)) and the sole rule (MP). Of course, in such axiomatic system of $\mathbf{S0.5}^\circ$ (resp. $\mathbf{S0.5}^\circ[\mathtt{D}]$; $\mathbf{S0.5}^\circ[\mathtt{T_q}]$), if $\varphi \notin \mathbf{PL}$, then from $\ulcorner\Box\varphi\urcorner$ we do not obtain φ, since $\mathbf{PL}, \mathrm{sub}(\mathtt{K}), \mathrm{sub}(\mathtt{D}), \mathrm{sub}(\mathtt{T_q}) \nvdash_{\mathbf{PL}} \Box\varphi \supset \varphi$.

$$\mathrm{REP}_{\mathbf{PL}} := \{\ulcorner\chi \equiv \chi[^{\varphi}/_{\psi}]\urcorner : \chi \in \mathrm{For}\ \&\ \ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}\},$$
$$\mathbf{PL}_{\mathrm{rte}} := \{\tau[^{\varphi_1}/_{\psi_1}, \dots, ^{\varphi_k}/_{\psi_k}] \in \mathrm{For} : \tau \in \mathbf{PL}\ \&\ \ulcorner\varphi_1 \equiv \psi_1\urcorner \in \mathbf{PL}, \dots, \ulcorner\varphi_k \equiv \psi_k\urcorner \in \mathbf{PL}\},$$

where $\tau[^{\varphi_1}/_{\psi_1}, \dots, ^{\varphi_k}/_{\psi_k}]$ is any formula that results from τ by replacing zero, one or more occurrences of φ_i, in τ, by ψ_i. Since $\ulcorner\chi \equiv \chi\urcorner \in \mathbf{PL}$, we have that: $\mathrm{REP}_{\mathbf{PL}} \subseteq \mathbf{PL}_{\mathrm{rte}}$ and $\Box\mathrm{REP}_{\mathbf{PL}} \subseteq \Box\mathbf{PL}_{\mathrm{rte}}$.

We will now focus on general properties of rte-systems.

LEMMA 1.16. *For any system $\boldsymbol{\Sigma}$ the following conditions are equiwalent*:

(a) *$\boldsymbol{\Sigma}$ is an rte-system,*
(b) $\mathbf{PL}_{\mathrm{rte}} \subseteq \boldsymbol{\Sigma}$*,*
(c) $\mathrm{REP}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$*,*
1. *$\boldsymbol{\Sigma}$ is closed under the following replacement*

$$\forall_{\varphi,\psi,\chi\in\mathrm{For}}\colon\ \textit{if } \ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}, \textit{ then } \ulcorner\Box\chi \equiv \Box\chi[^{\varphi}/_{\psi}]\urcorner \in \boldsymbol{\Sigma}.$$

PROOF: "(a) $\Rightarrow$ (b)" If $\ulcorner\varphi_i \equiv \psi_i\urcorner \in \mathbf{PL}$, $i = 1, \dots, k$, and $\tau \in \mathbf{PL} \subseteq \boldsymbol{\Sigma}$ then $\tau[^{\varphi_1}/_{\psi_1}] \in \boldsymbol{\Sigma}, \dots, \tau[^{\varphi_1}/_{\psi_1}, ..., ^{\varphi_k}/_{\psi_k}] \in \boldsymbol{\Sigma}$, by (rte). Thus, $\mathbf{PL}_{\mathrm{rte}} \subseteq \boldsymbol{\Sigma}$.

"(b) $\Rightarrow$ (c)" By the fact that $\mathrm{REP}_{\mathbf{PL}} \subseteq \mathbf{PL}_{\mathrm{rte}}$.

"(c) $\Rightarrow$ (a)" If $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$, then $\ulcorner\chi \equiv \chi[^{\varphi}/_{\psi}]\urcorner \in \mathrm{REP}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$. Moreover, if $\chi \in \boldsymbol{\Sigma}$, then $\chi[^{\varphi}/_{\psi}] \in \boldsymbol{\Sigma}$, by $\mathbf{PL}$.

"(c) $\Rightarrow$ (d)" Obvious.

"(d) $\Rightarrow$ (c)" Suppose that $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$. First we consider the possibility that $\chi = \varphi$. Then $\chi[^{\varphi}/_{\psi}] = \varphi$ or $\chi[^{\varphi}/_{\psi}] = \psi$.

Thus we may assume henceforth that $\chi \neq \varphi$. The proof proceeds by induction on the complexity of χ. We give it for the cases in which χ is ($*$) atomic; ($**$) $\ulcorner\neg\chi_1\urcorner$ or $\ulcorner\chi_1 \circ \chi_2\urcorner$, for $\circ = \vee, \wedge, \supset, \equiv$; and ($***$) a necessitation, $\ulcorner\Box\chi_1\urcorner$.

For ($*$): There is no replacement in this case. For ($***$): by the assumption.

For the inductive case ($**$) we assume, for induction, that the result holds for all sentences shorter than χ. So $\ulcorner\chi_1 \equiv \chi_1[^{\varphi}/_{\psi}]\urcorner \in \boldsymbol{\Lambda}$ and $\ulcorner\chi_2 \equiv \chi_2[^{\varphi}/_{\psi}]\urcorner \in \boldsymbol{\Lambda}$. It follows (by PL) that $\ulcorner\neg\chi_1 \equiv \neg\chi_1[^{\varphi}/_{\psi}]\urcorner \in \boldsymbol{\Lambda}$ and $\ulcorner(\chi_1 \circ \chi_2) \equiv (\chi_1 \circ \chi_2)[^{\varphi}/_{\psi}]\urcorner \in \boldsymbol{\Lambda}$, for $\circ = \vee, \wedge, \supset, \equiv$. ⊣

By lemmas 1.16, 1.6, 1.9 and 1.15 we obtain:

Corollary 1.17. *For any rte-system* $\boldsymbol{\Sigma}$:

1. $\mathrm{E}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$.
2. $\Box\top \in \boldsymbol{\Sigma}$ *iff* $\Box\mathbf{PL} \subseteq \boldsymbol{\Sigma}$.
3. *If* $\Box\top \in \boldsymbol{\Sigma}$ *and* $\mathrm{sub}(\mathtt{K}) \subseteq \boldsymbol{\Sigma}$, *then* $\boldsymbol{\Sigma}$ *is t-normal; consequently* $\mathrm{R}_{\mathbf{PL}} \subseteq \boldsymbol{\Sigma}$, $\mathrm{sub}(\mathtt{X}) \subseteq \boldsymbol{\Sigma}$ *and* $\mathrm{sub}(\mathtt{R}) \subseteq \boldsymbol{\Sigma}$.
4. *If* $\mathrm{sub}(\mathtt{X}) \subseteq \boldsymbol{\Sigma}$, *then* $\mathrm{sub}(\mathtt{K}) \subseteq \boldsymbol{\Sigma}$.

Of course, any modal system which includes some rte-system, is also an rte-system. So if $\boldsymbol{\Lambda}$ is an rte-logic, then $\boldsymbol{\Lambda}[\Phi]$ is.

Fact 1.18. *The set* $\mathbf{PL}_{\mathrm{rte}}$ *is the smallest rte-system and rte-logic.*

Proof: Of course, $\mathbf{PL} \subseteq \mathbf{PL}_{\mathrm{rte}}$. Let $\ulcorner\chi_1 \supset \chi_2\urcorner \in \mathbf{PL}_{\mathrm{rte}}$ and $\chi_1 \in \mathbf{PL}_{\mathrm{rte}}$, i.e., for some $\tau_0 \in \mathbf{PL}$, $\psi_0 \in$ For we have that: $\chi_1 = \tau_0[{}^{\varphi_1}/_{\psi_1}, \ldots, {}^{\varphi_k}/_{\psi_k}]$, $\ulcorner\tau_0 \supset \psi_0\urcorner \in \mathbf{PL}$, $\chi_2 = \psi_0[{}^{\varphi_{k+1}}/_{\psi_{k+1}}, \ldots, {}^{\varphi_{k+m}}/_{\psi_{k+m}}]$ and $\ulcorner\varphi_1 \equiv \psi_1\urcorner \in \mathbf{PL}$, $\ldots$, $\ulcorner\varphi_{k+m} \equiv \psi_{k+m}\urcorner \in \mathbf{PL}$. Hence $\psi_0 \in \mathbf{PL}$; so $\chi_2 \in \mathbf{PL}_{\mathrm{rte}}$. Thus, $\mathbf{PL}_{\mathrm{rte}}$ is a modal system. From Lemma 1.16, $\mathbf{PL}_{\mathrm{rte}}$ is the smallest rte-system.

For any uniform substitution s of formulae for propositional letters, $s(\tau[{}^{\varphi_1}/_{\psi_1}, \ldots, {}^{\varphi_k}/_{\psi_k}]) = s(\tau)[{}^{s(\varphi_1)}/_{s(\psi_1)}, \ldots, {}^{s(\varphi_k)}/_{s(\psi_k)}]$ and $s(\tau) \in \mathbf{PL}$. ⊣

Notice that $\mathbf{S0.5}^{\circ}$ (and so also $\mathbf{S0.5}^{\circ}$, $\mathbf{S0.5}^{\circ}[\mathtt{D}]$ and $\mathbf{S0.5}^{\circ}[\mathtt{T_q}]$) is not closed under (rte). For example, the formulae:

$$\begin{array}{ll} \text{a)} & \Box\Box p \supset \Box\Box\neg\neg p \\ \text{b)} & \Box\Box\neg\neg p \supset \Box\Box p \end{array} \qquad (\dagger)$$

do not belong to these logics (see e.g. Fact 3.6 in the second part).

Corollary 1.19. *For any rte-system* $\boldsymbol{\Sigma}$ *which includes* $\mathrm{M}_{\mathbf{PL}}$ *and has some formula of the form* $\ulcorner\Box\varphi\urcorner$ (*consequently,* $\Box\top \in \boldsymbol{\Sigma}$, *by Lemma 1.8*):

1. $\Box\mathrm{REP}_{\mathbf{PL}} \subseteq \Box\mathbf{PL}_{\mathrm{rte}} \subseteq \boldsymbol{\Sigma}$,
2. $\boldsymbol{\Sigma}$ *is closed under the following replacement*

$$\forall_{\varphi,\psi,\chi\in\mathrm{For}}\colon \textit{ if } \ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}, \textit{ then } \ulcorner\chi \rightarrowtail\!\!\!\prec \chi[{}^{\varphi}/_{\psi}]\urcorner \in \boldsymbol{\Sigma}. \qquad \text{(srte)}$$

Proof: 1. Let $\tau \in \mathbf{PL}$. By Corollary 1.17, $\Box\tau \in \boldsymbol{\Sigma}$. So if $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$, then $\Box\tau[{}^{\varphi}/_{\psi}] \in \boldsymbol{\Sigma}$, by (rte).

2. By 1, $\ulcorner\Box(\chi \supset \chi[{}^{\varphi}/_{\psi}])\urcorner$ and $\ulcorner\Box(\chi[{}^{\varphi}/_{\psi}] \supset \chi)\urcorner$ belong to $\boldsymbol{\Sigma}$. ⊣

Moreover, we obtain:

LEMMA 1.20. *For any rte-system* $\boldsymbol{\Sigma}$:

$$\textit{if } \operatorname{sub}(\Box(\mathtt{X})) \subseteq \boldsymbol{\Sigma}, \textit{ then } \operatorname{sub}(\Box(\mathtt{K})) \subseteq \boldsymbol{\Sigma}.$$

PROOF: If $\ulcorner\Box\big(\big(\Box(\top \supset \varphi) \wedge \Box(\varphi \supset \psi)\big) \supset \Box(\top \supset \psi)\big)\urcorner \in \boldsymbol{\Sigma}$, then $\ulcorner\Box\big(\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)\big)\urcorner \in \boldsymbol{\Sigma}$, by **PL** and two applications of (rte), since $\ulcorner\varphi \equiv (\top \supset \varphi)\urcorner \in \mathbf{PL}$ and $\ulcorner\psi \equiv (\top \supset \psi)\urcorner \in \mathbf{PL}$. ⊣

Let $\mathbf{S0.5^{\circ}_{rte}}$, $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{D}]$, $\mathbf{S0.5^{\circ}_{rte}}[\mathrm{T_q}]$ and $\mathbf{S0.5_{rte}}$ be, respectively, such versions of the logics $\mathbf{S0.5^{\circ}}$, $\mathbf{S0.5^{\circ}}[\mathtt{D}]$, $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{T_q}]$ and $\mathbf{S0.5}$ that are closed under (rte). Thus, $\mathbf{S0.5^{\circ}_{rte}}$ is the smallest t-normal rte-system; so $\mathbf{S0.5^{\circ}_{rte}} = \mathbf{PL}[\mathrm{REP_{PL}}, \mathtt{K}, \Box\top]$. The logics $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{D}]$, $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{T_q}]$ and $\mathbf{S0.5_{rte}}$ are the smallest t-normal rte-logics which contain ($\mathtt{D}$), ($\mathtt{T_q}$) and ($\mathtt{T}$), respectively. Thus, $\mathbf{S0.5_{rte}} = \mathbf{S0.5^{\circ}_{rte}}[\mathtt{T}] = \mathbf{PL}[\mathrm{REP_{PL}}, \mathtt{K}, \mathtt{T}, \Box\top]$ and $\mathbf{S0.5^{\circ}_{rte}}[\mathtt{D}] = \mathbf{PL}[\mathrm{REP_{PL}}, \mathtt{K}, \mathtt{D}, \Box\top]$. We have that $\mathbf{S0.5^{\circ}_{rte}} \subsetneq \mathbf{S0.5^{\circ}_{rte}}[\mathtt{D}] \subsetneq \mathbf{S0.5_{rte}}$, because ($\mathtt{D}$) $\notin \mathbf{S0.5^{\circ}_{rte}}$ and ($\mathtt{T}$) $\notin \mathbf{S0.5^{\circ}_{rte}}[\mathtt{D}]$. Moreover, we have that $\mathbf{S0.5^{\circ}_{rte}} \subsetneq \mathbf{S0.5^{\circ}_{rte}}[\mathtt{T_q}] \subsetneq \mathbf{S0.5_{rte}}$, because ($\mathtt{T_q}$) $\notin \mathbf{S0.5^{\circ}_{rte}}$ and ($\mathtt{T}$) $\notin \mathbf{S0.5^{\circ}_{rte}}[\mathtt{T_q}]$ (see [6]).

By Lemma 1.12, the logic $\mathbf{S0.5_{rte}}$ is closed under (RN_*) and (SMP). However for any $\varphi \in$ For: $\ulcorner\Box\varphi\urcorner \in \mathbf{S0.5^{\circ}_{rte}}$ iff $\varphi \in \mathbf{PL}_{\mathrm{rte}}$ iff $\ulcorner\Box\varphi\urcorner \in \mathbf{S0.5_{rte}}$ (see Fact 4.5 in the second part). So, by Lemma 1.16, $\mathbf{S0.5^{\circ}_{rte}}$ is also closed under (RN_*) and (SMP).

Let $\mathbf{C1_{rte}}$, $\mathbf{D1_{rte}}$, $\mathbf{C1_{rte}}[\mathtt{T_q}]$ and $\mathbf{E1_{rte}}$ be, respectively, such versions of the logics $\mathbf{C1}$, $\mathbf{D1}$, $\mathbf{C1}[\mathtt{T_q}]$ and $\mathbf{E1}$ that are closed under (rte). The logic $\mathbf{C1_{rte}}$ is the smallest t-regular rte-system; so $\mathbf{C1_{rte}} = \mathbf{PL}[\mathrm{R_{PL}}, \mathrm{REP_{PL}}]$. $\mathbf{D1_{rte}}$, $\mathbf{C1_{rte}}[\mathtt{T_q}]$ and $\mathbf{E1_{rte}}$ are smallest t-regular rte-logics which contain ($\mathtt{D}$), ($\mathtt{T_q}$) and ($\mathtt{T}$), respectively. We have that $\mathbf{C1_{rte}} \subsetneq \mathbf{D1_{rte}} \subsetneq \mathbf{E1_{rte}}$ and $\mathbf{C1_{rte}} \subsetneq \mathbf{C1_{rte}}[\mathtt{T_q}] \subsetneq \mathbf{E1_{rte}}$ (see [6]).

Finally notice that for the smallest rte-logic $\mathbf{PL}_{\mathrm{rte}}$ we have "valuation semantics". Let $\mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$ be the set of all valuations $V\colon \mathrm{For} \to \{0, 1\}$ from $\mathsf{Val}^{\mathsf{cl}}$ satisfying the following condition:

$$\forall_{\varphi,\psi,\chi\in\mathrm{For}}\colon \text{ if } \ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}, \text{ then } V(\chi) = V(\chi[^{\varphi}/_{\psi}]). \qquad (\star)$$

For the set $\mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$ we have a fact analogous to Lemma 1.1 for $\mathsf{Val}^{\mathsf{cl}}$.

LEMMA 1.21. 1. *$V \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$ iff for some $v\colon \mathrm{PAt} \to \{0,1\}$ such that*

$$\forall_{\varphi,\psi,\chi\in\mathrm{For}}\text{: if } \ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}, \text{ then } v(\Box\chi) = v(\Box\chi[{}^{\varphi}/_{\psi}]), \qquad (\star_{\mathrm{PAt}})$$

V is the unique extension of v by classical truth conditions for truth-value operators.

2. *For any $\varphi \in$ For: $\varphi \in \mathbf{PL}_{\mathrm{rte}}$ iff for any $v\colon \mathrm{PAt} \to \{0,1\}$ satisfying $(\star_{\mathrm{PAt}})$ we have that $V(\varphi) = 1$, where V is the unique extension of v by classical truth conditions for truth-value operators.*
3. *For any $\varphi \in$ For: $\varphi \in \mathbf{PL}_{\mathrm{rte}}$ iff for any $V \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$, $V(\varphi) = 1$.*

PROOF: 1. "$\Leftarrow$" Let $\chi, \varphi, \psi \in$ For such such $\ulcorner\varphi \equiv \psi\urcorner \in \mathbf{PL}$. By Lemma 1.1, $V \in \mathsf{Val}^{\mathsf{cl}}$ and $V(\varphi) = V(\psi)$.

First we consider the possibility that $\chi = \varphi$. Then $\chi[{}^{\varphi}/_{\psi}] = \psi$ (when there is no replacement) or $\chi[{}^{\varphi}/_{\psi}] = \varphi$ (when φ is replaced by ψ). So $V(\chi) = V(\chi[{}^{\varphi}/_{\psi}])$, by the assumption.

Thus we may assume henceforth that $\chi \neq \varphi$. The proof proceeds by induction on the complexity of χ. We give it for the cases in which χ is $(*)$ atomic; $(**)$ $\ulcorner\neg\chi_1\urcorner$ or $\ulcorner\chi_1 \circ \chi_2\urcorner$, for $\circ = \vee, \wedge, \supset, \equiv$; and $(***)$ a necessitation, $\ulcorner\Box\chi_1\urcorner$.

For $(*)$: There is no replacement. For $(***)$: For any $\chi_1 \in$ For we have that $V(\Box\chi_1) = v(\Box\chi_1)$. So we use the assumption $(\star_{\mathrm{PAt}})$.

For the inductive case $(**)$ we assume that the result holds for all sentences shorter than χ. So $V(\chi_1) = V(\chi_1[{}^{\varphi}/_{\psi}])$ and $V(\chi_2) = V(\chi_2[{}^{\varphi}/_{\psi}])$. We have: $V(\neg\chi_1) = V(\neg\chi_1[{}^{\varphi}/_{\psi}])$ and $V(\chi_1 \circ \chi_2) = V((\chi_1 \circ \chi_2)[{}^{\varphi}/_{\psi}])$, since $V \in \mathsf{Val}^{\mathsf{cl}}$.

"$\Rightarrow$" We put $v := V|_{\mathrm{PAt}}$. By the part "$\Leftarrow$", the unique extension of v by classical truth conditions for truth-value operators belongs to $\mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$ and it is equal to V.

2. "$\Leftarrow$" Suppose that φ is built by means of truth-value operators, different propositional letters $\alpha_1, \ldots, \alpha_n$ and different necessitations $\ulcorner\Box\chi_1\urcorner$, $\ldots$, $\ulcorner\Box\chi_m\urcorner$ $(n + m \geqslant 0)$.

If $m = 0$, i.e. φ is a classical formula, then $\varphi \in$ Taut. Moreover, $\varphi \in \mathbf{PL}$, if $m > 0$ but there is no $i, j = 1, \ldots, m$ such that $\chi_i = \chi_j[{}^{\psi}/_{\psi'}]$, for some $\psi, \psi' \in$ For such that $\ulcorner\psi \equiv \psi'\urcorner \in \mathbf{PL}$. Indeed, in none of both cases condition $(\star_{\mathrm{PAt}})$ is connected with φ, so this formula is true for an arbitrary valuation $v\colon \mathrm{PAt} \to \{0,1\}$.

Let us the assume that $m > 0$. We define the following equivalence relation in $\{\Box\chi_1, \ldots, \Box\chi_m\}$:

$$\Box\chi_i \; R \; \Box\chi_j \;\overset{\text{def}}{\Longleftrightarrow}\; \chi_i = \chi_j[{}^{\psi}/_{\psi'}],$$
$$\text{for some } \psi, \psi' \in \text{For such that } \ulcorner\psi \equiv \psi'\urcorner \in \mathbf{PL}.$$

If it is the identity relation in $\{\Box\chi_1, \ldots, \Box\chi_m\}$, then the second considered case holds.

Let $\|\varrho_1\|_R, \ldots, \|\varrho_k\|_R$ be different equivalence classes from $\{\Box\chi_1, \ldots, \Box\chi_m\}/_R$. For different formulae $\varrho_1, \ldots, \varrho_k$ we assign different propositional letters $\beta_1, \ldots, \beta_k$ (these letters are to be different as well from $\alpha_1, \ldots, \alpha_n$). All formulae from $\|\varrho_i\|_R$ are replaced by ϱ_i. We obtain the formula φ^*. Now, every ϱ_i is being replaced by β_i. In this way we obtain the classical formula φ^*_{cl}. By the assumption we have that $\varphi^*_{\text{cl}} \in \text{Taut}$. Replacing ϱ_i for β_i in φ^*_{cl} we obtain φ^*. Therefore $\varphi^* \in \mathbf{PL}$. The latest formula can be transformed into φ by suitable replacements (reverting to the initial ones) of formula ϱ_i. Thus $\varphi \in \mathbf{PL}_{\text{rte}}$.

3. "$\Rightarrow$" Let $\varphi \in \mathbf{PL}_{\text{rte}}$, i.e., there are $\tau \in \mathbf{PL}$ and $\psi_1 \ldots, \psi_k, \psi'_1, \ldots, \psi'_k \in$ For such that $\ulcorner\psi_1 \equiv \psi'_1\urcorner \in \mathbf{PL}, \ldots, \ulcorner\psi_k \equiv \psi'_k\urcorner \in \mathbf{PL}$ and $\varphi = \tau[{}^{\psi}/_{\psi'_1}, \ldots, {}^{\psi_k}/_{\psi'_k}]$. For any $V \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$ we have that $V(\tau) = 1$, because $\mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}} \subseteq \mathsf{Val}^{\mathsf{cl}}$. Thus, by $(\star)$, $V(\varphi) = V(\tau) = 1$.

"$\Leftarrow$" Let $\varphi \notin \mathbf{PL}_{\text{rte}}$. Then, by the part "$\Leftarrow$" of 2, for some $v\colon \text{PAt} \to \{0, 1\}$ which satisfies the condition $(\star_{\text{PAt}})$ we have that $V(\varphi) = 0$, where V is the unique extension of v by classical truth conditions for truth-value operators. Moreover, by 1, $V \in \mathsf{Val}^{\mathsf{cl}}_{\mathsf{rte}}$.

2. "$\Rightarrow$" By the part "$\Rightarrow$" of 1 and the part "$\Rightarrow$" of 3. ⊣

1.5. Strict classical logics. The logics S1, S0.9, S1° and S0.9°

After [1], we say that a logic $\boldsymbol{\Lambda}$ is *strict$_T$ classical* ("*traditionally strict classical*") iff $\Box\mathbf{PL} \subseteq \boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}$ is closed under "traditional replacement rule for strict equivalents":

$$\text{if } \ulcorner\varphi \succ\!\!\prec \psi\urcorner \in \boldsymbol{\Lambda} \text{ and } \chi \in \boldsymbol{\Lambda}, \text{ then } \chi[{}^{\varphi}/_{\psi}] \in \boldsymbol{\Lambda}. \qquad (\text{RRSE}_\text{T})$$

Moreover, a logic $\boldsymbol{\Lambda}$ is called *strict classical* iff $\Box\mathbf{PL} \subseteq \boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}$ is closed under the following replacement rule:

$$\text{if } \ulcorner\Box(\varphi \equiv \psi)\urcorner \in \boldsymbol{\Lambda} \text{ and } \chi \in \boldsymbol{\Lambda}, \text{ then } \chi[{}^{\varphi}/_{\psi}] \in \boldsymbol{\Lambda}. \qquad (\text{RRSE})$$

We obtain that for modal logics which contain (`K`) and/or (`X`), the above notions are equivalent. Firstly we notice that:

Lemma 1.22 ([1]). *Every strict$_T$ or strict classical logic is an rte-system.*

Secondly, by lemmas 1.11 and 1.22, and Corollary 1.17 we have that:

Lemma 1.23 ([1]). *For every logic $\boldsymbol{\Lambda}$ which contains* (K) *or* (X): *$\boldsymbol{\Lambda}$ is strict$_T$ classical iff $\boldsymbol{\Lambda}$ is strict classical.*

The logic **S0.9** (resp. **S1**) is the smallest strict classical logic which contains the formulae (T), $\Box$(T) and $\Box$(K) (resp. $\Box$(X)). For these logics see e.g. [1, 4, 6]. By lemmas 1.20 and 1.22, $\mathbf{S0.9} \subseteq \mathbf{S1}$. In [3] it was proved that $\mathbf{S0.9} \neq \mathbf{S1}$, since $\Box(\mathtt{X}) \notin \mathbf{S0.9}$ (see also e.g. [1, pp. 15–16]).

In [1] the Feys' logic $\mathbf{S1}^\circ$ from [2] is described as the smallest strict$_\mathrm{T}$ classical logic which contains the formulae (X) and $\Box$(X), and is closed under (SMP). In [8] the logic $\mathbf{S1}^\circ$ is described as the smallest strict$_\mathrm{T}$ classical logic which contains the formulae (X) and $\Box$(X), and is closed under (RN_*). By lemmas 1.12 and 1.22 both characterizations are equivalent.

Again by lemmas 1.20 and 1.22, and Corollary 1.17, $(\mathtt{K}), \Box(\mathtt{K}) \in \mathbf{S1}^\circ$. Since $(\mathtt{X}) \in \mathbf{S1}$ and **S1** is closed under (SMP), so $\mathbf{S1}^\circ \subseteq \mathbf{S1}$. Because $(\mathtt{T}), \Box(\mathtt{T}) \notin \mathbf{S1}^\circ$, so $\mathbf{S1}^\circ \neq \mathbf{S1}$ (see e.g. [1]).

Moreover, in [1] the logic $\mathbf{S0.9}^\circ$ is described as the smallest strict$_\mathrm{T}$ classical logic which contains the formulae (K) and $\Box$(K), and is closed under (SMP). We have $\mathbf{S0.9}^\circ \subseteq \mathbf{S1}^\circ$, because $(\mathtt{K}), \Box(\mathtt{K}) \in \mathbf{S1}^\circ$.

Since $(\mathtt{T}) \notin \mathbf{S0.9}^\circ$, $\Box(\mathtt{X}) \notin \mathbf{S0.9}$, $(\mathtt{K}) \in \mathbf{S0.9}$ and **S0.9** is closed under (SMP), so $\mathbf{S0.9}^\circ \subsetneq \mathbf{S0.9}$ and $\mathbf{S0.9}^\circ \subsetneq \mathbf{S1}^\circ$.

Notice that, by lemmas 1.12 and 1.22, the logics $\mathbf{S0.9}^\circ$, **S0.9**, $\mathbf{S1}^\circ$ and **S1** are also closed under (RN_*). We can describe the logic $\mathbf{S0.9}^\circ$ (resp. **S0.9**; $\mathbf{S1}^\circ$; **S1**) as the smallest logic which includes $\Box$Taut, is closed under (RN_*) and ($\mathrm{RRSE_T}$), and contains $\Box$(K) (resp. $\Box$(K) and $\Box$(T); $\Box$(X); $\Box$(X) and $\Box$(T)).

In the second part of this paper we shall prove that $\Box(\mathtt{K}), \Box(\mathtt{T}) \notin \mathbf{S0.5_{rte}}$, so $\mathbf{S0.5^\circ_{rte}} \subsetneq \mathbf{S0.9}^\circ$ and $\mathbf{S0.5_{rte}} \subsetneq \mathbf{S0.9}$.

In [1] the *Lewis version* $\mathbf{Lew}(\boldsymbol{\Lambda})$ of a logic $\boldsymbol{\Lambda}$ is understood as the smallest logic which includes $\boldsymbol{\Lambda}$ and contains the formula $\Box\top$, i.e. $\mathbf{Lew}(\boldsymbol{\Lambda}) := \boldsymbol{\Lambda}[\Box\top] = \mathbf{PL}[\boldsymbol{\Lambda}, \Box\top]$.

In [1] a logic is called *prenormal* iff it is congruential and contains the formula $\ulcorner\Box\top \supset (\mathtt{K})\urcorner$. Of course, every prenormal logic which contains $\Box\top$ is normal. In [1] were considered the logics **PK**, **PX**, **PKT** and **PXT** which are the smallest congruential logics respectively containing:

(K); (K) and (T); (X); (X) and (T). By Lemma 1.6, these logics contain (K), so also $\ulcorner\Box\top \supset \texttt{(K)}\urcorner$. Hence they are prenormal and we have that **PK** $\subseteq$ **PX** $\subseteq$ **PXT** and **PK** $\subseteq$ **PKT** $\subseteq$ **PXT**. In [1] it was proved that $\mathbf{S0.9}^\circ = \mathbf{Lew}(\mathbf{PK}) := \mathbf{PK}[\Box\top]$, $\mathbf{S0.9} = \mathbf{Lew}(\mathbf{PKT}) := \mathbf{PKT}[\Box\top]$, $\mathbf{S1}^\circ = \mathbf{Lew}(\mathbf{PX}) := \mathbf{PX}[\Box\top]$ and $\mathbf{S1} = \mathbf{Lew}(\mathbf{PXT}) := \mathbf{PXT}[\Box\top]$.

Finally, notice that the logics **S1**, **S0.9**, $\mathbf{S1}^\circ$ and $\mathbf{S0.9}^\circ$ are not congruential and that the formulae $\Box$(M), $\Box$(C) and

$$\Box p \prec \Box(p \vee q) \tag{1.1}$$

$$\Diamond(p \wedge q) \prec \Diamond p \tag{1.2}$$

are not members of **S1**, while the formulae (M), (C), '$\Box p \supset \Box(p \vee q)$' and '$\Diamond(p \wedge q) \supset \Diamond p$' belong to **C1**.

1.6. The logics $\mathbf{S2}^\circ$ and S2

We say the a logic $\boldsymbol{\Lambda}$ is closed under *Becker's rule* iff for any $\varphi, \psi \in$ For:

$$\text{if } \ulcorner\varphi \prec \psi\urcorner \in \boldsymbol{\Lambda}, \text{ then } \ulcorner\Box\varphi \prec \Box\psi\urcorner \in \boldsymbol{\Lambda}. \tag{RB}$$

In [4] the logic **S2** is described as the smallest modal logic which includes $\Box$Taut, contains the formulae (T), $\Box$(T), and $\Box$(K), and is closed under (RB). Of course, **S2** includes $\Box$**PL**, contains (K) and, by Lemma 1.12, it is closed under (RN_*) and (SMP).

Moreover, the logic $\mathbf{S2}^\circ$ is described in [8] as the smallest logic which includes $\Box$Taut, contains $\Box$(K), and is closed under (RB) and (RN_*). Of course, $\mathbf{S2}^\circ$ includes $\Box$**PL**, contains (K) and, by Lemma 1.12, it is closed under (SMP). So $\mathbf{S2}^\circ \subsetneq \mathbf{S2}$. For example $\texttt{(T)}, \Box\texttt{(T)} \notin \mathbf{S2}^\circ$.

In [4] Lemmon proved that $\Box\texttt{(X)} \in \mathbf{S2}$ and **S2** is closed under ($\mathrm{RRSE_T}$). His proof shows that also $\Box\texttt{(X)} \in \mathbf{S2}^\circ$ and $\mathbf{S2}^\circ$ is closed under ($\mathrm{RRSE_T}$). So we have that $\mathbf{S1}^\circ \subsetneq \mathbf{S2}^\circ$ and $\mathbf{S1} \subsetneq \mathbf{S2}$. Thus, **S2** and $\mathbf{S2}^\circ$ are $\mathrm{strict_T}$ and strict classical, but they are not congruential.

In [1] it was proved that $\mathbf{S2}^\circ = \mathbf{Lew}(\mathbf{C2}) := \mathbf{C2}[\Box\top]$ and $\mathbf{S2} = \mathbf{Lew}(\mathbf{E2}) := \mathbf{E2}[\Box\top]$. Moreover, for every $\varphi \in$ For:

$$\ulcorner\Box\varphi\urcorner \in \mathbf{S2}^\circ \quad \text{iff} \quad \varphi \in \mathbf{C2}, \tag{1.3}$$

$$\ulcorner\Box\varphi\urcorner \in \mathbf{S2} \quad \text{iff} \quad \varphi \in \mathbf{E2}. \tag{1.4}$$

Hence, the formulae $\Box$(M), $\Box$(C), (1.1) and (1.2) belong to $\mathbf{S2}^\circ$, because (M), (C), '$\Box p \supset \Box(p \vee q)$' and '$\Diamond(p \wedge q) \supset \Diamond p$' belong to **C1**.

2. Some new weak t-normal logics and t-normal rte-logics

In the present paper we examine some logics which are not strict classical, but these logics have the form $\boldsymbol{\Lambda}[\Box\Phi]$, where $\Phi \subseteq \mathbf{S0.5}$ and $\boldsymbol{\Lambda} = \mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{T_q}]$, $\mathbf{S0.5}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}[\mathtt{D}]$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}[\mathtt{T_q}]$, $\mathbf{S0.5_{rte}}$.

Remark 2.1. By Lemma 1.15, if a logic $\boldsymbol{\Lambda}$ is t-regular (resp. a t-regular rte-system) and $\Phi \neq \emptyset$, then $\boldsymbol{\Lambda}[\Box\Phi]$ is t-normal (resp. a t-normal rte-system).

For example, $\mathbf{C1}[\Box\Phi] = \mathbf{S0.5}^\circ[\Box\Phi]$, where $\Phi \neq \emptyset$. Similarly for t-regular logics $\mathbf{D1}$, $\mathbf{C1}[\mathtt{T_q}]$, $\mathbf{E1}$, $\mathbf{C1_{rte}}$, $\mathbf{D1_{rte}}$, $\mathbf{C1_{rte}}[\mathtt{T_q}]$, $\mathbf{E1_{rte}}$ and suitable t-normal logics $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{T_q}]$, $\mathbf{S0.5}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}[\mathtt{D}]$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}[\mathtt{T_q}]$, $\mathbf{S0.5_{rte}}$. ⊣

Remark 2.2. As we remember (see p. 42) the formulae (†) do not belong to $\mathbf{S0.5}$. The formula ($\dagger_{\mathrm{a}}$) belongs to $\mathbf{S0.5}[\Box\mathtt{K}, \Box(\Box p \supset \Box\neg\neg p)]$, where '$\Box p \supset \Box\neg\neg p$' $\in \mathbf{C1}$. But $\Box(\dagger)$ and

$$\begin{array}{ll} \text{a)} & \Box\Box\Box p \supset \Box\Box\Box\neg\neg p \\ \text{b)} & \Box\Box\Box\neg\neg p \supset \Box\Box\Box p \end{array} \qquad (\ddagger)$$

do not belong to $\mathbf{S0.5}[\Box\mathbf{S0.5}]$; so this logic is not an rte-system (see the second part). ⊣

In Section 3 for logics $\boldsymbol{\Lambda}[\Box\Phi]$, where $\boldsymbol{\Lambda} = \mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ[\mathtt{D}]$, $\mathbf{S0.5}^\circ[\mathtt{T_q}]$, $\mathbf{S0.5}$, we give simplified semantics formulated by means of some Kripke-style models. In Section 4 we give similar semantics for logics $\boldsymbol{\Lambda}[\Box\Phi]$, where $\boldsymbol{\Lambda} = \mathbf{S0.5}^\circ_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}[\mathtt{D}]$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}[\mathtt{T_q}]$, $\mathbf{S0.5_{rte}}$. In Section 5 we prove that considered logics are determined by some classes of these models.

Firstly notice that by Lemma 1.20 we obtain:

Corollary 2.1. *For any rte-logic* $\boldsymbol{\Lambda}$: $\boldsymbol{\Lambda}[\Box\Phi, \Box\mathtt{X}] = \boldsymbol{\Lambda}[\Box\Phi, \Box\mathtt{K}, \Box\mathtt{X}]$.

By facts from Section 1 and Corollary 2.1 we obtain:

Fact 2.2.
1. $\mathbf{S0.5}^\circ[\Box\mathtt{K}] \subseteq \mathbf{S0.5}^\circ_{\mathbf{rte}}[\Box\mathtt{K}] \subseteq \mathbf{S0.9}^\circ$.
2. $\mathbf{S0.5}[\Box\mathtt{K}, \Box\mathtt{T}] \subseteq \mathbf{S0.5_{rte}}[\Box\mathtt{K}, \Box\mathtt{T}] \subseteq \mathbf{S0.9}$.
3. $\mathbf{S0.5}^\circ[\Box\mathtt{K}, \Box\mathtt{X}] \subseteq \mathbf{S0.5}^\circ_{\mathbf{rte}}[\Box\mathtt{X}] \subseteq \mathbf{S1}^\circ$.
4. $\mathbf{S0.5}[\Box\mathtt{T}, \Box\mathtt{K}, \Box\mathtt{X}] \subseteq \mathbf{S0.5_{rte}}[\Box\mathtt{X}, \Box\mathtt{T}] \subseteq \mathbf{S1}$.

Moreover, we have:

LEMMA 2.3. *For any t-regular logic* $\boldsymbol{\Lambda}$ *and* $\Phi, \Psi \subseteq \mathrm{For}$, *if* $\Psi \models_{\mathbf{PL}} \Phi$, *then* $\boldsymbol{\Lambda}[\Box\Phi] \subseteq \boldsymbol{\Lambda}[\Box\Psi]$.

PROOF: Suppose that $\Psi \models_{\mathbf{PL}} \Phi$, i.e., for every $\varphi \in \Phi$ there is a subset $\{\psi_1, \dots, \psi_n\}$ of Ψ, $n \geqslant 0$, such that $\ulcorner(\psi_1 \wedge \cdots \wedge \psi_n) \supset \varphi\urcorner \in \mathbf{PL}$. Since $\boldsymbol{\Lambda}$ is t-regular, $\ulcorner(\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \supset \Box\varphi\urcorner \in \boldsymbol{\Lambda}$. Hence, $\Box\varphi \in \boldsymbol{\Lambda}[\Box\Psi]$, since $\Box\psi_1, \dots, \Box\psi_n \in \boldsymbol{\Lambda}[\Box\Psi]$. ⊣

By the above lemma we obtain:

COROLLARY 2.4. *For any r-regular logic* $\boldsymbol{\Lambda}$: $\boldsymbol{\Lambda}[\Box\Phi, \Box\mathtt{C}] \subseteq \boldsymbol{\Lambda}[\Box\Phi, \Box\mathtt{R}]$, $\boldsymbol{\Lambda}[\Box\Phi, \Box\mathtt{N}] \subseteq \boldsymbol{\Lambda}[\Box\Phi, \Box\mathtt{R}]$ *and* $\boldsymbol{\Lambda}[\Box\Phi, \Box\mathtt{C}, \Box\mathtt{N}] = \boldsymbol{\Lambda}[\Box\Phi, \Box\mathtt{R}]$.

From the facts (1.3) and (1.4) we have:

FACT 2.5. 1. *If* $\Phi \subseteq \mathbf{C2}$, *then* $\mathbf{S0.5}^{\circ}_{\mathbf{rte}}[\Box\Phi] \subseteq \mathbf{S2}^{\circ}$.
2. *If* $\Phi \subseteq \mathbf{E2}$, *then* $\mathbf{S0.5_{rte}}[\Box\Phi] \subseteq \mathbf{S2}$.

However in the present paper we are only interested in such a set $\Box\Phi$, as a set of new axioms, which satisfies condition $\Phi \subseteq \mathbf{S0.5}$. Notice that we have the following facts:

$$\mathbf{C1} = \mathbf{C2} \cap \mathbf{S0.5}^{\circ}, \tag{2.1}$$

$$\mathbf{C1} \subsetneq \mathbf{C2} \cap \mathbf{S0.5} \not\subseteq \mathbf{S0.5}^{\circ}\,, \tag{2.2}$$

$$\mathbf{E1} = \mathbf{E2} \cap \mathbf{S0.5}\,. \tag{2.3}$$

We have: $\mathbf{C1} \subsetneq \mathbf{C2}$, $\mathbf{C1} \subsetneq \mathbf{S0.5}^{\circ} \subsetneq \mathbf{S0.5}$, $\mathbf{E1} \subsetneq \mathbf{E2}$ and $\mathbf{E1} \subsetneq \mathbf{S0.5}$. The remaining facts we will obtain from the semantics presented in [6] (see Fact 3.12 in the second part of this paper).

Therefore the following corollary will be of crucial importance:

COROLLARY 2.6. 1. *If* $\Phi \subseteq \mathbf{C2} \cap \mathbf{S0.5}$, *then* $\mathbf{S0.5}^{\circ}_{\mathbf{rte}}[\Box\Phi] \subseteq \mathbf{S2}^{\circ}$.
2. *If* $\Phi \subseteq \mathbf{E1}$, *then* $\mathbf{S0.5_{rte}}[\Box\Phi] \subseteq \mathbf{S2}$.

In Section 6 (see Corollary 6.3 in the second part) we prove that in the subsequents in the above corollary the symbol '$\subseteq$' can be replaced by '$\subsetneq$'.

References

[1] Chellas, B. F., and K. Segerberg, *Modal logics in the vicinty of S1*, **Notre Dame Journal of Formal Logic** 37, 1 (1996), pp. 1–24.

[2] Feys, R., *Les systèmes formalisés des modalités aristotéliciennes*, **Revue Pilosophique de Louvain** 48 (1950), pp. 478–509. Also: R. Feys, **Modal Logics**. Louvain 1965: E. Nauwelaerta.

[3] Girle, R. A., S1 $\neq$ S0.9, **Notre Dame Journal of Formal Logic** 16 (1975), pp. 339–344.

[4] Lemmon, E. J., *New fundations for Lewis modal systems*, **The Journal of Symbolic Logic** 22, 2 (1957), pp. 176–186.

[5] Lewis, C. I., and C. H. Langford, **Symbolic Logic**, New York, 1932.

[6] Pietruszczak, A., *Simplified Kripke-style semantics for some very weak modal logics*, **Logic end Logical Philosophy** 18 (2009), pp. 271–296.

[7] Routley, R., *Decision procedure and semantics for* C1*,* E1 *and* S0.5°, **Logique et Analyse** 44 (1968), pp. 468–469.

[8] Zeman, J. J., *Lemmon-style bases for the systems S1°–S4°*, **The Journal of Symbolic Logic** 33, 3 (1968), pp. 458–461.

Nicolaus Copernicus University
Department of Logic
ul. Asnyka 2
87-100 Toruń, Poland
e-mail: pietrusz@uni.torun.pl