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Klaus Denecke
Hans-Jürgen Vogel (Editors)

General Algebra and Applications

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"59th Workshop on General Algebra",
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Contents

On some lattices of varieties related to changes of the type

K. Mruczek

Abstract

In several papers identities of some special structural forms were considered, e.g. regular identities (see [8]), normal identities (see [7] and [9]), externally compatible identities (see [2]). In those papers for a given variety V of type τ one chooses a subset S of the set $Id(V)$ of all identities of type τ satisfied in V such that S is an equational theory and one assignes to V a new variety $Mod(S)$. It seems interesting to give a full description of the lattice of all subvarieties of the variety $Mod(S)$.
The structure of the identities is connected with the type of the given varieties. The question was asked how much the diagram of the lattice of subvarieties of the variety defined by externally compatible identities of a given variety will be influenced by changing the type of algebras. In general cases the answer to this question seems to be very complicated, so we restrict overselves to the variety of Abelian groups with exponent q,where q is a prime.
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Key words: Abelian groups, identity, P-compatible identity, variety, lattice

1 Preliminaries

We consider a given type of algebras $\tau : F \to N$, where F is a set of fundamental operation symbols and N is a set of non-negative integers. Let P be a partition of F. An identity $\varphi \approx \psi$ is called P-compatible (see [9]) if it is of the form $x \approx x$ or of the form $f(\varphi_0, \dots, \varphi_{\tau(f)-1}) \approx g(\psi_0, \dots, \psi_{\tau(g)-1})$ where f and g belong to the same block $[f]_P$ of P and $\varphi_0, \dots, \varphi_{\tau(f)-1}, \psi_0, \dots, \psi_{\tau(g)-1}$ are terms of type τ. An identity $\varphi \approx \psi$ is externally compatible if and only if it is P-compatible, where P contains

singletons only. An identity $\varphi \approx \psi$ is normal if it is of the form $x \approx x$ or neither φ nor ψ is a variable.
We will use the following denotations:

> $P(\tau)$ - the set of all P-compatible identities of type τ,
>
> $P(V) = P(\tau) \cap \mathrm{Id}\ V$ - the set of all P-compatible identities of V,
>
> $Ex(\tau)$ - the set of all externally compatible identities of type τ, i.e. P-compatible for $P = \{\{f_i\} | i \in I\}$,
>
> $N(\tau)$ - the set of all normal identities of type τ, i.e. P-compatible for $P = \{\{f_i | i \in I\}\}$.

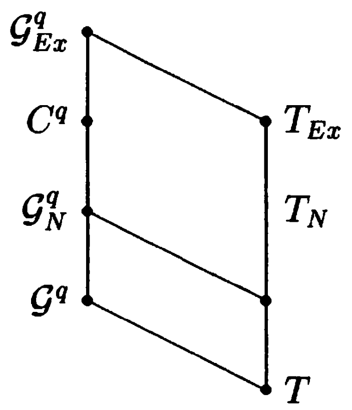
Obviously, $Ex(V) \subseteq P(V) \subseteq N(V) \subseteq Id(V)$ for any partition P.
If Σ is a set of identities of type τ then $Cn(\Sigma)$ denotes the closure of Σ under the rules of inference (1)-(5) (see [4] , p.170). It is easy to see that $P(\tau)$ and $P(V)$ are equational theories, i.e. closed under the operator C_n. By $Mod(\Sigma)$ we denote the class of all models of Σ that is, the class of all algebras of type τ satisfying all identities from Σ. So, if $\Sigma = P(V)$ then $Mod(\Sigma) = V_P$. A variety V such that $Id(V) = P(V)$ is called P-compatible. Let $\mathcal{L}(\mathcal{V})$ be the lattice of all subvarieties of V.
It is well known that the lattice of all equational theories of type τ is dually isomorphic to the lattice of all varieties of the same type. Let Σ be an equational theory. The lattice $\mathcal{L}(\pm)$ of all equational theories extending Σ is dually isomorphic to the lattice $\mathcal{L}(Mod(\Sigma))$.

2 The variety $\mathcal{G}^q_{Ex}$ of type τ_0

Let us fix the type τ_0, where $\tau_0 : \{\cdot, ^{-1}\} \to N$, $\tau_0(\cdot) = 2, \tau_0(^{-1}) = 1$. As usually, we denote $x^0 = x \cdot x^{-1}$, $x^n = x^{n-1} \cdot x$ for $n \in N$.
Of course, for the set $\{\cdot, ^{-1}\}$ we have only two partitions: $Ex = \{\{\cdot\}, \{^{-1}\}\}$, $N = \{\{\cdot, ^{-1}\}\}$.
Let $\mathcal{G}^{\text{II}}$ denote the variety of all Abelian groups of type τ_0 satisfying the identity $x^q \approx y^0$, where q is a prime and let $\mathcal{G}^p_{Ex}$denote the variety defined by all externally compatible identities of Abelian groups with exponent q of type τ_0. The identity $(x \cdot y)^{-1} \approx x^{-1} \cdot y^{-1}$ belongs to $Id(\mathcal{G}^q)$, but it is not externally compatible. It means that the variety $\mathcal{G}^q_{Ex}$ is larger then $\mathcal{G}^q$ and it may be interesting to characterize the lattice of all subvarieties of $\mathcal{G}^q_{Ex}$.
In [6] it was proved that the following identities:
$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$,
$x \cdot y \approx y \cdot x$,

$x^q \approx y^0,$
$(x \cdot y^0)^{-1} \approx x^{-1},$
$(x \cdot y) \cdot z^0 \approx x \cdot y.$
form an equational base of the variety $\mathcal{G}^q_{Ex}$. From the same paper we know that

Theorem 2.1 *If q is a prime then the lattice $\mathcal{L}(\mathcal{G}^q_{Ex})$ has the following diagram:*

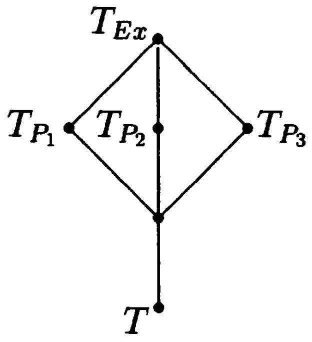


where $C^q = Mod(Ex(\mathcal{G}^q) \cup \{\cdot x^{-1} \approx (x \cdot x^{-1})^{-1}\})$ and T is the degenerated variety of type τ_0.

3 The lattice of all subvarieties of the variety $\mathcal{G}^q_{Ex}$ of type τ_1

All identities, equational theories and varieties we deal in this section are considered in type $\tau_1 : \{\cdot, ^{-1}, e\} \to N$, where $\tau_1(\cdot) = 2, \tau_1(^{-1}) = 1, \tau_1(e) = 0$. Put $P_0 = \{\{\cdot\}, \{^{-1}\}, \{e\}\}$, $P_1 = \{\{\cdot, ^{-1}\}, \{e\}\}$,, $P_2 = \{\{\cdot, e\}, \{^{-1}\}\}$, $P_3 = \{\{^{-1}, e\}, \{\cdot\}\}$, $P_4 = \{\{\cdot, ^{-1}, e\}\}$. An identity is P_0-compatible if and only if it is externally compatible, so the partition P_0 will be denoted by Ex, and analogously the partition P_4 will be denoted by N. We see that Ex, P_1, P_2, P_3 and N are all partitions of $\{\cdot, ^{-1}, e\}$.
Let us denote by T the degenerated variety. It is easy to verify that the set $\{x \cdot y \approx z \cdot t, x^{-1} \approx y^{-1}\}$ is an equational base of T_{Ex} and the sets: $\{x \cdot y \approx z^{-1}\}$, $\{x \cdot y \approx e\}$, $\{x^{-1} \approx e\}$, $\{x \cdot y \approx z^{-1} \approx e\}$ are equational bases of $T_{P_1}, T_{P_2}, T_{P_3}, T_N$, respectively.
We have the following lemma:

Lemma 3.1 *The lattice $\mathcal{L}(T_{Ex})$ af all subvarieties of the variety T_{Ex} has the following diagram:*



Let $(\Pi_F + 1)^d$ be the lattice dual to the lattice $\Pi_F + 1$ of all partitions of F with an additional greatest element 1.

Theorem 3.2 *The function $h : (\Pi_F + 1)^d \to \mathcal{L}(\mathcal{G}^q_{Ex})$ defined as follows:*
$h(1) = \mathcal{G}^q$
$h(P) = \mathcal{G}^q_P$
is order-preserving and injective.

Proof. Consider $P_i, P_j \in \Pi_F$, where $i, j \in \{0, \ldots, 4\}$ and $P_i < P_j$. Since $P_i(\mathcal{G}^q) \subseteq P_j(\mathcal{G}^q)$, we obtain $\mathcal{G}^q_{P_j} \subseteq \mathcal{G}^q_{P_i}$. Obviously $1 > P_i$ for $i \in \{0, \ldots, 4\}$ and this implies $P_i(\mathcal{G}^q) \subseteq Id(\mathcal{G}^q)$. Hence $\mathcal{G}^q \subseteq \mathcal{G}^q_{P_i}$. It follows immediately that h is order-preserving.
It is easy to show that for any $f, g \in F$ there exists an identity $\varphi \approx \psi$ in $Id(\mathcal{G}^q)$ such that f and g are the most external fundamental operation symbols in the terms φ and ψ, respectively. So we have that h is injection. ■
We can observe that the function h for the variety $\mathcal{G}^q_{Ex}$ of type τ_0 is a lattice embedding (this is a consequence of a more general result of K. Hałkowska see[5]) For the type τ_1: $P_2 \vee P_3 = P_N$, and it is obvious that $\mathcal{G}^q_N \subseteq \mathcal{G}^q_{P_2} \wedge \mathcal{G}^q_{P_3}$ (this is a consequence of Theorem 3.2), but $\mathcal{G}^q_{P_2} \wedge \mathcal{G}^q_{P_3} \neq \mathcal{G}^q_N$.

Theorem 3.3 *The function $h : (\Pi_F + 1)^d \to \mathcal{L}(\mathcal{G}^q_{Ex})$ defined as follows:*
$h(1) = \mathcal{G}^q$
$h(P) = Mod(Ex(\mathcal{G}^q) \cup E_P)$,
where $E_P = \{f(e, \ldots, e) \approx g(e, \ldots, e) : f, g \in \{\cdot, ^{-1}, e\}, g \in [f]_P\}$ is a lattice embedding.

Proof. Let us note that the following identities are true:
$Cn(Ex(\mathcal{G}^q) \cup E_{Ex}) = Cn(Ex(\mathcal{G}^q))$,
$Cn(Ex(\mathcal{G}^q) \cup E_{P_1}) = Cn(Ex(\mathcal{G}^q) \cup \{e \cdot e \approx e^{-1}\})$,

$Cn(Ex(\mathcal{G}^q) \cup E_{P_2}) = Cn(Ex(\mathcal{G}^q) \cup \{e \approx e \cdot e\})$,
$Cn(Ex(\mathcal{G}^q) \cup E_{P_3}) = Cn(Ex(\mathcal{G}^q) \cup \{e \approx e^{-1}\})$,
$Cn(Ex(\mathcal{G}^q) \cup E_N) = Cn(Ex(\mathcal{G}^q) \cup \{e \approx e \cdot e, e \approx e^{-1}\})$.
As an immediate consequence of these identities we have the following:
$Mod(Cn(Ex(\mathcal{G}^q) \cup E_{Ex})) = \mathcal{G}^q_{Ex}$
$Mod(Cn(Ex(\mathcal{G}^q) \cup E_{P_2})) = \mathcal{G}^q_{P_2}$
$Mod(Cn(Ex(\mathcal{G}^q) \cup E_{P_3})) = \mathcal{G}^q_{P_3}$.
Let C^q, C^q_N denote the classes $Mod(Cn(Ex\mathcal{G}^q \cup E_{P_1}))$, $Mod(Cn(Ex\mathcal{G}^q \cup E_N))$, respectively. It is clear that h is order-preserving and injective. We have to show that:
$C^q \wedge \mathcal{G}^q_{P_2} = C^q_N$,
$C^q \wedge \mathcal{G}^q_{P_3} = C^q_N$,
$\mathcal{G}^q_{P_2} \wedge \mathcal{G}^q_{P_3} = C^q_N$.
Obviously $C^q \wedge \mathcal{G}^q_{P_2} = Mod(Cn((Ex(\mathcal{G}^q) \cup \{e \cdot e \approx e^{-1}\}) \cup (Ex(\mathcal{G}^q) \cup \{e \approx e^{-1}\})))$. Since $Mod(Cn((Ex(\mathcal{G}^q) \cup \{e \cdot e \approx e^{-1}\}) \cup (Ex(\mathcal{G}^q) \cup \{e \approx e^{-1}\}))) = Mod(Cn(Ex(\mathcal{G}^q) \cup \{e \cdot e \approx e^{-1}, e \approx e^{-1}\}))$ so $C^q \wedge \mathcal{G}^q_{P_2} = C^q_N$. We act similarly in the remaining cases. ■
In [3] it was proved that if E is a finite set of identities of type τ_0 then there exists a set E_1 of one variable identities of type τ_0 such that $Cn(Ex(\mathcal{G}^q) \cup E) = Cn(Ex(\mathcal{G}^q) \cup E_1)$.
It is not difficult to check, that a theorem analogously to this theorem is true for type τ_1. So, we have the next theorem:

Theorem 3.4 *Let τ_1 be a type of Abelian groups with exponent q and $\mathcal{A}$ be a free algebra in the variety $\mathcal{G}^q_{Ex}$ with a one element set of generators. Then $Id(\mathcal{A}) = Id(\mathcal{G}^q_{Ex})$.*

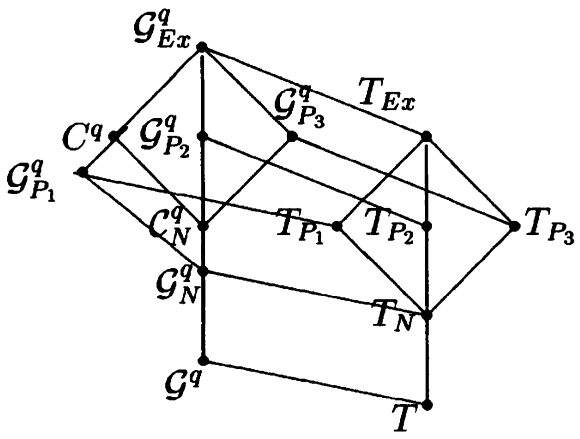
In the next lemma we construct a free algebra in the variety $\mathcal{G}^q_{Ex}$.

Lemma 3.5 *The algebra $\mathcal{A} = \{\{x, e, e \cdot e, e^{-1}, e \cdot x, e \cdot x^2, \ldots, e \cdot x^{q-1}, (x^{-1})^{-1}, ((x^2)^{-1})^{-1}, \ldots, ((x^{q-1})^{-1})^{-1}\}, \cdot, ^{-1}, e\}$ is free in $\mathcal{G}^q_{Ex}$ with one generator x.*

From the last theorem we obtain the following:

Theorem 3.6 *If V is a subvariety of the variety $\mathcal{G}^q_{Ex}$ of type τ_1 then V is one of the classes: $T, T_N, T_{P_1}, T_{P_2}, T_{P_3}, T_{Ex}, C^q_N, \mathcal{G}^q, \mathcal{G}^q_{P_1}, \mathcal{G}^q_{P_2}, \mathcal{G}^q_{P_3}, \mathcal{G}^q_N, C^q, \mathcal{G}^q_{Ex}$.*

Corollary 3.7 *The lattice $\mathcal{L}(\mathcal{G}^q_{Ex})$ of all subvarieties of the variety $\mathcal{G}^q_{Ex}$ of type τ_1 has the following diagram:*



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